PEGN 620A: Naturally Fractured Reservoir
Homework 10 : Solution

Given Equations

1. Fracture pressure equation:

$$\nabla\cdot\{k_{f,eff}[\lambda_{tf}^{n}\nabla P_{of}^{n+1}-(\lambda_{wf}^{n}\gamma_{w}^{n}+\lambda_{of}^{n}\gamma_{o}^{n})\nabla D-\lambda_{wf}^{n}\nabla P_{cwof}^{n}]\}-\tau_{t}^{n+1}+\hat{q}_{tf}^{n+1}=(\phi c_t)_f\frac{\partial P_{of}}{\partial t} \quad (1)$$

2. Total transfer function:

$$\tau_{t}^{n+1}=\sigma k_m[\lambda_{tf}^{n}(P_{of}^{n+1}-P_{om}^{n+1})+\frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{n}\gamma_{w}^{n}+\lambda_{om/f}^{n}\gamma_{o}^{n})(h_{wf}^{n}-h_{wm}^{n})+\lambda_{wf/m}^{n}(P_{cwom}^{n}-P_{cwof}^{n})] \quad (2)$$

3. Total matrix flow:

$$\tau_{t}^{n+1}=(\phi c_t)_m\frac{P_{om}^{n+1}-P_{om}^{n}}{\Delta t} \quad (3)$$

4. Saturation equation:

   (a) Option 1: IMPES

$$\nabla\cdot\left\{k_{f,eff}\lambda_{wf}^{n}\left[\nabla P_{of}^{[n+1]}-\gamma_{w}^{n}\nabla D-\nabla P_{cwof}^{n}\right]\right\}-\tau_{w}^{[n+1]}+\hat{q}_{wf}=\phi_f S_{wf}^{n+1}(c_\phi+c_w)_f\frac{P_{of}^{[n+1]}-P_{of}^{n}}{\Delta t}+\phi_f\frac{S_{wf}^{n+1}-S_{wf}^{n}}{\Delta t} \quad (4)$$

   (b) Option 2: Partially Implicit

$$\nabla\cdot\left\{k_{f,eff}\lambda_{wf}^{n}\left[\nabla P_{of}^{[n+1]}-\gamma_{w}^{n}\nabla D-\frac{\partial P_{cwof}^{n}}{\partial S_{wf}}\nabla S_{wf}^{n+1}\right]\right\}-\tau_{w}^{[n+1]}+\hat{q}_{wf}=\phi_f S_{wf}^{n+1}(c_\phi+c_w)_f\frac{P_{of}^{[n+1]}-P_{of}^{n}}{\Delta t}+\phi_f\frac{S_{wf}^{n+1}-S_{wf}^{n}}{\Delta t} \quad (5)$$

5. Water transfer

$$\tau_{w}^{[n+1]}=\sigma k_m\left[\lambda_{wf}^{n}(P_{of}^{[n+1]}-P_{om}^{[n+1]})+\frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{n}\gamma_{w}^{n})(h_{wf}^{n}-h_{wm}^{n})+\lambda_{wf/m}^{n}(P_{cwom}^{n}-P_{cwof}^{n})\right] \quad (6)$$

6. Water matrix flow:

$$\tau_w^{[n+1]} = \phi_m S_{wm}^{n+1}(c_\phi + c_w)_m \frac{P_{om}^{[n+1]} - P_{om}^n}{\Delta t} + \phi_m \frac{S_{wm}^{n+1} - S_{wm}^n}{\Delta t} \tag{7}$$

Re-call HW 9, Pressure equation can be discretized for 1D problem and rearrange as follows

$$DP_{of,i-1}^{n+1} + EP_{of,i}^{n+1} + FP_{of,i+1}^{n+1} = RHS \tag{8}$$

Using Tayler's series expansion as $P_{of,i}^{n+1} = P_{of,i}^n + \Delta t \frac{\partial P_{of,i}}{\partial t}$,

Discretize above equation we get $P_{of,i}^{n+1} = P_{of,i}^n + \Delta t \frac{P_{of,i}^{n+1} - P_{of,i}^n}{\Delta t}$.

