Agenda

1. Review for Exam II: Wed. 12-12:50
   BH 243

Thought for the day...
I intend to live forever... so far so good! See...
Exam II Review

Griffiths Ch. 10 & 11 + relativity

1. Electrodynamics: Maxwell's Eqs $\Rightarrow$ potentials

\[ D^2 A^\mu - \partial^\mu \xi = -\rho \tilde{J}^\mu \]

\[ A^\mu = (\vec{A}, \vec{V}) \quad \xi = \partial\mu A^\mu \quad \tilde{J}^\mu = (\vec{J}, \varphi) \]

\[ D^2 = \partial\mu J^\mu \]
II. 4-vectors & relativity

Lorentz 4-vector: transforms as $\chi^\mu$

$\chi^\mu = (\vec{\chi}, ct)$

4-vector contractions $\rightarrow$ Lorentz scalars

$\chi^\mu \chi_\mu = \vec{\chi} \cdot \vec{\chi} - c^2 t^2$ Invariant

Examples:

4-momentum: $p^\mu = (\vec{p}, c, E)$

Invariant: $p^\mu p_\mu = p^2 - E^2 = -(mc^2)^2$

4-vector potential: $A^\mu = (\vec{A}, V/c)$

Invariant: Lorentz Gauge - $\mathcal{L} = \frac{1}{2} A^\mu A_\mu = 0$
IV. Calculating $A^\mu$ from sources:

Retarded potential

\[ A^\mu(\vec{r}, t) = \frac{\kappa}{4\pi} \int d^3 \vec{r}' \frac{J^\mu(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \]

\[ t_R = t - \frac{1}{c} \frac{\vec{r} \cdot \vec{v}}{c} \]

Satisfies Maxwell's Equation in Lorentz gauge

\[ \square A^\mu = -\mu J^\mu \quad \& \quad \mathcal{L} = \delta_{\mu\nu} F_{\mu\nu}^2 = 0 \]
IV. Working with Retarded potentials

A. Example: instant switch-on of current

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{\mathbf{I}(t)}{r} \, dt \]

\[ I(t) = \begin{cases} 0; & t < 0 \\ I_0; & \frac{1}{c} \sqrt{t} > t > 0 \\ I_0; & t > \frac{1}{c} \sqrt{t} > 0 \end{cases} \]
IV. B. Working with retarded potentials

moving point charge: Liénard-Wiechert

\[ V(\vec{r}, t) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{|\vec{r}|} \right) \frac{1}{1 - \hat{n} \cdot \hat{v} / c} \]

\[ (\hat{A} = \vec{v} V/c^2) \]

\[ t_R = t - |\vec{r} - \vec{r}'(t_R)| / c \]
III b. Using Lorentz-Invariant potential

\[ \vec{E} + \vec{B} \text{ - fields:} \]

\[ \vec{E} = \frac{\beta}{4\pi \varepsilon} \frac{1}{(\vec{v} - \vec{C})^3} \left\{ (1 - \frac{\vec{v} \cdot \vec{C}}{C^2}) (\vec{\alpha} - \vec{\nu}) \right\} \]

\[ \left( \vec{B} = \frac{\vec{\alpha} \times \vec{E}}{c} \right) \]

Note: \( \vec{r} \) - scaling behaviour

Special case: constant velocity: \( \vec{r}' = \vec{v} t \) (\( \vec{a} = 0 \))

\[ \vec{E} \rightarrow \frac{1}{4\pi \varepsilon} \frac{\vec{R}}{R^2} \frac{1 - \frac{v^2}{C^2}}{(1 - \frac{v^2}{C^2} \sin^2 \theta)^{3/2}} \]

\[ \vec{R} = \vec{r} - \vec{v} t \text{ (instantaneous relative position)} \]
I. Generation of EM radiation

A. Oscillating electric dipole

\[ \mathbf{d}(t) = \mathbf{d} \cos \omega t \]

B. Oscillating magnetic dipole

\[ \mathbf{I}(t) = \mathbf{I}_0 \cos \omega t \]
V. Generating EM radiation:

Hierarchy of length scales

\[ d \ll x \ll r \]

\[ \bar{A} = \frac{\mu_0}{4\pi} \int b \, dp \, \frac{\dot{I}(t)}{|r|} \]

