PEGN 620A: Naturally Fractured Reservoir
Homework 12:Solution

1. OPTION3:FULLY IMPLICIT ITERATIVE

   Given Equations

   (a) Fracture pressure equation:

$$\nabla\cdot\{k_{f,eff}[\lambda_{tf}^{[n+1]}\nabla P_{of}^{n+1}-(\lambda_{wf}^{[n+1]}\gamma_{w}^{[n+1]}+\lambda_{of}^{[n+1]}\gamma_{o}^{[n+1]})\nabla D-\lambda_{wf}^{[n+1]}\nabla P_{cwof}^{[n+1]}]\}-\tau_{t}^{n+1}+\hat{q}_{tf}^{n+1}=(\phi c_{t}^{[n+1]})_{f}\frac{\partial P_{of}}{\partial t} \tag{1}$$

   (b) Total transfer function:

$$\tau_{t}^{n+1}=\sigma k_{m}[\lambda_{tf}^{[n+1]}(P_{of}^{n+1}-P_{om}^{n+1})+\frac{\sigma_{z}}{\sigma}(\lambda_{wf/m}^{[n+1]}\gamma_{w}^{[n+1]}+\lambda_{om/f}^{[n+1]}\gamma_{o}^{[n+1]})(h_{wf}^{[n+1]}-h_{wm}^{[n+1]})+\lambda_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]}-P_{cwof}^{[n+1]})] \tag{2}$$

   (c) Total matrix flow:

$$\tau_{t}^{n+1}=(\phi c_{t}^{[n+1]})_{m}\frac{P_{om}^{n+1}-P_{om}^{n}}{\Delta t} \tag{3}$$

   (d) Saturation equation:

$$\nabla\cdot\left\{k_{f,eff}\lambda_{wf}^{[n+1]}\left[\nabla P_{of}^{[n+1]}-\gamma_{w}^{[n+1]}\nabla D-\frac{\partial P_{cwof}^{n}}{\partial S_{wf}}\nabla S_{wf}^{n+1}\right]\right\}-\tau_{w}^{[n+1]}+\hat{q}_{wf}^{[n+1]}=\phi_{f}S_{w}^{n+1}(c_{\phi}+c_{w})_{f}\frac{P_{of}^{[n+1]}-P_{of}^{n}}{\Delta t}+\phi_{f}\frac{S_{wf}^{n+1}-S_{wf}^{n}}{\Delta t} \tag{4}$$

   (e) Water transfer

$$\tau_{w}^{n+1}=\sigma k_{m}\left[\lambda_{wf}^{[n+1]}(P_{of}^{[n+1]}-P_{om}^{[n+1]})+\frac{\sigma_{z}}{\sigma}(\lambda_{wf/m}^{[n+1]}\gamma_{w}^{[n+1]})(h_{wf}^{[n+1]}-h_{wm}^{[n+1]})+\lambda_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]}-P_{cwof}^{[n+1]})\right] \tag{5}$$

   (f) Water matrix flow:

$$\tau_{w}^{[n+1]}=\phi_{m}S_{wm}^{n+1}(c_{\phi}+c_{w})_{m}\frac{P_{om}^{[n+1]}-P_{om}^{n}}{\Delta t}+\phi_{m}\frac{S_{wm}^{n+1}-S_{wm}^{n}}{\Delta t} \tag{6}$$

DISCRETIZE EQUATIONS IN 1D PROBLEM

STEP1: Solving Fracture pressure equation:

Discretize and rearrange Eq (3) we get

$$\tau_t^{n+1} = (\phi c_t^{[n+1]})_m \frac{P_{i,om}^{n+1} - P_{i,om}^n}{\Delta t}$$
$$P_{i,om}^{n+1} = \tau_t^{n+1} \frac{\Delta t}{(\phi c_t^{[n+1]})_m} + P_{i,om}^n$$

Substitute $P_{i,om}^{n+1}$ into Eq (2) and rearrange

$$\tau_t^{n+1} = \beta\sigma k_m[\lambda_{tf}^{[n+1]}(P_{i,of}^{n+1} - (\tau_t^{n+1}\frac{\Delta t}{(\phi c_t)_m} + P_{i,om}^n)) + \frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{om/f}^{[n+1]}\gamma_o^{[n+1]})(h_{wf}^{[n+1]} - h_{wm}^{[n+1]}) + \lambda_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]} - P_{cwof}^{[n+1]})]$$

$$\tau_t^{n+1}(1 + \frac{\beta\sigma k_m \lambda_{tf}^{[n+1]}\Delta t}{(\phi c_t)_m}) = \beta\sigma k_m[\lambda_{tf}^{[n+1]}(P_{i,of}^{n+1} - P_{i,om}^n) + \frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{om/f}^{[n+1]}\gamma_o^{[n+1]})(h_{wf}^{[n+1]} - h_{wm}^{[n+1]}) + \lambda_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]} - P_{cwof}^{[n+1]})]$$

$$\tau_t^{n+1} = \frac{\beta\sigma k_m}{(1 + \frac{\beta\sigma k_m \lambda_{tf}^{[n+1]}\Delta t}{(\phi c_t)_m})}[\lambda_{tf}^{[n+1]}(P_{i,of}^{n+1} - P_{i,om}^n) + \frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{om/f}^{[n+1]}\gamma_o^{[n+1]})(h_{wf}^{[n+1]} - h_{wm}^{[n+1]}) + \lambda_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]} - P_{cwof}^{[n+1]})] \tag{7}$$

Multiply Eq (1) by V_R, where $V_R = \Delta x_i \Delta y_i \Delta z_i$ and rearrange

$$\left(\begin{array}{c} \overbrace{V_R\nabla\cdot k_{f,eff}\lambda_{tf}^{[n+1]}\nabla P_{of}^{n+1}}^{Pressure-term} - \overbrace{V_R\nabla\cdot k_{f,eff}\lambda_{wf}^{[n+1]}\nabla P_{cwof}^{[n+1]}}^{Capillary-term} \\ -\underbrace{V_R\nabla\cdot k_{f,eff}(\lambda_{wf}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{of}^{[n+1]}\gamma_o^{[n+1]})\nabla D}_{Gravity-term} - V_R\tau_t^{n+1} + q_{tf}^{n+1} \end{array}\right) = \overbrace{V_R(\phi c_t^{[n+1]})_f\frac{\partial P_{of}}{\partial t}}^{Accumulation-term} \tag{8}$$

Discretize Eq (8) for 1-D problem term by term and rearrange

- Pressure term

$$\beta\Delta y_i\Delta z_i\left[(k_{f,eff}\lambda_{tf}^{[n+1]})_{i+\frac{1}{2}}(\frac{P_{of,i+1}^{n+1} - P_{of,i}^{n+1}}{\Delta x_{i+\frac{1}{2}}}) - (k_{f,eff}\lambda_{tf}^{[n+1]})_{i-\frac{1}{2}}(\frac{P_{of,i}^{n+1} - P_{of,i-1}^{n+1}}{\Delta x_{i-\frac{1}{2}}})\right] \tag{9}$$

  Rearrange

$$T_{x,i+\frac{1}{2}}^{[n+1]}P_{of,i+1}^{n+1} + T_{x,i-\frac{1}{2}}^{[n+1]}P_{of,i-1}^{n+1} - (T_{x,i+\frac{1}{2}}^{[n+1]} + T_{x,i-\frac{1}{2}}^{[n+1]})P_{of,i}^{n+1} \tag{10}$$

