As you know, the nucleus is composed of neutrons and protons which are spin-1/2 fermions. Let $N$ equal the number of neutrons, $Z$ equal the number of protons, and $A$ be the sum, $A = N + Z$. Nucleons bind together with short range forces which means they pack together like hard spheres; so the nucleon density is approximately constant with the volume proportional to $A$. The nuclear radius then scales like the cube root of the volume, $R = r_0 A^{\frac{1}{3}}$. A global fit to nuclear data gives $r_0 \simeq 1.2\ \text{fm}$ which is constant for all nuclei. In this problem treat the nucleus as two degenerate Fermi gases, one of protons and one of neutrons. (Data: $\hbar c = 197.3\ \text{MeV} \cdot \text{fm}, m_p \simeq m_n = 939\ \text{MeV}/c^2$, Lead (Pb): $A = 208, Z = 82,$ and $N = 126$)

(a) Find the proton density for lead (i.e. number of protons per cubic femtometer, $\text{fm}^{-3}$). 

Solution: The proton density, $n_p$, is the number of protons (82) divided by the volume: $n_p = Z/V = Z/(4\pi r_0^2 A) = 82/(4\pi (1.2\ \text{fm})^3 208) = 0.0.0545\ \text{fm}^{-3}$.

(b) What is the Fermi momentum (in MeV/c) of the protons in lead? Solution: The Fermi momentum (times $c$) given by: 

$$p_F c = \hbar c (3\pi^2 n_p)^{\frac{1}{3}} = (197.3\ \text{MeV} \cdot \text{fm})(3\pi^2 0.0545\ \text{fm}^{-3})^{\frac{1}{3}} = 226.7\ \text{MeV}.$$

(c) What is the Fermi energy (in MeV) of the protons in lead? 

Solution: Using the result from part (b), the Fermi energy is: $E_{F,p} = \frac{p_F^2}{2m} = \frac{(p_F c)^2}{2mc^2} = (226.7\ \text{MeV})^2/(2 \cdot 939\ \text{MeV}) = 27.36\ \text{MeV}$.

(d) Examine the $Z$-dependence of your result from part (c) and re-scale your proton result to find the Fermi energy of the neutrons in lead without re-doing the calculation.

Solution: The energy depends on the momentum squared which in turn depends on the density to the one third power. Thus: $E_{F,n} = E_{F,p}(\frac{N}{Z})^{\frac{1}{3}} = 36.43\ \text{MeV}$.