Let's define $\Delta_t P_{of} = P_{of,i}^{n+1} - P_{of,i}^n$, thus $P_{of,i}^{n+1} = P_{of,i}^n + \Delta_t P_{of}$. Using this definition we can rearrange Eq (8) as follows

$$D\Delta_t P_{of,i-1} + E\Delta_t P_{of,i} + F\Delta_t P_{of,i+1} = RHS - DP_{of,i-1}^n - EP_{of,i}^n - FP_{of,i+1}^n \tag{9}$$

where

$$D = T_{x,i-\frac{1}{2}}^n$$
$$F = T_{x,i+\frac{1}{2}}^n$$
$$E = -\left(T_{x,i+\frac{1}{2}}^n + T_{x,i-\frac{1}{2}}^n + V_i \frac{(\phi c_t)_f}{\Delta t} + V_R \frac{\alpha \sigma k_m \lambda_{tf}^n}{1 + \frac{\alpha \sigma k_m \lambda_{tf}^n \Delta t}{(\phi c_t)_m}} \right)$$

$$RHS = TransferTerm + GravityTerm + CapillaryTerm - V_R \frac{(\phi c_t)_f}{\Delta t} P_{i,of}^n - q_{tf}$$

$$TransferTerm = V_R \frac{\alpha \sigma k_m}{(1 + \frac{\alpha \sigma k_m \lambda_{tf}^n \Delta t}{(\phi c_t)_m})} \left[-\lambda_{tf}^n P_{i,om}^n + \frac{\sigma_z}{\sigma}(\lambda_{wf/m}^n \gamma_w^n + \lambda_{om/f}^n \gamma_o^n)(h_{wf}^n - h_{wm}^n) + \lambda_{wf/m}^n (P_{cwom}^n - P_{cwof}^n) \right]$$

$$GravityTerm = (T^n_{xw}\gamma^n_w + T^n_{xo}\gamma^n_o)_{i+\frac{1}{2}}(D_{i+1} - D_i) - (T^n_{xw}\gamma^n_w + T^n_{xo}\gamma^n_o)_{i+\frac{1}{2}}(D_i - D_{i-1})$$

$$CapillaryTerm = T^n_{xw,i+\frac{1}{2}}(P^n_{cwof,i+1} - P^n_{cwof,i}) - T^n_{xw,i-\frac{1}{2}}(P^n_{cwof,i} - P^n_{cwof,i-1})$$

$$T^n_{x,i+\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{tf}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T^n_{x,i-\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{tf}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T^n_{xw,i+\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{wf}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T^n_{xw,i-\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{wf}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T^n_{xo,i+\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{of}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T^n_{xo,i-\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{of}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$\alpha = 0.006328$$

A OPTION 1:IMPES

To solve saturation equation, multiply Eq (4) by V_R, where $V_R = (\Delta x \Delta y \Delta z)_{i,j,k}$ and rearrange as follows

$$\left(\begin{array}{c} \overbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^n \nabla P_{of}^{[n+1]}}^{Pressure-term} - \overbrace{V_R \nabla \cdot k_{f,eff} (\lambda_{wf}^n \gamma_w^n) \nabla D}^{Gravity-term} \\ - \underbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^n \nabla P_{cwof}^n}_{Capillary-term} - V_R \tau_w^{[n+1]} + q_{wf}^{[n+1]} \end{array} \right) = \overbrace{V_R \phi_f \left(S_w^{n+1} (c_\phi + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^n}{\Delta t} + \frac{S_w^{n+1} - S_w^n}{\Delta t} \right)}^{Accumulation-term} \tag{10}$$

Discretize Eq (10) for 1D problem term by term we get

- Pressure Term
$$T_{wx,i+\frac{1}{2}}^n \left(P_{of,i+1}^{[n+1]} - P_{of,i}^{[n+1]} \right) - T_{wx,i-\frac{1}{2}}^n \left(P_{of,i}^{[n+1]} - P_{of,i-1}^{[n+1]} \right)$$
- Gravity Term
$$(T_{wx}^n \gamma_w)_{i+\frac{1}{2}} (D_{i+1} - D_i) - (T_{wx}^n \gamma_w)_{i-\frac{1}{2}} (D_i - D_{i-1})$$
- Capillary Term
$$T_{wx,i+\frac{1}{2}}^n \left(P_{cwof,i+1}^{[n+1]} - P_{cwof,i}^{[n+1]} \right) - T_{wx,i-\frac{1}{2}}^n \left(P_{cwof,i}^{[n+1]} - P_{cwof,i-1}^{[n+1]} \right)$$
- Accumulation Term
$$S_{wf}^{n+1} V_R \left(\phi_f (c_\phi + c_w)_f \frac{P_{of}^{n+1} - P_{of}^n}{\Delta t} + \frac{\phi_f}{\Delta t} \right) - \phi_f V_R \frac{S_{wf}^n}{\Delta t}$$