- Gravity term

$$\beta \Delta y_i \Delta z_i \left[(k_{f,eff}(\lambda_{wf}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{of}^{[n+1]}\gamma_o^{[n+1]}))_{i+\frac{1}{2}} (\frac{D_{i+1} - D_i}{\Delta x_{i+\frac{1}{2}}}) - (k_{f,eff}(\lambda_{wf}^{[n+1]}\gamma_w^{[n+1]} + \lambda_{of}^{[n+1]}\gamma_o^{[n+1]}))_{i-\frac{1}{2}} (\frac{D_i - D_{i-1}}{\Delta x_{i-\frac{1}{2}}}) \right] \tag{11}$$

Rearrange

$$(T_{xw}^{[n+1]}\gamma_w^n + T_{xo}^{[n+1]}\gamma_o^n)_{i+\frac{1}{2}}(D_{i+1} - D_i) - (T_{xw}^{[n+1]}\gamma_w^n + T_{xo}^{[n+1]}\gamma_o^n)_{i+\frac{1}{2}}(D_i - D_{i-1}) \tag{12}$$

- Capillary term

$$\beta \Delta y_i \Delta z_i \left[(k_{f,eff}\lambda_{wf}^{[n+1]})_{i+\frac{1}{2}} (\frac{P_{cwof,i+1}^{[n+1]} - P_{cwof,i}^{[n+1]}}{\Delta x_{i+\frac{1}{2}}}) - (k_{f,eff}\lambda_{wf}^{[n+1]})_{i-\frac{1}{2}} (\frac{P_{cwof,i}^{[n+1]} - P_{cwof,i-1}^{[n+1]}}{\Delta x_{i-\frac{1}{2}}}) \right] \tag{13}$$

Rearrange

$$T_{xw,i+\frac{1}{2}}^{[n+1]}(P_{cwof,i+1}^{[n+1]} - P_{cwof,i}^{[n+1]}) - T_{xw,i-\frac{1}{2}}^{[n+1]}(P_{cwof,i}^{[n+1]} - P_{cwof,i-1}^{[n+1]}) \tag{14}$$

- Accumulation term

$$V_i(\phi c_t^{[n+1]})_f \frac{P_{i,of}^{n+1} - P_{i,of}^n}{\Delta t} \tag{15}$$

Substitute Eq(7), (10), (12), (14) and (15) into Eq (8) and rearrange

$$FP_{of,i+1}^{n+1} + DP_{of,i-1}^{n+1} + EP_{of,i}^{n+1} = RHS \tag{16}$$

Let's define $\Delta_t P_{of} = P_{of,i}^{n+1} - P_{of,i}^n$, thus $P_{of,i}^{n+1} = P_{of,i}^n + \Delta_t P_{of}$. Using this definition we can rearrange Eq (16) as follows

$$D\Delta_t P_{of,i-1} + E\Delta_t P_{of,i} + F\Delta_t P_{of,i+1} = RHS - DP_{of,i-1}^n - EP_{of,i}^n - FP_{of,i+1}^n \tag{17}$$

where,

$$D = T_{x,i-\frac{1}{2}}^{[n+1]}$$

$$F = T_{x,i+\frac{1}{2}}^{[n+1]}$$

$$E = -\left(T_{x,i+\frac{1}{2}}^{[n+1]} + T_{x,i-\frac{1}{2}}^{[n+1]} + V_i \frac{(\phi c_t)_f}{\Delta t} + V_R \frac{\beta \sigma k_m \lambda_{tf}^{[n+1]}}{1 + \frac{\beta \sigma k_m \lambda_{tf}^{[n+1]} \Delta t}{(\phi c_t^{[n+1]})_m}} \right)$$

$$RHS = TransferTerm + GravityTerm + CapillaryTerm - V_R \frac{(\phi c_t^{[n+1]})_f}{\Delta t} P_{i,of}^n - q_{tf}^{[n+1]}$$

$$TransferTerm = V_R \frac{\beta \sigma k_m}{(1 + \frac{\beta \sigma k_m \lambda_{tf}^{[n+1]} \Delta t}{(\phi c_t^{[n+1]})_m})} \left[\begin{array}{c} -\lambda_{tf}^{[n+1]} P_{i,om}^{[n+1]} + \frac{\sigma_z}{\sigma} (\lambda_{wf/m}^{[n+1]} \gamma_w^{[n+1]} + \lambda_{om/f}^{[n+1]} \gamma_o^{[n+1]}) \\ (h_{wf}^{[n+1]} - h_{wm}^{[n+1]}) + \lambda_{wf/m}^{[n+1]} (P_{cwom}^{[n+1]} - P_{cwof}^{[n+1]}) \end{array} \right] \tag{18}$$

$$GravityTerm = (T_{xw}^{[n+1]} \gamma_w^{[n+1]} + T_{xo}^{[n+1]} \gamma_o^{[n+1]})_{i+\frac{1}{2}} (D_{i+1} - D_i) - (T_{xw}^{[n+1]} \gamma_w^{[n+1]} + T_{xo}^{[n+1]} \gamma_o^{[n+1]})_{i+\frac{1}{2}} (D_i - D_{i-1})$$

$$CapillaryTerm = T_{xw,i+\frac{1}{2}}^{[n+1]} (P_{cwof,i+1}^{[n+1]} - P_{cwof,i}^{[n+1]}) - T_{xw,i-\frac{1}{2}}^{[n+1]} (P_{cwof,i}^{[n+1]} - P_{cwof,i-1}^{[n+1]})$$

$$T_{x,i+\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{tf}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{x,i-\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{tf}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T_{xw,i+\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{wf}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{xw,i-\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{wf}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T_{xo,i+\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{of}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{xo,i-\frac{1}{2}}^{[n+1]} = \beta (k_{f,eff} \lambda_{of}^{[n+1]} \frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$\beta = 0.006328$$

STEP2:Solving for Matrix Pressure

From Eq(7), we can calculate τ_{tf}^{n+1}. $P_{i,om}^{n+1}$ can be solved by re-arrange Eq(2) as follows

$$P_{om}^{n+1} = -\frac{\tau_t^{n+1}}{\sigma k_m \lambda_{tf}^{[n+1]}} + P_{of}^{n+1} + \frac{\sigma_z}{\sigma}(f_{wf/m}^{[n+1]}\gamma_w^{[n+1]} + f_{om/f}^{[n+1]}\gamma_o^{[n+1]})(h_{wf}^{[n+1]} - h_{wm}^{[n+1]}) + f_{wf/m}^{[n+1]}(P_{cwom}^{[n+1]} - P_{cwof}^{[n+1]}) \tag{19}$$

where

$$f_{wf/m}^{[n+1]} = \frac{\lambda_{wf/m}^{[n+1]}}{\lambda_{tf}^{[n+1]}}$$

$$f_{om/f}^{[n+1]} = \frac{\lambda_{om/f}^{[n+1]}}{\lambda_{tf}^{[n+1]}}$$

STEP3:Solving for Fracture Saturation

To solve saturation equation, multiply Eq (4) by V_R, where $V_R = (\Delta x \Delta y \Delta z)_{i,j,k}$ and rearrange as follows

$$\left(\begin{array}{c} \overbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^{[n+1]} \nabla P_{of}^{[n+1]}}^{Pressure-term} - \overbrace{V_R \nabla \cdot k_{f,eff} (\lambda_{wf}^{[n+1]} \gamma_w^{[n+1]}) \nabla D}^{Gravity-term} \\ - \underbrace{V_R \nabla \cdot k_{f,eff} \lambda_{wf}^{[n+1]} \frac{\partial P_{cwof}^{[n+1]}}{\partial S_{wf}} \nabla S_{wf}^{n+1}}_{Capillary-term} - V_R \tau_w^{[n+1]} + q_{wf}^{[n+1]} \end{array} \right) = \overbrace{V_R \phi_f \left(S_{wf}^{n+1} (c_\phi + c_w)_f \frac{P_{of}^{[n+1]} - P_{of}^n}{\Delta t} + \frac{S_{wf}^{n+1} - S_{wf}^n}{\Delta t} \right)}^{Accumulation-term} \tag{20}$$

