1. Suppose an air bubble of initial volume $10 \text{ cm}^3 \ (10^{-5} \text{ m}^3)$ and temperature of 20 C (298 K) rises from the bottom of a 5 m deep lake. Assume the water temperature is a uniform 20 C. The pressure depends on depth according to the relation: $P = P_0 + \rho gd$, where $P_0$ is the pressure at the surface, $P_0 = 1.0 \times 10^5 \text{ Pa}$, $\rho = 10^3 \text{ kg/m}^3$ is the density of water, $g = 9.8 \text{ m/s}^2$ is the acceleration of gravity, and $d$ is the depth. Show all work. ($k_B = 1.38 \times 10^{-23} \text{ J/K}, R = 8.31 \text{ J/K-mol}$)

(a) What is the pressure at the bottom of the lake?

(b) How many air molecules are in the air bubble?

(c) Suppose the bubble rises fast enough such that no heat has time to enter the bubble before it reaches the surface; that is, its expansion is adiabatic. What is the volume of the air bubble when it reaches the surface?

(d) For the situation described in part (c), what is the temperature of the air in the bubble when it reaches the surface?
2. Consider two magnetic systems in thermal contact with each other but otherwise thermally isolated. The first, system A, consists of four spin-1/2 atoms. Each atom in system A can have two possible values for its magnetic quantum number: \( m_A = +1/2, -1/2 \). The second, system B, consists of three spin-1 atoms. Each of these atoms can have three possible values for its magnetic quantum number: \( m_B = +1, 0, -1 \).

(a) How many microstates does each system have separately and together?

(b) Define the macroscopic magnetization for each system as the net value of the magnetic quantum number, i.e. \( M_A = [(+1/2) n_A(+1/2) + (-1/2) n_A(-1/2)] \) and \( M_B = [(+1) n_B(+1) + (0) n_B(0) + (-1) n_B(-1)] \), where \( n_A(m_A) \) (i.e. the function, “\( n_A\)-of-\( m_A \)” ) is the number of atoms in system A with magnetic quantum number, \( m_A = \pm 1/2 \), and similarly for system B but with \( m_B = -1, 0, +1 \). Let \( \Omega_A(N_A = 4, M_A) \) be the multiplicity for system A with magnetization \( M_A \) and similarly for system B. Let’s treat the case where the total magnetization is fixed at \( M = M_A + M_B = +1 \). Finish filling out the multiplicity table below for the \( M = +1 \) case where \( M = M_A + M_B \) for each row adds up to \( M = +1 \). (The \( M_B = -3 \) and \( M_B = -2 \) cases don’t contribute to the \( M = +1 \) case under study because it would require a magnetization for system A that cannot be reached with 4 spin-1/2 atoms.)

\[
\begin{array}{cccc}
M_A & \Omega_A(4, M_A) & M_B(= 1 - M_A) & \Omega_B(3, M_B) & \Omega_A(4, M_A)\Omega_B(3, M_B = 1 - M_A) \\
-4 & 0 \text{ (not possible)} & -3 & 1 & 0 \times 1 = 0 \\
-3 & 0 \text{ (not possible)} & -2 & 3 & 0 \times 3 = 0 \\
+2 & & -1 & & \\
+1 & & & & \\
0 & & +1 & & \\
-1 & & +2 & & \\
-2 & & +3 & &
\end{array}
\]

(c) What is the probability that system A has \( M_A = +1 \)? Explain your reasoning.

(d) What is the equilibrium value of \( M_A \)? Explain your reasoning.
3. (a) Consider a closed gas cylinder of fixed total volume, \( V \). A moveable leak-proof piston separates the cylinder into two systems, A and B. The piston is a good thermal conductor so the two systems are in thermal equilibrium (i.e. \( T_A = T_B \)) at all times. Recall that when two systems are in mechanical equilibrium, the composite multiplicity is maximal with respect to variation of the volume of either one of the systems. Suppressing the energy and particle number dependence of the multiplicity, the entropy of the two systems is:

\[
S_{AB}(V, V_A) = S_A(V_A) + S_B(V_B) = k \ln \Omega_A(V_A) + \ln \Omega_B(V_B = V - V_A),
\]

where \( V = V_A + V_B \) is the total volume (fixed). Show that when the two systems are in mechanical equilibrium under these conditions, the entropies satisfy the condition: \( \frac{\partial S_A}{\partial V_A} = \frac{\partial S_B}{\partial V_B} \). (Note that the systems are always in thermal equilibrium so the internal energies are not changed as the volume changes.)

(b) Recall that the pressure is the “thing” that is the same when two systems are in mechanical equilibrium. Based on the result from part (a), one can define the “entropic pressure”, \( P_A \), for system A to be:

\[
P_A = T_A \frac{\partial S_A}{\partial V_A}.
\]

Use this definition to derive an expression for the pressure of a monatomic ideal gas as a function of volume and temperature. The Sakur-Tetrode expression for the multiplicity of \( N_A \) gas atoms of mass, \( m \), confined to volume, \( V_A \), is

\[
\Omega(N_A, V_A, U_A) = \left[ \frac{V_A}{N_A} \left( \frac{4\pi m U_A}{3h^2 N_A} \right)^{\frac{3}{2}} e^{\frac{U_A}{2kT}} \right]^{N_A},
\]

where \( h \) is Planck’s constant, \( V_A \) is the volume, and, \( U_A \) is the internal energy which, by the equipartion theorem, can be written, \( U_A = \frac{3}{2} N_A kT \). Show all work.