Physics 422: Nuclear Physics
Exam I: KEY
March 15, 2006

1. Calculate the threshold kaon laboratory kinetic energy necessary for the reaction:

$$\pi^- + p \to \Delta^0(1232)$$

where \( m_\Delta = 1232 \text{ MeV}/c^2 \), \( m_p = 938 \text{ MeV}/c^2 \), and \( m_\pi = 139 \text{ MeV}/c^2 \).

Solution: The invariant four-momentum squared (Mandelstam s) in the final state is \( s_f = m_\Delta^2 \).

The four-momentum in the initial state in the lab frame is: \( p_i = (E_i, m_p, 0, 0) \). Thus, \( s_i = p_i^2 = (E_i + m_p)^2 - p_i^2 \). Expanding and using the Einstein triangle relation \( (E^2 = m^2 + p^2) \) gives \( s_i = 2E_im_p + m_p^2 + m_\pi^2 \). Setting \( s_i = s_f \) gives \( E_i = (m_\Delta^2 - m_p^2 - m_\pi^2)/(2m_p) = 329.8 \text{ MeV} \). Thus the threshold kinetic energy is \( T_\pi = E_i - m_\pi = 190.8 \text{ MeV} \).

2. Elastic \( p^{40}\text{Ca} \) scattering was studied at Los Alamos. A copy of one of the cross section plots is attached. The kinetic energy of incident protons is 497 MeV. The rest mass of the proton is 938 MeV/c^2. Let the nuclear density be parametrized as \( \rho(r) = \rho_0/(1 + e(r-c)/a) \).

(a) What is the incident laboratory momentum of the protons in MeV/c?

Solution: Using the Einstein triangle relation gives \( p_p = \sqrt{(T_p + m_p)^2 - m_p^2} = 1086 \text{ MeV} \).

(b) Find the half-density radius, \( c \), of \( ^{40}\text{Ca} \) from this data.

Solution: For intermediate energy the scattering is predominately diffractive in nature: \( \sigma \simeq \sigma_0 |\cos(qc)\exp(-\pi qa)|^2 \) where \( q = 2p_p \sin \theta/2 \). Thus the minima occur when the phase of the cosine term goes through \( \pi \). From the graph one can see minima at \( \theta_1 = 10^\circ \) \( (q_1 = 189.3 \text{ MeV}/c) \) and \( \theta_2 = 28^\circ \) \( (q_2 = 525.45 \text{ MeV}/c) \). Thus, we find \( c = 2\pi/(q_2 - q_1) = .00187\hbar c = 3.69 \text{ fm} \).

(c) Find the diffusivity parameter, \( a \), of \( ^{40}\text{Ca} \) from this data.

Solution: From the above analysis we have the ratio of cross sections at local maxima (where the cosine term equal unity): \( \sigma_1/\sigma_2 = \exp(-2\pi a(q_1 - q_2)) \). From the graph we see maxima at \( \theta_1 = 13^\circ \) \( (q_1 = 245.9 \text{ MeV}/c) \) with \( \sigma_1 \simeq 48 \text{ mb}/sr \) and \( \theta_2 = 23^\circ \) \( (q_2 = 433.0 \text{ MeV}/c) \) with \( \sigma_2 \simeq 1.8 \text{ mb}/sr \). Thus, \( a = \hbar c \ln(\sigma_1/\sigma_2)/(2\pi(q_1 - q_2)) = .55 \text{ fm} \).

3. Estimate the lifetime of a nucleon (proton/neutron) from the net geothermal heat flux \( (\dot{\Phi}_\oplus = 0.1 \text{ W/m}^2) \) of the earth. Assume that half the rest energy of the nucleon goes into heating and the remainder escapes via neutrino emission. (Data: \( M_N = 1.67 \times 10^{-27} \text{ kg}, M_\oplus = 6.0 \times 10^{24} \text{ kg}, R_\oplus = 6.4 \times 10^{6} \text{ m} \).)

Solution: Ignoring the mass of the electron, the mass of the earth is entirely due to nucleons (protons/neutrons; so the total number of nucleons composing the earth is \( N_\oplus = 3.6 \times 10^{81} \). If nucleons decay with a mean lifetime of \( \tau \), then \( N(t) = N_\oplus e^{-t/\tau} \). If half of the rest energy goes into heat, then the total power generated is \( P_\oplus = -\frac{1}{2} \frac{dN}{dt} m_N c^2 = \frac{1}{\tau} N_\oplus m_N c^2 \). If the heat is radiated isotropically, then the heat flux at \( t = 0 \) is \( \dot{\Phi}_\oplus = P_\oplus/(4\pi R_\oplus^2) \).
Thus, \( \tau = \left( \frac{N_m^2 m N c^2}{8\pi \Phi \R^2} \right) = 5.25 \times 10^{27} \ \text{s} = 1.66 \times 10^{20} \ \text{y} \).