Discretize Eq (20) for 1D problem term by term we get

- Pressure Term

$$T_{wx,i+\frac{1}{2}}^{[n+1]} \left(P_{of,i+1}^{[n+1]} - P_{of,i}^{[n+1]} \right) - T_{wx,i-\frac{1}{2}}^{[n+1]} \left(P_{of,i}^{[n+1]} - P_{of,i-1}^{[n+1]} \right)$$

- Gravity Term

$$\left(T_{wx}^{[n+1]} \gamma_w \right)_{i+\frac{1}{2}} (D_{i+1} - D_i) - \left(T_{wx}^{[n+1]} \gamma_w \right)_{i-\frac{1}{2}} (D_i - D_{i-1})$$

- Capillary Term

$$\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i+\frac{1}{2}}^{[n+1]}\left(S_{wf,i+1}^{n+1}-S_{wf,i}^{n+1}\right)-\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{[n+1]}\left(S_{wf,i}^{n+1}-S_{wf,i-1}^{n+1}\right)$$

- Accumulation Term

$$S_{wf}^{n+1}V_R\left(\phi_f(c_{\phi_f}+c_w)_f\frac{P_{of}^{[n+1]}-P_{of}^{n}}{\Delta t}+\frac{\phi}{\Delta t}\right)-\phi_f V_R\frac{S_{wf}^{n}}{\Delta t}$$

Substitute all terms into Eq (20) we get

$$DS_{wf,i-1}^{n+1}+ES_{wf,i}^{n+1}+FS_{wf,i+1}^{n+1}=RHS \tag{21}$$

Let's define $\Delta_t S_w = S_{w,i}^{n+1} - S_{w,i}^{n}$, thus $S_{w,i}^{n+1} = S_{w,i}^{n} + \Delta_t S_w$. Using this definition we can rearrange Eq (21) as follows

$$D\Delta_t S_{wf,i-1}+E\Delta_t S_{wf,i}+F\Delta_t S_{wf,i+1}=RHS-DS_{wf,i-1}^{n}-ES_{wf,i}^{n}-FS_{wf,i+1}^{n} \tag{22}$$

where

$$D=\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{[n+1]}$$

$$F=\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i+\frac{1}{2}}^{[n+1]}$$

$$E=-\left(\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{[n+1]}+\left(T_{wx}\frac{\partial P_{cwof}}{\partial S_{wf}}\right)_{i+\frac{1}{2}}^{[n+1]}-V_R\left(\phi_f(c_{\phi}+c_w)_f\frac{P_{of}^{[n+1]}-P_{of}^{n}}{\Delta t}+\frac{\phi_f}{\Delta t}\right)\right)$$

$$RHS=PressureTerm-GravityTerm+\phi_f V_R\frac{S_{wf}^{n}}{\Delta t}-V_R\tau_w^{[n+1]}+q_w$$

$$PressureTerm=T_{wx,i+\frac{1}{2}}^{[n+1]}\left(P_{of,i+1}^{[n+1]}-P_{of,i}^{[n+1]}\right)-T_{wx,i-\frac{1}{2}}^{[n+1]}\left(P_{of,i}^{[n+1]}-P_{of,i-1}^{[n+1]}\right)$$

$$GravityTerm = \left(T_{wx}^{[n+1]}\gamma_w\right)_{i+\frac{1}{2}}(D_{i+1} - D_i) - \left(T_{wx}^{[n+1]}\gamma_w\right)_{i-\frac{1}{2}}(D_i - D_{i-1})$$

$$T_{xw,i+\frac{1}{2}}^{[n+1]} = \beta(k_{f,eff}\lambda_{wf}^{[n+1]}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{xw,i-\frac{1}{2}}^{[n+1]} = \beta(k_{f,eff}\lambda_{wf}^{[n+1]}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$T_{xo,i+\frac{1}{2}}^{[n+1]} = \beta(k_{f,eff}\lambda_{of}^{[n+1]}\frac{\Delta y \Delta z}{\Delta x})_{i+\frac{1}{2}}$$

$$T_{xo,i-\frac{1}{2}}^{[n+1]} = \beta(k_{f,eff}\lambda_{of}^{[n+1]}\frac{\Delta y \Delta z}{\Delta x})_{i-\frac{1}{2}}$$

$$\beta = 0.006328$$

$$\frac{\partial P_{cwo}}{\partial S_w} = \begin{cases} dP_{cwo,max}, & S_w \leq S_{w,pcmax} \\ dP_{cwo,min}, & S_w \geq S_{w,pcmin} \\ dP_{cwo1}, & S_{w,pcmax} < S_w \leq S_{wx} \\ dP_{cwo2} & S_{wx} < S_w < S_{w,pcmin} \end{cases}$$

where,

$$dP_{cwo,max} = -\frac{\alpha}{S_{w,pcmax} - S_{wr}}$$

$$dP_{cwo,min} = -\frac{\alpha}{1 - S_{w,pcmin} - S_{owr}}$$

$$dP_{cwo1} = -\frac{\alpha}{S_w - S_{wr}}$$

$$dP_{cwo2} = -\frac{\alpha}{1 - S_w - S_{owr}}$$

$$S_{w,pcmax} = S_{wr} + (S_{wx} - S_{wr})\, e^{-\frac{P_{cow,max}}{\alpha}}$$

$$S_{w,pcmin} = 1 - S_{owr} - (1 - S_{wx} - S_{owr})\, e^{\frac{P_{cow,min}}{\alpha}}$$

$$P_{cow1} = \alpha ln \left[\frac{1 - S_w - S_{owr}}{1 - S_{wx} - S_{owr}} \right]$$

,

$$P_{cow2} = -\alpha ln \left[\frac{S_w - S_{wr}}{S_w x - S_{wr}} \right]$$

STEP4:Solving for Matrix Saturation

From Eq(6)we can solve S_{wm}^{n+1} as follows

$$S_{wm}^{n+1} = \frac{\tau_w^{[n+1]} + \phi_m \frac{S_{wm}^n}{\Delta t}}{\phi_m \left((c_\phi + c_w)_m \frac{P_{om}^{[n+1]} - P_{om}^n}{\Delta t} + \frac{1}{\Delta t} \right)} \tag{23}$$

2. OPTION4:FULLY IMPLICIT SIMULTANEOUS

Given Equations

(a) Water saturation equation:

$$\nabla\cdot\left\{k_{f,eff}\lambda_{wf}^{n+1}\left[\nabla P_{of}^{n+1}-\gamma_w^n\nabla D-\nabla P_{cwof}^{n+1}\right]\right\}-\tau_{wf}^{n+1}+\hat{q}_{wf}^{n+1}=\phi_f\left[S_{wf}^{n+1}(c_\phi+c_w)_f\frac{\partial P_{of}}{\partial t}+\frac{\partial S_{wf}}{\partial t}\right] \tag{24}$$

(b) Water transfer function:

$$\tau_w^{n+1}=\sigma k_m\left[\lambda_{wf/m}^{n+1}(P_{of}^{n+1}-P_{om}^{n+1})+\frac{\sigma_z}{\sigma}(\lambda_{wf/m}^{n+1}\gamma_w^n)(h_{wf}^{n+1}-h_{wm}^{n+1})+\lambda_{wf/m}^{n+1}(P_{cwom}^{n+1}-P_{cwof}^{n+1})\right] \tag{25}$$