Substitute all terms into Eq (10) we get

$$PressureTerm - GravityTerm - CapillaryTerm - V_R \tau_w^{[n+1]} + q_{wf}^{[n+1]} = S_{wf}^{n+1} V_R \left(\phi_f (c_\phi + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^n}{\Delta t} + \frac{\phi_f}{\Delta t} \right) - \phi_f V_R \frac{S_{wf}^n}{\Delta t}$$

Rearrange above equation we get

$$S_{wf,i}^{n+1} = \frac{PressureTerm - GravityTerm - CapillaryTerm - V_R \tau_w^{[n+1]} + q_{wf}^{[n+1]} + \phi_f V_R \frac{S_{wf,i}^n}{\Delta t}}{V_R \left(\phi_f (c_\phi + c_w)_f \frac{P_{of}^{n+1} - P_{of}^n}{\Delta t} + \frac{\phi_f}{\Delta t} \right)} \tag{11}$$

where

$$PressureTerm = T^n_{wx,i+\frac{1}{2}}\left(P^{[n+1]}_{of,i+1} - P^{[n+1]}_{of,i}\right) - T^n_{wx,i-\frac{1}{2}}\left(P^{[n+1]}_{of,i} - P^{[n+1]}_{of,i-1}\right)$$

$$GravityTerm = \left(T^n_{wx}\gamma_w\right)_{i+\frac{1}{2}}\left(D_{i+1} - D_i\right) - \left(T^n_{wx}\gamma_w\right)_{i-\frac{1}{2}}\left(D_i - D_{i-1}\right)$$

$$CapillaryTerm = T^n_{wx,i+\frac{1}{2}}\left(P^{[n+1]}_{cwof,i+1} - P^{[n+1]}_{cwof,i}\right) - T^n_{wx,i-\frac{1}{2}}\left(P^{[n+1]}_{cwof,i} - P^{[n+1]}_{cwof,i-1}\right)$$

B OPTION 2: PARTIALLY IMPLICIT

To solve saturation equation, multiply Eq (5) by V_R, where $V_R = (\Delta x \Delta y \Delta z)_{i,j,k}$ and rearrange as follows

$$\left(\begin{array}{l} \overbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^{n} \nabla P_{of}^{[n+1]}}^{Pressure-term} - \overbrace{V_R \nabla \cdot k_{f,eff} (\lambda_{wf}^{n} \gamma_w^n) \nabla D}^{Gravity-term} \\ - \underbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^{n} \frac{\partial P_{cwof}^{n}}{\partial S_{wf}} \nabla S_{wf}^{n+1}}_{Capillary-term} - V_R \tau_w^{[n+1]} + q_{wf}^{[n+1]} \end{array} \right) = \overbrace{V_R \phi_f \left(S_{wf}^{n+1} (c_\phi + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^{n}}{\Delta t} + \frac{S_{wf}^{n+1} - S_{wf}^{n}}{\Delta t} \right)}^{Accumulation-term} \tag{12}$$

Discretize Eq (12) for 1D problem term by term we get

- Pressure Term

$$T_{wx,i+\frac{1}{2}}^{n} \left(P_{of,i+1}^{[n+1]} - P_{of,i}^{[n+1]} \right) - T_{wx,i-\frac{1}{2}}^{n} \left(P_{of,i}^{[n+1]} - P_{of,i-1}^{[n+1]} \right)$$

- Gravity Term

$$(T_{wx}^{n} \gamma_w)_{i+\frac{1}{2}} (D_{i+1} - D_i) - (T_{wx}^{n} \gamma_w)_{i-\frac{1}{2}} (D_i - D_{i-1})$$

- Capillary Term

$$\left(T_{wx} \frac{\partial P_{cwof}}{\partial S_{wf}} \right)_{i+\frac{1}{2}}^{n} \left(S_{wf,i+1}^{n+1} - S_{wf,i}^{n+1} \right) - \left(T_{wx} \frac{\partial P_{cwof}}{\partial S_{wf}} \right)_{i-\frac{1}{2}}^{n} \left(S_{wf,i}^{n+1} - S_{wf,i-1}^{n+1} \right)$$

- Accumulation Term

$$S_{wf}^{n+1} V_R \left(\phi_f (c_{\phi_f} + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^{n}}{\Delta t} + \frac{\phi}{\Delta t} \right) - \phi_f V_R \frac{S_{wf}^{n}}{\Delta t}$$

Substitute all terms into Eq (12) we get

$$D S_{wf,i-1}^{n+1} + E S_{wf,i}^{n+1} + F S_{wf,i+1}^{n+1} = RHS \tag{13}$$

Using Tayler's series expansion as $S_{w,i}^{n+1} = S_{w,i}^{n} + \Delta t \frac{\partial S_{w,i}}{\partial t}$.