4. Consider the pionic break up of the deuteron which proceeds via the hadronic interaction,
\[ \pi^0 + d \to p + n. \]

At very low pion energy the relative angular momentum of the pion and deuteron in the initial state is zero \((L_i = 0)\). (Data: \((J^*T)_{\text{deuteron}} = 1^+0, \ (J^*T)_{\text{pion}} = 0^-1\).) Use the generalized Pauli exclusion principle applied to the final proton and neutron (i.e. treating them as identical particles including isospin) to find the spin, isospin, and relative angular momentum of the final state.

Solution: The initial state has total angular momentum 1 (deuteron has spin 1), parity (-1) (pion is a pseudoscalar), and isospin 1. The hadronic interaction conserves all these quantities. Treat the final state as a product of space state times spin state times isospin state. The parity and permutation properties of the space state are determined by the relative angular momentum: \((-1)^L\). Since the initial state is isovector \((t = 1)\) then the final isospin must also be \(t = 1\). Recall that the triplet states are symmetric with respect to permutation of particle labels. Thus the generalized Pauli principle requires that the space times spin state must be antisymmetric. Consider the possible cases:

1. \(3^S_1 \ (L = 0 \ \text{and} \ s = 1)\): Both of these are permutation symmetric; so this case is Pauli forbidden.

2. \(1^P_1 \ (L = 1 \ \text{and} \ s = 0)\): The singlet state is antisymmetric and so is the orbital angular momentum state, thus this combination is net symmetric and is Pauli forbidden.

3. \(3^P_1 \ (L = 1 \ \text{and} \ s = 1)\): The triplet state is permutation symmetric which when multiplied by the antisymmetric \(L = 1\) state leads to a permutation antisymmetric state which is allowed.

4. \(3^D_1 \ (L = 2 \ \text{and} \ s = 1)\): Both of these are permutation symmetric and so this case is Pauli forbidden.

Since no other orbital angular momentum values can be combined with the spin to give a total angular momentum of one, all cases are exhausted. By permutation considerations, the only possible allowed case is \(3^P_1\) (case 3.).

5. Consider pion-nucleon scattering which proceeds via the strong interaction. The pion is an isospin triplet \((t_\pi = 1)\) and the nucleon is an isospin doublet \((t_N = 1/2)\); thus the total multiplicity is \(3 \times 2 = 6\).

(a) Identify the values of the allowed total isospin and show that the total multiplicity is still six.

Solution: Adding isospin one to isospin 1/2 can result in a total isospin of 3/2 (stretched state) or 1/2 (jack-knife state). Thus the total multiplicity is \((2 \cdot \frac{3}{2} + 1) + (2 \cdot \frac{1}{2} + 1) = 4 + 2 = 6\).

(b) Use the Clebsch-Gordon tables to write the the \(|\pi^+n\rangle\)-state and the \(|\pi^0p\rangle\)-state in terms of the states of good total isospin. (Use bra-ket notation: \(|(t_1 t_2) \ t \ t_3\rangle\). For example, \(|\pi^+p\rangle = |1 + 1\rangle |\frac{1}{2} + \frac{1}{2}\rangle = |(1 \frac{1}{2}) 3 \frac{3}{2} + 3 \frac{3}{2}\rangle\).
Solution: From the CG-Tables:

$$|\pi^+ n\rangle = \sqrt{\frac{1}{3}} \left| \frac{3}{2} \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} \left| \frac{1}{2} \frac{1}{2} \right\rangle$$

$$|\pi^0 p\rangle = \sqrt{\frac{2}{3}} \left| \frac{3}{2} \frac{1}{2} \right\rangle - \sqrt{\frac{1}{3}} \left| \frac{1}{2} \frac{1}{2} \right\rangle$$

(c) Assume the scattering amplitude is dominated by the $T = 1/2$-channel (a fictional case), find the cross section ratio,

$$\frac{\sigma(\pi^+ + n \rightarrow \pi^0 + p)}{\sigma(\pi^+ + n \rightarrow \pi^+ + n)}.$$

Solution: Since the hadronic interaction conserves isospin only transitions between the same values of isospin are allowed (e.g. $\frac{3}{2} \rightarrow \frac{3}{2}$ or $\frac{1}{2} \rightarrow \frac{1}{2}$). By assumption, we may neglect the first case. Thus,

$$\sigma(\pi^+ + n \rightarrow \pi^0 + p) \propto \frac{2}{9} |\langle \frac{1}{2} \frac{1}{2} | H | \frac{1}{2} \frac{1}{2} \rangle|^2$$

$$\sigma(\pi^+ + n \rightarrow \pi^+ + n) \propto \frac{4}{9} |\langle \frac{1}{2} \frac{1}{2} | H | \frac{1}{2} \frac{1}{2} \rangle|^2$$

Thus the ratio is $1/2$. 