(c) Water matrix flow:

$$\tau_w^{n+1}=\phi_m\left[S_{wm}^{n+1}(c_\phi+c_w)_m\frac{\partial P_{om}}{\partial t}+\frac{\partial S_{wm}}{\partial t}\right] \tag{26}$$

(d) Oil saturation equation:

$$\nabla\cdot\left\{k_{f,eff}\lambda_{of}^{n+1}\left[\nabla P_{of}^{n+1}-\gamma_o^n\nabla D\right]\right\}-\tau_{of}^{n+1}+\hat{q}_{of}^{n+1}=\phi_f\left[S_{of}^{n+1}(c_\phi+c_o)_f\frac{\partial P_{of}}{\partial t}+\frac{\partial S_{of}}{\partial t}\right] \tag{27}$$

(e) Oil transfer function:

$$\tau_o^{n+1}=\sigma k_m\left[\lambda_{om/f}^{n+1}(P_{of}^{n+1}-P_{om}^{n+1})+\frac{\sigma_z}{\sigma}(\lambda_{om/f}^{n+1}\gamma_o^n)(h_{wf}^{n+1}-h_{wm}^{n+1})\right] \tag{28}$$

(f) Oil matrix flow:

$$\tau_o^{n+1}=\phi_m\left[S_{om}^{n+1}(c_\phi+c_o)_m\frac{\partial P_{om}}{\partial t}+\frac{\partial S_{om}}{\partial t}\right] \tag{29}$$

Using Newton-Raphson to expand terms in Eq(25)

$$\lambda_{wf/m}^{n+1}=\lambda_{wf/m}^{l+1}=\lambda_{wf/m}^{l}+\left(\frac{\partial\lambda_{wf/m}}{\partial S_{wf}}\right)^l\delta S_{wf}$$

$$\lambda_{om/f}^{n+1} = \lambda_{om/f}^{l+1} = \lambda_{om/f}^{l} + \left(\frac{\partial \lambda_{om/f}}{\partial S_{wm}}\right)^{l} \delta S_{wm}$$

$$h_{wf}^{n+1} = h_{wf}^{l+1} = h_{wf}^{l} + \left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^{l} \delta S_{wf}$$

$$h_{wm}^{n+1} = h_{wm}^{l+1} = h_{wm}^{l} + \left(\frac{\partial h_{wm}}{\partial S_{wm}}\right)^{l} \delta S_{wm}$$

$$P_{cwof}^{n+1} = P_{cwof}^{l+1} = P_{cwof}^{l} + \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^{l} \delta S_{wf}$$

$$P_{cwom}^{n+1} = P_{cwom}^{l+1} = P_{cwom}^{l} + \left(\frac{\partial P_{cwom}}{\partial S_{wm}}\right)^{l} \delta S_{wm}$$

$$P_{of}^{n+1} = P_{of}^{l+1} = P_{of}^{l+1} + \delta P_{of}$$

$$P_{om}^{n+1} = P_{om}^{l+1} = P_{om}^{l+1} + \delta P_{om}$$

$$S_{wf}^{n+1} = S_{wf}^{l+1} = S_{wf}^{l+1} + \delta S_{wf}$$

Substitute all expanded terms into Eq(25) and rearrange we get

$$V_R \tau_w^{n+1} = \left\{ \begin{array}{c} V_R \tau_w^l + \beta\sigma V_R k_m \lambda_{wf/m}^l \delta P_{of,i} - \beta\sigma V_R k_m \lambda_{wf/m}^l \delta P_{om,i} \\ + \left[\frac{1}{\lambda_{wf/m}^l} \left(\frac{\partial \lambda_{wf/m}}{\partial S_{wf}}\right)^l V_R \tau_w^l + \beta\sigma V_R k_m \lambda_{wf/m}^l \left(\frac{\sigma_z}{\sigma} \gamma_w^n \left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^l - \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^l \right) \right] \delta S_{wf,i} \\ + \beta\sigma V_R k_m \lambda_{wf/m}^l \left[-\frac{\sigma_z}{\sigma} \gamma_w^n \left(\frac{\partial h_{wm}}{\partial S_{wm}}\right)^l + \left(\frac{\partial P_{cwom}}{\partial S_{wm}}\right)^l \right] \delta S_{wm,i} \end{array} \right\} \tag{30}$$

where $\beta = 0.006328$

Multiply Eq(24) by V_R and discretize for 1D problem we get

$$\left[\begin{array}{c} T^{n+1}_{xw,i-\frac{1}{2}} P^{n+1}_{of,i-1} - \left(T^{n+1}_{xw,i-\frac{1}{2}} + T^{n+1}_{xw,i+\frac{1}{2}}\right) P^{n+1}_{of,i} + T^{n+1}_{xw,i+\frac{1}{2}} P^{n+1}_{of,i+1} \\ -T^{n+1}_{xw,i-\frac{1}{2}} \gamma^n_w (D_{i-1} - D_i) - T^{n+1}_{xw,i+\frac{1}{2}} \gamma^n_w (D_{i+1} - D_i) \\ -T^{n+1}_{xw,i-\frac{1}{2}} P^{n+1}_{cwof,i-1} + \left(T^{n+1}_{xw,i-\frac{1}{2}} + T^{n+1}_{xw,i+\frac{1}{2}}\right) P^{n+1}_{cwof,i} \\ -T^{n+1}_{xw,i+\frac{1}{2}} P^{n+1}_{cwof,i+1} - V_R \tau^{n+1}_{wf} + q^{n+1}_{wf} \end{array}\right] = \phi_f \left[S^{n+1}_{wf} (c_\phi + c_w)_f \frac{P^{n+1}_{of} - P^n_{of}}{\Delta t} + \frac{S^{n+1}_{wf} - S^n_{wf}}{\Delta t}\right] \tag{31}$$

Using Newton-Raphson to expand each term in Eq(31)

$$T^{n+1}_{xw} = T^{l+1}_{xw} = T^l_{xw} + \left(\frac{\partial T_{xw}}{\partial S_{wf}}\right)^l \delta S_{wf}$$

where, $\frac{\partial T_{xw}}{\partial S_{wf}} = \beta k_f \frac{\Delta y \Delta z}{\Delta x} \frac{\partial \lambda_{wf}}{\partial S_{wf}}$

We can expand sink/source terms as follows

Option:1 Pressure-Controlled Well

$$q^{n+1}_{wf} = WI \lambda^{n+1}_{wf} \left(P_{well} - (P^{n+1}_{of,i})\right) \tag{32}$$

$$= WI \left(\lambda^l_{wf} + \left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^l \delta S_{wf}\right) \left(P_{well} - P^l_{of,i} - \delta P_{of,i}\right) \tag{33}$$

$$= \left\{\begin{array}{c} WI \lambda^l_{wf} \left(P_{well} - P^l_{of,i}\right) - WI \lambda^l_{wf} \delta P_{of} \\ +WI \left(P_{well} - P^l_{of,i}\right) \left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^l \delta S_{wf} \end{array}\right\} \tag{34}$$

Option:2 Total Rate-Controlled Well

$$q^{n+1}_{wf} = q_{tf} - WI \lambda^{n+1}_{of} \left(P^{n+1}_{well} - (P^{n+1}_{of,i})\right)$$

$$(35)$$

$$= q_{tf} - WI\left(\lambda_{of}^{l} + \left(\frac{\partial\lambda_{of}}{\partial S_{wf}}\right)^{l}\delta S_{wf}\right)\left(P_{well}^{l} - P_{of,i}^{l} + \delta P_{well} - \delta P_{of,i}\right) \quad (36)$$

$$= \left\{ \begin{array}{c} q_{tf} - WI\lambda_{of}^{l}\left(P_{well}^{l} - P_{of,i}^{l}\right) - WI\lambda_{of}^{l}\left(\delta P_{well} - \delta P_{of,i}\right) \\ -WI\left(P_{well}^{l} - P_{of,i}^{l}\right)\left(\frac{\partial\lambda_{of}}{\partial S_{wf}}\right)^{l}\delta S_{wf} \end{array} \right\} \quad (37)$$

