Work alone or in small groups; however each student must turn in his/her own exercise sheet. In preparation for our treatment of electromagnetic potentials using Lorentz 4-vectors, we need to review the standard results of special relativity and recast them in terms of 4-vectors. We will only consider the case where the relative velocity, \( v \), of the two inertial frames of reference is along the x-axis. To keep the units of each component of the four vector the same, we multiply the time by c. Start with the Lorentz transformations:

\[
\begin{align*}
x' &= \gamma (x - \beta ct) \\
y' &= y \\
z' &= z \\
c't' &= \gamma (ct - \beta x)
\end{align*}
\]

where \( \beta = v/c \) and \( \gamma = 1/\sqrt{1-\beta^2} \). The idea of using 4-vectors is to give a matrix representation to the Lorentz transformations. Define the so-called contravariant and covariant 4-vectors:

\[
x^\mu = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}, \quad x_\mu = \begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}
\]

These provide a convenient way to construct the Lorentz invariant:

\[
x^\mu x_\mu = (x^\mu)^T x_\mu = (\vec{x} \cdot \vec{c} - ct)^2
\]

where we have adopted the Einstein summation convention whereby any repeated index is implicitly summed over.

**Exercise 1:** Show that \( x^\prime x_\mu = x^\mu x_\mu \).

\[
\chi^2 = (x' - \beta t)^2 = \chi^2 - 2\beta x' - (\beta^2 t')^2
\]

\[
= \chi^2 \left[ \chi^2 - 2\beta x (ct) + \beta^2 (ct)^2 - (ct)^2 - 2\beta \frac{x}{ct} + \beta^2 \frac{x^2}{(1-\beta^2)} \right]
\]

\[
= \chi^2 \left[ \chi^2 (1-\beta^2) - (ct)^2 (1-\beta^2) \right] = \chi^2 - (ct)^2
\]

In terms of 4-vectors the Lorentz transformation can be written:

\[
x^\mu = \Lambda^\mu_\nu x^\nu, \quad \text{where } \Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta \gamma & 0 & 0 & \gamma \end{pmatrix}
\]

The derivative operations also make up a 4-vector: \( \partial_\mu = \frac{\partial}{\partial x^\mu} \) and \( \partial^\mu = \frac{\partial}{\partial x_\mu} \). (Note the contra- and co-variant forms.)

**Exercise 2:**

a. What is the invariant: \( \partial_\mu x^\mu ? \)

\[
\frac{\partial}{\partial x^\mu} \Rightarrow \left( \begin{array}{c} \nabla \cdot \vec{X} \\ \frac{\partial}{\partial ct} \end{array} \right) \left( \begin{array}{c} \vec{X} \\ ct \end{array} \right) = \nabla \cdot \vec{X} + \frac{\partial}{\partial ct} = \nabla \cdot \vec{X}
\]

b. What is the invariant \( \partial^\mu \partial_\mu ? \)

\[
\partial^\mu \Rightarrow \left( \begin{array}{c} \nabla \cdot \vec{X} \\ -\frac{1}{c^2} \frac{\partial}{\partial ct} \end{array} \right) \Rightarrow \partial^\mu \partial_\mu \Rightarrow \left( \begin{array}{c} \nabla \cdot \vec{X} - \frac{1}{c^2} \frac{\partial}{\partial ct} \end{array} \right) = \nabla^2 \frac{1}{c^2} \frac{\partial}{\partial ct} = \square^2
\]
The velocity addition formulas are given (without derivation) by:

\[
\begin{align*}
  u'_x &= \frac{u_x - v}{1 - u_xv/c^2} \\
  u'_y &= \frac{u_y}{\gamma(1 - u_xv/c^2)}
\end{align*}
\]

The relativistic energy and momentum of a point particle are given by

\[
E = \gamma mc^2, \quad \vec{p} = \gamma \vec{m}
\]

where here \( \gamma = 1/\sqrt{1 - u^2/c^2} \) and \( m \) is the rest mass.

**Exercise 3:** Prove the Einstein triangle relation: \( E^2 = (pc)^2 + (mc^2)^2 \).

\[
\left( \sqrt{m c^2} \right)^2 = \left( \sqrt{p_x c} \right)^2 + \left( \sqrt{mc^2} \right)^2 = \gamma^2 \left( \sqrt{p_x c} \right)^2 \left( 1 - \frac{u^2}{c^2} \right)
\]

\[
\left( \sqrt{m c^2} \right)^2 - \frac{u^2}{c^2} \left( \sqrt{mc^2} \right)^2 = \frac{1}{1 - \frac{u^2}{c^2}} \left( \sqrt{mc^2} \right)^2 \left( 1 - \frac{u^2}{c^2} \right) = (mc^2)
\]

QED

It turns out that the energy and momentum (times \( c \)) satisfy the same Lorentz transformation formulas as that of the space-time transformation Eq. [1], namely,

\[
\begin{align*}
  p'_x c &= \gamma(p_x c - \beta E) \\
  p'_y c &= p_y c \\
  p'_z c &= p_z c \\
  E' &= \gamma(E - \beta p_x c)
\end{align*}
\]

where the momentum has been multiplied by \( c \) to give all the components energy units.

