Due Sept. 12

The goal of this project is to create a numerical code to calculate the structure of white dwarf stars. The fundamental physics of the white dwarf star involves the balance of the inward force due to gravitational attraction with the outward force due to the degenerate electron pressure. This later phenomenon results from the application of the Pauli exclusion principle and a simple model for the allowed quantum states of the electrons which allows one to calculate the electron energy. In standard thermodynamic fashion the pressure is the negative derivative of the energy with respect to volume. As discussed in class, the resulting equations are a pair of coupled first order ordinary differential equations:

\[ \frac{dm}{dr} = 4\pi r^2 \rho(r) \]  
\[ \frac{d\rho}{dr} = -\frac{Gm(r) 3m_e}{r^2} \left( \frac{64\pi^2 m_e^5 \rho(r)}{9Y_e} \right)^{1/3} \]

where \( m(r) \) is the mass out to radius \( r \), \( \rho(r) \) is the mass density, \( m_e \) (\( m_p \)) is the electron (proton) mass, \( Y_e \) is “metallicity” (i.e. the ratio of protons to the sum of protons plus neutrons), \( G \) is Newton’s gravitational constant, and \( h \) is Planck’s constant.

(a) (20 points) From basic physics considerations one can show that the relevant length and mass scales in this problem are:

\[ \rho_0 = \frac{8\pi m_p}{3Y_e} \left( \frac{m_e c}{h} \right)^3 \]  
\[ R_0 = \sqrt{\left( \frac{h^2}{3m_e} \right) \left( \frac{9Y_e}{64\pi^2 m_p^5 \rho_0} \right)^{1/3} \left( \frac{1}{4\pi G} \right)} \]

First show that these quantities have the correct dimensions (10 points), and then use them to define dimensionless quantities \( \{x, \tilde{\rho}, \tilde{m}\} \): \( r = R_0 x \), \( m(r) = M_0 \tilde{m}(x) \), \( \rho(r) = \rho_0 \tilde{\rho}(x) \), and show that Eq. (1-2) take the form (10 points):

\[ \frac{d\tilde{m}}{dx} = x^2 \tilde{\rho}(x) \]  
\[ \frac{d\tilde{\rho}}{dx} = -\frac{\tilde{m}^{1/3}}{x^2} \]

b. (50 points) Write a computer code using to calculate the density profile of a white dwarf star given as input the scaled central density \( (\tilde{\rho}(0)) \) and metallicity \( (Y_e) \) (30 points). The density will decrease with \( x \) and at some point go to zero; this determines the radius (and mass) of the star which can then be compared to data. Demonstrate that your code is working properly by using \( Y_e = 1 \) and \( \tilde{\rho}(0) = 1 \) and obtaining a star with a mass of \( 5.43 \times 10^{30} \) kg and a radius of \( 1.99 \times 10^7 \) m (20 points).

c. (10 points) Test the convergence of your code by examining the sensitivity of your results to changes in the (arbitrary) numerical parameters (such as the number of grid points).

d. (20 points) Most white dwarf stars have such high electron densities that the equation of state must be treated relativistically. The mass equation is unchanged, but the density equation picks up an additional density-dependent term:

\[ \frac{d\tilde{m}}{dx} = x^2 \tilde{\rho}(x) \]  
\[ \frac{d\tilde{\rho}}{dx} = -\frac{\tilde{m}^{1/3}}{x^2} \sqrt{1 + \tilde{\rho}^{2/3}} \]

Add the relativistic correction to your code (10 points). Use it to calculate radius of Sirius B assuming \( Y_e = 0.5 \) (Carbon) and repeat the calculation for \( Y_e = 0.464 \) (Iron). Adjust the central density to give the observed mass of \( 2.09 \times 10^{30} \) kg. What is your conclusion as to the make up of Sirius B if the observed radius is \( 5.15 \times 10^6 \) m? (10 points)