For the total-rate control well, we need one more equation to solve for well pressure. Let's introduce the following equation

$$q_{tf} = q_{of}^{n+1} + q_{wf}^{n+1} \quad (38)$$

$$= WI\lambda_{of}^{n+1}\left(P_{well}^{n+1} - P_{of,i}^{n+1}\right) + WI\lambda_{wf}^{n+1}\left(P_{well}^{n+1} - P_{of,i}^{n+1}\right) \quad (39)$$

$$= \left\{ \begin{array}{c} WI\left(\lambda_{of}^{l} + \left(\frac{\partial\lambda_{of}}{\partial S_{wf}}\right)^{l}\delta S_{wf}\right)\left(P_{well}^{l} - P_{of,i}^{l} + \delta P_{well} - \delta P_{of,i}\right) \\ +WI\left(\lambda_{wf}^{l} + \left(\frac{\partial\lambda_{wf}}{\partial S_{wf}}\right)^{l}\delta S_{wf}\right)\left(P_{well}^{l} - P_{of,i}^{l} + \delta P_{well} - \delta P_{of,i}\right) \end{array} \right\} \quad (40)$$

$$= \left\{ \begin{array}{c} WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)\left(P_{well}^{l} - P_{of,i}^{l}\right) + WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)\delta P_{well} \\ -WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)\delta P_{of,i} + WI\left(P_{well}^{l} - P_{of,i}^{l}\right)\left(\left(\frac{\partial\lambda_{of}}{\partial S_{wf}}\right)^{l} + \left(\frac{\partial\lambda_{wf}}{\partial S_{wf}}\right)^{l}\right)\delta S_{wf} \end{array} \right\} \quad (41)$$

We can set above equation into matrix form as

$$qEpf\delta P_{of,i} + qEpw\delta P_{well} + qEsf\delta S_{wf,i} = qRHS \quad (42)$$

where,

$$qEpf = -WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)$$

$$qEpw = WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)$$

$$qEsf = WI\left(P_{well}^{l} - P_{of,i}^{l}\right)\left(\left(\frac{\partial \lambda_{of}}{\partial S_{wf}}\right)^{l} + \left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^{l}\right)$$

$$qRHS = q_{tf} - WI\left(\lambda_{of}^{l} + \lambda_{wf}^{l}\right)\left(P_{well}^{l} - P_{of,i}^{l}\right)$$

Substitute all expanded terms into Eq(31) and multiply the both sides by V_R.Assuming that multiplication of δ and δ terms is negligible, we can rearrange Eq(31) as follows

$$LHS = RHS \tag{43}$$

where,

$$LHS = \left\{ \begin{array}{c} T^l_{xw,i-\frac{1}{2}} P^l_{of,i-1} - (T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}}) P^l_{of,i} + T^l_{xw,i+\frac{1}{2}} P^l_{of,i+1} \\ -T^l_{xw,i-\frac{1}{2}} \gamma^n_w (D_{i-1} - D_i) - T^l_{xw,i+\frac{1}{2}} \gamma^n_w (D_{i+1} - D_i) \\ -T^l_{xw,i-\frac{1}{2}} P^l_{cwof,i-1} + \left(T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}} \right) P^l_{cwof,i} - T^l_{xw,i+\frac{1}{2}} P^l_{cwof,i+1} \\ +T^l_{xw,i-\frac{1}{2}} \delta P_{of,i-1} - (T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}}) \delta P_{of,i} + T^l_{xw,i+\frac{1}{2}} \delta P_{of,i+1} \\ -T^l_{xw,i-\frac{1}{2}} \left(\frac{\partial P_{cwof}}{\partial S_{wf}} \right)^l_{i-1} \delta S_{wf,i-1} - T^l_{xw,i+\frac{1}{2}} \left(\frac{\partial P_{cwof}}{\partial S_{wf}} \right)^l_{i+1} \delta S_{wf,i+1} \\ +(T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}}) \left(\frac{\partial P_{cwof}}{\partial S_{wf}} \right)^l_{i} \delta S_{wf,i} \\ + \left(P^l_{of,i-1} - P^l_{of,i} - P^l_{cwof,i-1} + P^l_{cwof,i} - \gamma^n_w (D_{i-1} - D_i) \right) \left(\frac{\partial T_{xw}}{\partial S_{wf}} \right)^l_{i-\frac{1}{2}} \delta S_{wf,i-\frac{1}{2}} \\ + \left(P^l_{of,i+1} - P^l_{of,i} - P^l_{cwof,i+1} + P^l_{cwof,i} - \gamma^n_w (D_{i+1} - D_i) \right) \left(\frac{\partial T_{xw}}{\partial S_{wf}} \right)^l_{i+\frac{1}{2}} \delta S_{wf,i+\frac{1}{2}} \\ -V_R \tau^{n+1}_{wf} + q^{n+1}_{wf} \end{array} \right\} \quad (44)$$

$$RHS = V_R \frac{\phi_f}{\Delta t} \left\{ \begin{array}{c} S^l_{wf,i} (c_\phi + c_w)_f (P^l_{of,i} - P^n_{of,i}) + (S^l_{wf,i} - S^n_{wf,i}) \\ +S^l_{wf,i} (c_\phi + c_w)_f \delta P_{of,i} \\ + \left((c_\phi + c_w)_f (P^l_{of,i} - P^n_{of,i}) + 1 \right) \delta S_{wf,i} \end{array} \right\} \quad (45)$$

Substitute Eq(30) and (32) into Eq(43) and rearrange

$$\left[\begin{array}{c} D_{wpf} \delta P_{of,i-1} + E_{wpf} \delta P_{of,i} + F_{wpf} \delta P_{of,i+1} \\ +D_{wsf} \delta S_{wf,i-1} + E_{wsf} \delta S_{wf,i} + F_{wsf} \delta S_{wf,i+1} \\ +D_{wsf12} \delta S_{wf,i-\frac{1}{2}} + F_{wsf12} \delta S_{wf,i+\frac{1}{2}} + E_{wsm} \delta S_{wm,i} + E_{wpm} \delta P_{om,i} \end{array} \right] = RHS_w \quad (46)$$

where

$$D_{wpf} = T^l_{xw,i-\frac{1}{2}},\; F_{wpf} = T^l_{xw,i-\frac{1}{2}}$$

$$D_{wsf12} = \left(P^l_{of,i-1} - P^l_{of,i} - P^l_{cwof,i-1} + P^l_{cwof,i} - \gamma^n_w (D_{i-1} - D_i)\right) \left(\frac{\partial T_{xw}}{\partial S_{wf}}\right)^l_{i-\frac{1}{2}}$$

$$F_{wsf12} = \left(P^l_{of,i+1} - P^l_{of,i} - P^l_{cwof,i+1} + P^l_{cwof,i} - \gamma^n_w (D_{i+1} - D_i)\right) \left(\frac{\partial T_{xw}}{\partial S_{wf}}\right)^l_{i+\frac{1}{2}}$$

$$D_{wsf} = -T^l_{xw,i-\frac{1}{2}} \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^l_{i-1}$$

$$F_{wsf} = -T^l_{xw,i+\frac{1}{2}} \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^l_{i+1}$$