Example V.C: prob. 11.12

Source wave field of point (radiation zone)

\[ |r| \approx r \left[ 1 - \frac{b}{r} \sin \theta \cos (\phi - \phi') \right] \]

\[ t_r \approx t - \frac{r}{c} + \frac{b}{c} \sin \theta \cos (\phi - \phi') \]

Taylor...

\[ I(t_0) \approx I(t_r) + \frac{dI(t)}{dt} \frac{b}{c} \sin \theta \cos (\phi - \phi') \]

\[ \bar{A} \to \frac{\mu_0}{4\pi} \int b \, dp \left[ I(t_r) + \frac{dI(t_r)}{dt} \frac{b}{c} \sin \theta \cos (\phi - \phi') \right] \times \left[ \cos \phi \hat{y} - \sin \phi \hat{z} \right] \]

\[ \bar{A} \to \frac{\mu_0}{4\pi c} \frac{b^2}{r} \dot{I}(t_r) \sin \theta \hat{\phi} \]
\[ I c \ - \ (P n 6 \ 11.12) \]

\[ \vec{A} = \frac{\mu_0}{4\pi} \frac{\dot{m}(t_r)}{r} \sin \theta \ \hat{\phi} \]

\[ \vec{B} = \nabla \times \vec{A} = -\frac{\partial A\phi}{\partial r} \ \hat{\theta} + \frac{1}{r} \frac{\partial A\phi}{\partial \theta} \ \hat{r} \]

\[ \frac{\partial A\phi}{\partial r} = \# \ \frac{2}{r} \ \frac{\dot{m}(t - \gamma c)}{r} \]

\[ \rightarrow \quad \# \ (-\frac{1}{c}) \ \frac{\dot{m}(t_r)}{r} \]
II D. Power radiated:

Poynting vector: \( \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \)

Example II C: \( \vec{B} = \frac{\mu_0}{4\pi} \frac{\text{inc} \text{ld} \text{v} \text{sin} \theta}{r} \)
\( \vec{E} = c \vec{B} \times \hat{r} \)

\( \Rightarrow \quad \vec{S} = \frac{c}{\mu_0} |\vec{B}| \frac{1}{r} \hat{r} \quad \rightarrow \frac{\text{sin}^2 \theta}{r^2} \)

\( P = \oint \vec{dA} \cdot \vec{S} \quad \Rightarrow \quad d\vec{A} = r^2 \text{sin} \theta \text{d}\theta \text{d}\phi \hat{r} \)

\( P = \frac{\varepsilon}{\mu_0} \left( \frac{\mu_0}{4\pi} m \right)^2 \frac{1}{r^2} \int r^2 \text{sin} \theta \text{d}\theta \text{d}\phi \cdot \text{sin}^2 \theta \)

\( P = \frac{2\pi m^4}{3\mu_0} \)
II. Math skills needed

1. Taylor's Theorem: \( f(x+a) = f(x) + f'(x) a + \ldots \)

\[ |\Delta| = \left[ (\vec{r} - \vec{r}')^2 \right]^{\frac{1}{2}} \]

\[ = (\vec{r}^2 - 2\vec{r}.\vec{r}' + \vec{r}'^2)^{\frac{1}{2}} \]

\[ = r\left[ 1 - 2\frac{\vec{r}.\vec{r}'}{r} + \frac{\vec{r}'^2}{r^2} \right]^{\frac{1}{2}} \]

\[ \simeq r\left( 1 - \frac{1}{2}\frac{\vec{r}.\vec{r}'}{r} \right) \]

\[ \frac{1}{|\Delta|} \simeq \frac{1}{r(1 - \frac{1}{2}\frac{r'.r}{r})} \simeq \frac{1}{r} \left( 1 + \frac{r'.r}{r^2} \right) \]

\[ \text{Eq. 2:} \quad \cos(\theta/\lambda) \simeq 1 + \frac{d/\lambda}{2} \quad \sin(\theta/\lambda) \simeq \frac{d/\lambda}{2} - \frac{(d/\lambda)^3}{3!} \]

\[ \text{Trig identities:} \quad \sin(a+b) = \sin a \cos b + \cos a \sin b \]

\[ \cos(a+b) = \cos a \cos b - \sin a \sin b \]