Numerical Solution of first order ODEs:

General form: \( \frac{dy}{dx} = f(x, y) \)

\[ \begin{pmatrix} \frac{dy}{dx} \\ \frac{d^2 y}{dx^2} \\ \vdots \\ \frac{d^n y}{dx^n} \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} \]

Simple, familiar example: Newton's Law of motion:

\( \begin{pmatrix} \frac{d\vec{x}}{dt} \\ \frac{d\vec{v}}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\vec{F}(\vec{x}, t)}{m} \end{pmatrix} \)

Specific solution requires boundary conditions; e.g.,

\( x(0) = x_0 \)
\( v(0) = v_0 \)

In typical numerical analysis fashion we must define a domain of interest (time interval).

Discretizing in uniform space:

\( h = \frac{(t_f - t_i)}{N} \)
Under the marker assumption of "smoothness", use Taylor expansion about the last known value, i.e.

Starting:

\[ x_{i+1} = x_i + \frac{dx}{dt} \cdot h + \frac{1}{2} \frac{d^2 x}{dt^2} \cdot h^2 + \ldots \]
\[ v_{i+1} = v_i + \frac{dv}{dt} \cdot h + \frac{1}{2} \frac{d^2 v}{dt^2} \cdot h^2 + \ldots \]

Truncate to leading order:

\[ x_{i+1} = x_i + v_i \cdot h + O(h^2) \]
\[ v_{i+1} = v_i + F(x_i, v_i, 0) \cdot h + O(h^3) \]

Now that we "know" \( x_i, v_i \) we can get the next point. Repeatedly, we have for the \((i+1)\)-th step:

\[ x_{i+1} = x_i + v_i \cdot h \]
\[ v_{i+1} = v_i + \frac{F(x_i, v_i, t_i)}{m} \cdot h \]

"Euler" (damped, explicit, integration).
Problems with Euler.

Consider spring-mass system:

\[ \frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{k}{m} x \]

For definiteness, take \( x(t=0) = x_0, \quad v(t=0) = 0 \).

The exact solution is:

\[ x(t) = x_0 \cos \omega t \]
\[ v(t) = \omega x_0 \sin \omega t \]

where

\[ \omega = \sqrt{\frac{k}{m}} \]

This system conserves the total energy:

\[ E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \]

\[ = \frac{1}{2} m \left( \frac{k}{m} \right) x_0^2 \sin^2 \omega t + \frac{1}{2} k x_0^2 \cos^2 \omega t \]

\[ = \frac{1}{2} k x_0^2 \left( \sin^2 \omega t + \cos^2 \omega t \right) = \frac{1}{2} k x_0^2 \]

(independent of \( t \))
Examine the Euler algorithm at $t = 1$ (one time step):

$$
\begin{align*}
    x &= x_0 + v_0 \cdot h = x_0 \\
    v &= v_0 - \frac{k}{m} x_0 \cdot h = -\frac{k}{m} x_0 \cdot h
\end{align*}
$$

The energy is not conserved:

$$
\begin{align*}
    E(t = h) &= \frac{1}{2} m \left( -\frac{k}{m} x_0 \cdot h \right)^2 + \frac{1}{2} k x_0^2 \\
    &= E_0 + \frac{1}{2} \left( \frac{k}{m} x_0^2 \right) h^2
\end{align*}
$$

The algorithm gets $(x, v)$ current to $O(h^2)$, but the energy diverges as $O(h^2)$. 
Higher order Euler - \( \frac{dy}{dx} = f(x,y) \)

\[ y(1) = y_0 + y'(0) h + y''(0) \frac{h^2}{2} + \ldots \]

\[ = y(0) + f(0,y_0) h + \frac{d}{dx} f(x,y) \frac{h^2}{2} + \ldots \]

\[ \frac{df}{dx} \bigg|_{x=0} = \frac{2f}{2x} + \frac{df}{dy} \frac{dy}{dx} \bigg|_{x=0} = \left[ \frac{d^2 f}{dx^2} + \frac{df}{dy} \right] \]

Useful if \( f \) is known analytically.

Otherwise use finite element approximations.

Specifically, at \( x = i+1 \):

\[ y_{i+1} = y_i + \frac{h}{2} \left( f_i + f_{i+1} \right) \]

\[ + \left( \frac{f_i - f_{i-1}}{h} + \frac{f_{i+1} - f_i}{h} \right) + \left( \frac{y_i - y_{i-1}}{h^2} \right) \frac{h^2}{2} \]

Better approach - implicit.
Implicit Multistep approach.

Formally, we can write the integration step exactly:

\[ y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y) \, dx \]

This is not yet useful because we need \( f \) at points we haven't evaluated yet. But we can use Taylor to use information outside the integration region to evaluate \( f \) inside:

Easiest non-trivial case - linear

\[ f(x_i \leq x \leq x_{i+1}) = f_i + \left( f_i - f_{i-1} \right) (x - x_i) + \cdots \]

Now do the integral to get 2nd order "Adams-Bashforth"

\[ y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} [f_i + \frac{(f_i - f_{i-1}) (x - x_i)}{1}] \, dx \]

\[ = y_i + f_i \frac{h}{2} + \frac{f_i - f_{i-1}}{2} \left( x_{i+1} - x_i \right) \]

\[ = y_i + f_i \frac{h}{2} + \frac{f_i - f_{i-1}}{2} \left( x_{i+1} - x_i \right) \]