$$E_{wsm} = -\beta\sigma V_R k_m \lambda^l_{wf/m} \left[-\frac{\sigma_z}{\sigma}\gamma^n_w \frac{\partial h_{wm}}{\partial S_{wm}} + \frac{\partial P_{cwom}}{\partial S_{wm}}\right]$$

$$E_{wpm} = \beta\sigma V_R k_m \lambda^l_{wf/m}$$

$$E_{wsf} = \left\{ \begin{array}{c} -\frac{1}{\lambda^l_{wf/m}} \left(\frac{\partial \lambda_{wf/m}}{\partial S_{wf}}\right)^l V_R \tau^l_w - \beta\sigma V_R k_m \lambda^l_{wf/m} \left(\frac{\sigma_z}{\sigma}\gamma^n_w \left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^l - \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^l\right) \\ -V_R \frac{\phi_f}{\Delta t}\left[(c_\phi + c_w)_f (P^l_{of,i} - P^n_{of,i}) + 1\right] + (T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}}) \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^l_i + Source_{Ewsf} \end{array} \right\} \tag{47}$$

$$E_{wpf} = -\beta\sigma V_R k_m \lambda^l_{wf/m} - V_R \frac{\phi_f}{\Delta t} S^l_{wf,i} (c_\phi + c_w)_f - (D_{wpf} + F_{wpf}) + Source_{Ewpf}$$

$$RHS_w = \begin{bmatrix} +V_R \frac{\phi_f}{\Delta t}\left(S^l_{wf,i}(c_\phi + c_w)_f (P^l_{of,i} - P^n_{of,i}) + (S^l_{wf,i} - S^n_{wf,i}) \right) \\ \\ +V_R \tau^l_w + term_{pres} + term_{grav} + term_{cap} + Source_{wRHS} \end{bmatrix} \tag{48}$$

$$V_R \tau^l_w = \beta \sigma V_R k_m \lambda^l_{wf/m} \left[(P^l_{of,i} - P^l_{om,i}) + \frac{\sigma_z}{\sigma} \gamma^n_w (h^l_{wf} - h^l_{wm}) + (P^l_{cwom,i} - P^l_{cwof,i}) \right]$$

$$term_{pres} = -D_{wpf} P^l_{of,i-1} + (D_{wpf} + F_{wpf}) P^l_{of,i} - F_{wpf} P^l_{of,i+1}$$

$$term_{grav} = T^l_{xw,i-\frac{1}{2}} \gamma^n_w (D_{i-1} - D_i) + T^l_{xw,i+\frac{1}{2}} \gamma^n_w (D_{i+1} - D_i)$$

$$term_{cap} = T^l_{xw,i-\frac{1}{2}} P^l_{cwof,i-1} - \left(T^l_{xw,i-\frac{1}{2}} + T^l_{xw,i+\frac{1}{2}} \right) P^l_{cwof,i} + T^l_{xw,i+\frac{1}{2}} P^l_{cwof,i+1}$$

$$Source_{Ewsf} = \begin{cases} WI \left(P_{well} - P^l_{of,i} \right) \left(\frac{\partial \lambda_{wf}}{\partial S_{wf}} \right)^l, & \text{if Pressure-Controlled well} \\ \\ -WI \left(P^l_{well} - P^l_{of,i} \right) \left(\frac{\partial \lambda_{of}}{\partial S_{wf}} \right)^l, & \text{if Total Rate-Controlled well} \end{cases}$$

$$Source_{Ewell} = \begin{cases} 0, & \text{if Pressure-Controlled well} \\ \\ -WI \lambda^l_{of}, & \text{if Total Rate-Controlled well} \end{cases}$$

$$Source_{Ewpf} = \begin{cases} -WI \lambda^l_{wf}, & \text{if Pressure-Controlled well} \\ WI \lambda^l_{of}, & \text{if Total Rate-Controlled well} \end{cases}$$

$$Source_{wRHS} = \begin{cases} -WI\lambda_{wf}^{l}\left(P_{well} - P_{of,i}^{l}\right), & \text{if Pressure-Controlled well} \\ -q_{tf} + WI\lambda_{of}^{l}\left(P_{well}^{l} - P_{of,i}^{l}\right) & \text{if Total Rate-Controlled well} \end{cases}$$

Using Newton-Raphson to expand each term in Eq(26)

$$P_{om}^{n+1} = P_{om}^{l+1} = P_{om}^{l+1} + \delta P_{om}$$

$$S_{wm}^{n+1} = S_{wm}^{l+1} = S_{wm}^{l+1} + \delta S_{wm}$$

Substitute all expanded terms into Eq(26) and multiply the both sides by V_R. Assuming that multiplication of δ and δ terms is negligible, we can rearrange Eq(26) as follows

$$V_R\tau_w^{n+1} = \left\{ \begin{array}{c} V_R\frac{\phi_f}{\Delta t}\left[S_{wm,i}^{l}(c_\phi + c_w)_m(P_{om,i}^{l} - P_{om,i}^{n}) + (S_{wm,i}^{l} - S_{wm,i}^{n})\right] \\ +V_R\frac{\phi_f}{\Delta t}S_{wm,i}^{l}(c_\phi + c_w)_m\delta P_{om,i} \\ +V_R\frac{\phi_f}{\Delta t}\left((c_\phi + c_w)_m(P_{om,i}^{l} - P_{om,i}^{n}) + 1\right)\delta S_{wm,i} \end{array} \right\} \tag{49}$$

Using the fact that Eq(49)=Eq(30), thus we can rearange as follows

$$E_{\tau wpf}\delta P_{of,i} + E_{\tau wpm}\delta P_{om,i} + E_{\tau wsf}\delta S_{wf,i} + E_{\tau wsm}\delta S_{wm,i} = RHS_{\tau w} \tag{50}$$

where,

$$E_{\tau wpf} = \beta\sigma V_R k_m \lambda_{wf/m}^{l}$$

$$E_{\tau wpm} = -\beta\sigma V_R k_m \lambda_{wf/m}^{l} - V_R\frac{\phi_m}{\Delta t}S_{wm,i}^{l}(c_\phi + c_w)_m$$

$$E_{\tau wsf} = \frac{1}{\lambda_{wf/m}^{l}}\left(\frac{\partial \lambda_{wf/m}}{\partial S_{wf}}\right)^{l} V_R\tau_w^{l} + \beta\sigma V_R k_m \lambda_{wf/m}^{l}\left(\frac{\sigma_z}{\sigma}\gamma_w^{n}\left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^{l} - \left(\frac{\partial P_{cwof}}{\partial S_{wf}}\right)^{l}\right)$$

$$E_{\tau wsm} = \beta\sigma V_R k_m \lambda_{wf}^l \left(-\frac{\sigma_z}{\sigma}\gamma_w^n \left(\frac{\partial h_{wm}}{\partial S_{wm}}\right)^l + \left(\frac{\partial P_{cwom}}{\partial S_{wm}}\right)^l \right) - V_R \frac{\phi_m}{\Delta t} \left((c_\phi + c_w)_m (P_{om,i}^l - P_{om,i}^n) + 1 \right)$$

$$RHS_{\tau w} = V_R \frac{\phi_m}{\Delta t} \left[S_{wm,i}^l (c_\phi + c_w)_m (P_{om,i}^l - P_{om,i}^n) + (S_{wm,i}^l - S_{wm,i}^n) \right] - V_R \tau_w^l$$

$$V_R \tau_w^l = \beta\sigma V_R k_m \lambda_{wf/m}^l \left[(P_{of,i}^l - P_{om,i}^l) + \frac{\sigma_z}{\sigma}\gamma_w^n (h_{wf}^l - h_{wm}^l) + (P_{cwom,i}^l - P_{cwof,i}^l) \right]$$