**Exercise 4:** For the special case where \( \vec{u}' = (0, u'_y, 0, 0) \), use the velocity addition formulas, Eq(4), to show that \( p'_y = p_y \).

Here, one must distinguish between the boost factor \( \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \) and the inertial mass factor \( \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \).

By Eq.(4): \( u'_y = \frac{1}{\gamma} \frac{u_y}{\gamma} = \frac{u_y'}{\sqrt{1 - \frac{u^2}{c^2}}} \)

\[
\Rightarrow \quad p'_y = \frac{m \gamma'}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{m u_y'}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)^2 - (\gamma \gamma')^2}} = \frac{1}{\gamma'} m u_y'
\]

QED.
This means that the energy-momentum 4-vector,

\[ p^\mu = \begin{pmatrix} p_x c \\ p_y c \\ p_z c \\ E \end{pmatrix}, \tag{6} \]

transforms exactly as \( x^\mu \). Any vector for which this is true we call a Lorentz vector.

**Exercise 5**: What is the invariant \( p^\mu p_\mu \)?

\[
\begin{align*}
p^\mu p_\mu &= \left( p_x c \right)^2 - E^2 = -(mc^2)^2 \quad \text{(by Ex. 3)}
\end{align*}
\]

Let's conclude with a look at energy-momentum conservation in relativistic collisions. The Lorentz transformation is linear; so any sum of Lorentz vectors will still be a Lorentz vector. Specifically, the total 4-momentum of a collision is a Lorentz vector, and, since the energy and momentum are conserved, the total Lorentz 4-momentum will be conserved as well. As an example, consider the laboratory production of a pion by colliding a high energy proton into a hydrogen (proton) target. Our goal is to find the threshold laboratory kinetic energy needed to produce a pion.

In the center of momentum frame this would be very easy, because at threshold after the collision all the particles would be at rest. Let \( E_{cm} = m_p c^2 + T_{cm} \) be the initial energy of each of the protons where \( m_p \) is the proton rest mass. The total energy is \( 2E_{cm} \) (and in the cm frame, of course, the total momentum is zero.) Therefore, at threshold we would have \( 2E_{cm} = 2m_p c^2 + m_\pi c^2 \); so \( T_{cm} = \frac{1}{2}m_\pi c^2 \) as we would have guessed. We could use \( E_{cm} \) to find \( v_{cm} \) and then use the Lorentz transformations to find \( p_L \), but an easier approach is to invoke Lorentz invariance using the square of the total 4-momentum. In the cm frame the total 4-momentum squared is \( (2E_{cm})^2 = (2m_p c^2 + m_\pi c^2)^2 \). We now proceed to calculate the same quantity in the lab frame.

Suppressing the y- and z-directions, let the four momentum of the two protons be:

\[
p_{1L}^\alpha = \begin{pmatrix} p_{1L} c \\ E_L \end{pmatrix}, \quad \text{and} \quad p_{2L}^\alpha = \begin{pmatrix} 0 \\ m_p c^2 \end{pmatrix},
\]

where particle 1 is the projectile proton, particle 2 is the target proton, and \( p_L \) and \( E_L \) are the relativistic momentum and energy of the projectile proton. Thus, the total 4-momentum in the lab frame is given by:

\[
p_L^\mu = p_{2L}^\mu + p_{1L}^\mu = \begin{pmatrix} p_{1L} c \\ E_L + m_p c^2 \end{pmatrix}
\]

**Exercise 6**: Now it's your turn. Square the total 4-momentum in the lab frame and, since it is a Lorentz invariant, we can set it equal to the same quantity we already calculated in the cm frame. Use the Einstein triangle relation to solve for \( p_L \) and from that, calculate the lab kinetic energy, \( T_L = E_L - m_p c^2 \). Plug in the values \( m_p c^2 = 938 \text{ MeV} \) and \( m_\pi c^2 = 135 \text{ MeV} \), to find \( T_L \) in MeV. This is the minimum energy that the designers had to give the proton beam at the Los Alamos Meson Physics Facility to create a secondary pion beam.