Discretize above equation we get $S_{w,i}^{n+1} = S_{w,i}^{n} + \Delta t \frac{S_{w,i}^{n+1} - S_{w,i}^{n}}{\Delta t}$.

Let's define $\Delta_t S_w = S_{w,i}^{n+1} - S_{w,i}^{n}$, thus $S_{w,i}^{n+1} = S_{w,i}^{n} + \Delta_t S_w$. Using this definition we can rearrange Eq (13) as follows

$$D\Delta_t S_{wf,i-1} + E\Delta_t S_{wf,i} + F\Delta_t S_{wf,i+1} = RHS - DS_{wf,i-1}^{n} - ES_{wf,i}^{n} - FS_{wf,i+1}^{n} \quad (14)$$

where

$$D = \left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{n}$$

$$F = \left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i+\frac{1}{2}}^{n}$$

$$E = -\left(\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{n} + \left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i+\frac{1}{2}}^{n} - V_R\left(\phi_f(c_\phi + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^{n}}{\Delta t} + \frac{\phi_f}{\Delta t}\right)\right)$$

$$RHS = PressureTerm - GravityTerm + \phi_f V_R \frac{S_{wf}^{n}}{\Delta t} - V_R \tau_w^{[n+1]} + q_w^{[n+1]}$$

$$PressureTerm = T_{wx,i+\frac{1}{2}}^{n}\left(P_{of,i+1}^{[n+1]} - P_{of,i}^{[n+1]}\right) - T_{wx,i-\frac{1}{2}}^{n}\left(P_{of,i}^{[n+1]} - P_{of,i-1}^{[n+1]}\right)$$

$$GravityTerm = (T_{wx}^{n}\gamma_w)_{i+\frac{1}{2}}(D_{i+1} - D_i) - (T_{wx}^{n}\gamma_w)_{i-\frac{1}{2}}(D_i - D_{i-1})$$

$$T_{xw,i+\frac{1}{2}}^{n} = \alpha(k_{f,eff}\lambda_{wf}^{n}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{xw,i-\frac{1}{2}}^{n} = \alpha(k_{f,eff}\lambda_{wf}^{n}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T_{xo,i+\frac{1}{2}}^{n} = \alpha(k_{f,eff}\lambda_{of}^{n}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T^n_{xo,i-\frac{1}{2}} = \alpha(k_{f,eff}\lambda^n_{of}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$\alpha = 0.006328$$

$$\frac{\partial P_{cwo}}{\partial S_w} = \begin{cases} dP_{cwo,max}, & S_w \leq S_{w,pcmax} \\ dP_{cwo,min}, & S_w \geq S_{w,pcmin} \\ dP_{cwo1}, & S_{w,pcmax} < S_w \leq S_{wx} \\ dP_{cwo2} & S_{wx} < S_w < S_{w,pcmin} \end{cases}$$

where,

$$dP_{cwo,max} = -\frac{\alpha}{S_{w,pcmax} - S_{wr}}$$

$$dP_{cwo,min} = -\frac{\alpha}{1 - S_{w,pcmin} - S_{owr}}$$

$$dP_{cwo1} = -\frac{\alpha}{S_w - S_{wr}}$$

$$dP_{cwo2} = -\frac{\alpha}{1 - S_w - S_{owr}}$$

$$S_{w,pcmax} = S_{wr} + (S_{wx} - S_{wr})\, e^{-\frac{P_{cow,max}}{\alpha}}$$

$$S_{w,pcmin} = 1 - S_{owr} - (1 - S_{wx} - S_{owr})\, e^{\frac{P_{cow,min}}{\alpha}}$$

$$P_{cow1} = \alpha ln\left[\frac{1 - S_w - S_{owr}}{1 - S_{wx} - S_{owr}}\right]$$

,

$$P_{cow2} = -\alpha ln\left[\frac{S_w - S_{wr}}{S_w x - S_{wr}}\right]$$