$$\left(\frac{\partial T_{xwf}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^l = \beta k_{f,eff} \left(\frac{\Delta y \Delta z}{\Delta x}\right)_{i-\frac{1}{2}} \left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^l$$

$$\frac{\partial \lambda_{wf}}{\partial S_{wf}} = \frac{k_{rwf}^* n_{wf}}{\mu_w (1 - S_{owrf} - S_{wrf})} \left(\frac{S_{wf} - S_{wrf}}{1 - S_{owrf} - S_{wrf}}\right)^{n_{wf}-1}$$

$$\frac{\partial \lambda_{om}}{\partial S_{wm}} = -\frac{k_{rowm}^* n_{om}}{\mu_o (1 - S_{owrm} - S_{wrm})} \left(\frac{1 - S_{om} - S_{wrm}}{1 - S_{owrm} - S_{wrm}}\right)^{n_{om}-1}$$

$$\frac{\partial \lambda_{wf}}{\partial S_{wf}} = \frac{k_{rwf}^* n_{wf}}{\mu_w (1 - S_{owrf} - S_{wrf})} \left(\frac{S_{wf} - S_{wrf}}{1 - S_{owrf} - S_{wrf}}\right)^{n_{wf}-1}$$

$$\beta = 0.006328$$

Using Newton-Raphson to expand terms in Eq(28), substitute all expanded terms into Eq(28) and rearrange we get

$$V_R\tau_o^{n+1} = \left\{ \begin{array}{c} V_R\tau_o^l + \beta\sigma V_R k_m \lambda_{om/f}^l \delta P_{of,i} - \beta\sigma V_R k_m \lambda_{om/f}^l \delta P_{om,i} \\ + \left[\frac{1}{\lambda_{om/f}^l} \left(\frac{\partial \lambda_{om/f}}{\partial S_{wm}} \right)^l V_R \tau_o^l - \beta\sigma V_R k_m \lambda_{om/f}^l \frac{\sigma_z}{\sigma} \gamma_o^n \left(\frac{\partial h_{wm}}{\partial S_{wm}} \right)^l \right] \delta S_{wm,i} \\ -\beta\sigma V_R k_m \lambda_{om/f}^l \frac{\sigma_z}{\sigma} \gamma_o^n \left(\frac{\partial h_{wf}}{\partial S_{wf}} \right)^l \delta S_{wf,i} \end{array} \right\} \tag{51}$$

where $\beta = 0.006328$

Multiply Eq(27) by V_R and discretize for 1D problem we get

$$\left[\begin{array}{c} T_{xo,i-\frac{1}{2}}^{n+1} P_{of,i-1}^{n+1} - \left(T_{xo,i-\frac{1}{2}}^{n+1} + T_{xo,i+\frac{1}{2}}^{n+1} \right) P_{of,i}^{n+1} + T_{xo,i+\frac{1}{2}}^{n+1} P_{of,i+1}^{n+1} \\ -T_{xo,i-\frac{1}{2}}^{n+1} \gamma_o^n (D_{i-1} - D_i) - T_{xo,i+\frac{1}{2}}^{n+1} \gamma_o^n (D_{i+1} - D_i) \\ -V_R \tau_o^{n+1} + q_{of}^{n+1} \end{array} \right] = \phi_f \left[S_{of}^{n+1} (c_\phi + c_o)_f \frac{P_{of}^{n+1} - P_{of}^n}{\Delta t} + \frac{S_{of}^{n+1} - S_{of}^n}{\Delta t} \right] \tag{52}$$

Using Newton-Raphson to expand each term in Eq(52)

$$T_{xo}^{n+1} = T_{xo}^{l+1} = T_{xo}^l + \left(\frac{\partial T_{xo}}{\partial S_{wf}} \right)^l \delta S_{wf}$$

where, $\frac{\partial T_{xo}}{\partial S_{wf}} = \beta k_f \frac{\Delta y \Delta z}{\Delta x} \frac{\partial \lambda_{of}}{\partial S_{wf}}$

We can expand sink/source terms as follows

Option:1 Pressure-Controlled Well

$$q_{of}^{n+1} = WI \lambda_{of}^{n+1} \left(P_{well} - (P_{of,i}^{n+1}) \right) \tag{53}$$

$$= WI \left(\lambda_{of}^l + \left(\frac{\partial \lambda_{of}}{\partial S_{wf}} \right)^l \delta S_{wf} \right) \left(P_{well} - P_{of,i}^l - \delta P_{of,i} \right) \tag{54}$$

$$= \left\{ \begin{array}{c} WI\lambda_{of}^{l}\left(P_{well}-P_{of,i}^{l}\right)-WI\lambda_{of}^{l}\delta P_{of} \\ +WI\left(P_{well}-P_{of,i}^{l}\right)\left(\frac{\partial \lambda_{of}}{\partial S_{wf}}\right)^{l}\delta S_{wf} \end{array} \right\} \tag{55}$$

Option:2 Total Rate-Controlled Well

$$q_{of}^{n+1} = q_{tf}-WI\lambda_{wf}^{n+1}\left(P_{well}^{n+1}-(P_{of,i}^{n+1})\right) \tag{56}$$

$$= q_{tf}-WI\left(\lambda_{wf}^{l}+\left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^{l}\delta S_{wf}\right)\left(P_{well}^{l}-P_{of,i}^{l}+\delta P_{well}-\delta P_{of,i}\right) \tag{57}$$

$$= \left\{ \begin{array}{c} q_{tf}-WI\lambda_{wf}^{l}\left(P_{well}^{l}-P_{of,i}^{l}\right)-WI\lambda_{wf}^{l}\left(\delta P_{well}-\delta P_{of,i}\right) \\ -WI\left(P_{well}^{l}-P_{of,i}^{l}\right)\left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^{l}\delta S_{wf} \end{array} \right\} \tag{58}$$

Substitute all expanded terms into Eq(52) and multiply the both sides by V_R.Assuming that multiplication of δ and δ terms is negligible, we can rearrange Eq(52) as follows

$$\left[\begin{array}{c} D_{opf}\delta P_{of,i-1}+E_{opf}\delta P_{of,i}+F_{opf}\delta P_{of,i+1} \\ +D_{osf12}\delta S_{wf,i-\frac{1}{2}}+E_{osf}\delta S_{wf,i}+F_{osf12}\delta S_{wf,i+\frac{1}{2}} \\ +E_{osm}\delta S_{wm,i}+E_{opm}\delta P_{om,i} \end{array} \right] = RHS_o \tag{59}$$

where

$$D_{opf}=T_{xo,i-\frac{1}{2}}^{l},\ F_{opf}=T_{xo,i-\frac{1}{2}}^{l}$$

$$D_{osf12}=\left(P_{of,i-1}^{l}-P_{of,i}^{l}-\gamma_{o}^{n}(D_{i-1}-D_{i})\right)\left(\frac{\partial T_{xo}}{\partial S_{wf}}\right)_{i-\frac{1}{2}}^{l}$$

$$F_{osf12} = \left(P^l_{of,i+1} - P^l_{of,i} - \gamma^n_o (D_{i+1} - D_i)\right) \left(\frac{\partial T_{xo}}{\partial S_{wf}}\right)^l_{i+\frac{1}{2}}$$

$$E_{osm} = -\frac{1}{\lambda^l_{om/f}} \left(\frac{\partial \lambda_{om/f}}{\partial S_{wm}}\right)^l V_R \tau^l_o + \beta\sigma V_R k_m \lambda^l_{om/f} \frac{\sigma_z}{\sigma} \gamma^n_o \left(\frac{\partial h_{wm}}{\partial S_{wm}}\right)^l$$

$$E_{opm} = \beta\sigma V_R k_m \lambda^l_{om/f}$$

$$E_{osf} = -\beta\sigma V_R k_m \lambda^l_{om/f} \frac{\sigma_z}{\sigma} \gamma^n_o \left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^l + V_R \frac{\phi_f}{\Delta t} \left[(c_\phi + c_o)_f (P^l_{of,i} - P^n_{of,i}) + 1\right] + Source_{Eosf}$$

$$E_{opf} = -\beta\sigma V_R k_m \lambda^l_{om/f} - V_R \frac{\phi_f}{\Delta t} S^l_{of,i} (c_\phi + c_o)_f - (D_{opf} + F_{opf}) + Source_{Eopf}$$

$$RHS_o = \begin{bmatrix} V_R \frac{\phi_f}{\Delta t} \left(S^l_{of,i}(c_\phi + c_o)_f (P^l_{of,i} - P^n_{of,i}) + (S^l_{of,i} - S^n_{of,i})\right) \\ + V_R \tau^l_o + term_{pres} + term_{grav} + Source_{oRHS} \end{bmatrix} \tag{60}$$

$$V_R \tau^l_o = \beta\sigma V_R k_m \lambda^l_{om/f} \left[(P^l_{of,i} - P^l_{om,i}) + \frac{\sigma_z}{\sigma} \gamma^n_o (h^l_{wf} - h^l_{wm})\right]$$

$$term_{pres} = -D_{opf} P^l_{of,i-1} + (D_{opf} + F_{opf}) P^l_{of,i} - F_{opf} P^l_{of,i+1}$$

$$term_{grav} = T^l_{xo,i-\frac{1}{2}} \gamma^n_o (D_{i-1} - D_i) + T^l_{xo,i+\frac{1}{2}} \gamma^n_o (D_{i+1} - D_i)$$

$$Source_{Eosf} = \begin{cases} WI\left(P_{well} - P^l_{of,i}\right)\left(\frac{\partial \lambda_{of}}{\partial S_{wf}}\right)^l, & \text{if Pressure-Controlled well} \\ \\ -WI\left(P^l_{well} - P^l_{of,i}\right)\left(\frac{\partial \lambda_{wf}}{\partial S_{wf}}\right)^l, & \text{if Total Rate-Controlled well} \\ \end{cases}$$

$$Source_{Ewell} = \begin{cases} 0, & \text{if Pressure-Controlled well} \\ -WI\lambda^l_{wf}, & \text{if Total Rate-Controlled well} \\ \end{cases}$$

$$Source_{Ewpf} = \begin{cases} -WI\lambda^l_{of}, & \text{if Pressure-Controlled well} \\ WI\lambda^l_{wf}, & \text{if Total Rate-Controlled well} \end{cases}$$

$$Source_{wRHS} = \begin{cases} -WI\lambda^l_{of}\left(P_{well} - P^l_{of,i}\right), & \text{if Pressure-Controlled well} \\ \\ -q_{tf} + WI\lambda^l_{wf}\left(P^l_{well} - P^l_{of,i}\right) & \text{if Total Rate-Controlled well} \end{cases}$$

Using Newton-Raphson to expand each term in Eq(29) and substitute all expanded terms into Eq(29). Multiply both sides by V_R. Assuming that multiplication of δ and δ terms is negligible, we can rearrange Eq(29) as follows

$$V_R\tau_o^{n+1} = \left\{ \begin{array}{c} V_R\frac{\phi_f}{\Delta t}\left[S^l_{om,i}(c_\phi + c_o)_m(P^l_{om,i} - P^n_{om,i}) + (S^l_{om,i} - S^n_{om,i})\right] \\ \\ +V_R\frac{\phi_f}{\Delta t}S^l_{wo,i}(c_\phi + c_o)_m\delta P_{om,i} \\ \\ -V_R\frac{\phi_f}{\Delta t}\left((c_\phi + c_o)_m(P^l_{om,i} - P^n_{om,i}) + 1\right)\delta S_{wm,i} \end{array} \right\} \quad (61)$$

Using the fact that Eq(61)=Eq(51), thus we can rearange as follows

$$E_{\tau opf}\delta P_{of,i} + E_{\tau opm}\delta P_{om,i} + E_{\tau osf}\delta S_{wf,i} + E_{\tau osm}\delta S_{wm,i} = RHS_{\tau o} \tag{62}$$

where,

$$E_{\tau opf} = \beta\sigma V_R k_m \lambda^l_{om/f}$$

$$E_{\tau opm} = -\beta\sigma V_R k_m \lambda^l_{om/f} - V_R \frac{\phi_m}{\Delta t} S^l_{om,i}(c_\phi + c_o)_m$$

$$E_{\tau osf} = \beta\sigma V_R k_m \lambda^l_{om/f} \frac{\sigma_z}{\sigma} \gamma^n_o \left(\frac{\partial h_{wf}}{\partial S_{wf}}\right)^l$$

$$E_{\tau osm} = \frac{1}{\lambda^l_{om/f}} \left(\frac{\partial \lambda_{om/f}}{\partial S_{wm}}\right)^l V_R \tau^l_o - \beta\sigma V_R k_m \lambda^l_{om/f} \frac{\sigma_z}{\sigma} \gamma^n_o \left(\frac{\partial h_{wm}}{\partial S_{wm}}\right)^l + V_R \frac{\phi_m}{\Delta t} \left((c_\phi + c_o)_m (P^l_{om,i} - P^n_{om,i}) + 1\right)$$

$$RHS_{\tau o} = V_R \frac{\phi_m}{\Delta t} \left[S^l_{om,i}(c_\phi + c_o)_m (P^l_{om,i} - P^n_{om,i}) + (S^l_{om,i} - S^n_{om,i})\right] - V_R \tau^l_o$$

$$V_R \tau^l_o = \beta\sigma V_R k_m \lambda^l_{om/f} \left[(P^l_{of,i} - P^l_{om,i}) + \frac{\sigma_z}{\sigma} \gamma^n_o (h^l_{wf} - h^l_{wm})\right]$$

$$\left(\frac{\partial T_{xof}}{\partial S_{wf}}\right)^l_{i-\frac{1}{2}} = \beta k_{f,eff} \left(\frac{\Delta y \Delta z}{\Delta x}\right)_{i-\frac{1}{2}} \left(\frac{\partial \lambda_{of}}{\partial S_{wf}}\right)^l_{i-\frac{1}{2}}$$

$$\frac{\partial \lambda_{of}}{\partial S_{wf}} = -\frac{k^*_{rowf} n_{of}}{\mu_o (1 - S_{owrf} - S_{wrf})} \left(\frac{1 - S_{of} - S_{wrf}}{1 - S_{owrf} - S_{wrf}}\right)^{n_{of}-1}$$

$$\frac{\partial \lambda_{om}}{\partial S_{wm}} = -\frac{k^*_{rowm} n_{om}}{\mu_o (1 - S_{owrm} - S_{wrm})} \left(\frac{1 - S_{om} - S_{wrm}}{1 - S_{owrm} - S_{wrm}}\right)^{n_{om}-1}$$

$$\frac{\partial \lambda_{wf}}{\partial S_{wf}} = \frac{k^*_{rwf} n_{wf}}{\mu_w (1 - S_{owrf} - S_{wrf})} \left(\frac{S_{wf} - S_{wrf}}{1 - S_{owrf} - S_{wrf}} \right)^{n_{wf}-1}$$

$$\beta = 0.006328$$