Nuclear Fundamentals: Lesson 2

Agenda

I. Semiempirical Mass Formula

II. Radioactivity
How can we understand the systematics of the binding energetics of nuclei?

Nuclear binding due to nearest neighbor force only (short range)

Nuclear saturation $\implies$ Volume term

$$\text{BE(Volume)} = a_v A$$

Overcounting at surface $\implies$ Surface term

$$\text{BE(Surface)} = -a_s A^{2/3}$$
Coulomb repulsion of protons \implies \text{Coulomb term}

Long range coulomb repulsion means every pair of protons contributes to the energy. Coulomb force falls like

\[ \frac{1}{R} \propto A^{-1/3} \]

\[ \text{BE(Coulomb)} = -a_c Z^2 A^{-1/3} \]

\[ U = + \frac{Q^2}{\kappa R} \left( \frac{\theta}{\pi} \right) \]
Pauli exclusion principle $\implies$ Symmetry term

Energy preferred

$\text{BE(Symmetry)} = -a_{\text{sym}} \frac{(N-Z)^2}{A}$
Spin-pairing preferred $\implies$ Pair term

\[ \begin{array}{c}
\downarrow \\
\uparrow
\end{array} \quad \text{Energy preferred} \quad \begin{array}{c}
\uparrow \quad \uparrow \\
\downarrow
\end{array} \quad \text{NOT preferred}
\]

\[ \text{BE(Pair)} = \begin{cases} 
+\delta & \text{even-even} \\
0 & \text{even-odd or odd-even} \\
-\delta & \text{odd-odd}
\end{cases} \]
Semiempirical mass formula summary:

\[ BE = a_v A - a_s A^{2/3} - a_{\text{sym}} (N-Z)^2 A^{-1} - a_c A^{-2/3} + \delta_{\text{pair}} \]

Volume: \( a_v \approx 15 - 16 \text{ MeV} \) \[ 15.56 \text{ MeV} \]

Surface: \( a_s \approx 13 - 18 \text{ MeV} \) \[ 17.23 \text{ MeV} \]

Symmetry: \( a_{\text{sym}} \approx 19 - 23 \text{ MeV} \) \[ 23.285 \text{ MeV} \]

Coulomb: \( a_c \approx 0.6 - 0.7 \text{ MeV} \) \[ 0.697 \text{ MeV} \]

Pair: \( \delta \approx 12.0 \text{ MeV} \) \[ 12.0 \text{ MeV} \]
Semiempirical Mass Formula

- The nuclear binding energy is defined by:
  \[ \text{BE} = Z \text{ M(proton)} + N \text{ M(neutron)} - \text{M(A,Z)} \]

- (Note the assumption of the relativistic equivalence of mass and energy:
  \( E=mc^2 \) which allows one to measure mass in MeV.)

- The mass of any stable nucleus can be calculated from the mass excess
  which is listed in Appendix C of Meyerhof. By convention the ATOMIC
  mass excesses are tabulated instead of the NUCLEAR mass excess. Thus
  the mass of any ATOM (including the electrons) is given by:

  \[ \text{Matom(A,Z)} = (\text{A-MassExcess}) \times u, \text{ where } u \text{ is the atomic mass unit:} \]
  \( u=931.49432 \text{ MeV (1994 number).} \)

- The semiempirical mass formula, Meyerhof, Eq.(2-127)
  without the shell term is:

  \[
  \text{In}[6]:= \\
  \text{BE}[\text{A}, \text{Z}] := \text{avol} \text{ A} - \text{asurf} \text{ A}^{(2/3)} - \text{ac} \text{ Z(Z-1)/A}^{(1/3)} - \\
  \text{asym} \text{ (A-2 Z)^2/A} + \\
  \text{delta}(1-\text{Mod}[\text{A}, 2])(1-\text{Mod}[\text{Z}, 2]-\text{Mod}[\text{A-Z}, 2])/\text{A}^{(1/2)}
  \]

  Note the use of \( \text{Mod}[\text{A}, 2] \) etc to satisfy the pairing condition (Meyerhof p. 41)

- Typical values for the constants in MeV:

  \[
  \text{In}[2]:= \\
  \text{const} = \{\text{avol} \to 15.8, \text{asurf} \to 18, \text{asym} \to 23.5, \text{ac} \to 0.72, \text{delta} \to 11\}
  \]

  \[
  \text{Out}[2]= \\
  \{\text{avol} \to 15.8, \text{asurf} \to 18, \text{asym} \to 23.5, \text{ac} \to 0.72, \text{delta} \to 11\}
  \]
Example 1: binding energy of Gold-197 (A=197, Z=79):

\[ \text{In[7]:=} \]
\[ \text{Print}[\text{N[BE[197,79]/.\text{const}]\text{ MeV}}] \]
\[ 1559.25 \text{ MeV} \]

Using appendix C of Meyerhof:
\[ \text{BE} = (Z \text{ Mhyd} + N \text{ Mneut})-(A-.033448)^* u = 1558.89 \text{ MeV} \]

Example 2: binding energy of Tungsten-180 (A=180, Z=74):

\[ \text{In[8]:=} \]
\[ \text{Print}[\text{N[BE[180,74]/.\text{const}]\text{ MeV}}] \]
\[ 1448.44 \text{ MeV} \]

Using appendix C of Meyerhof:
\[ \text{BE} = (Z \text{ Mhyd} + N \text{ Mneut})-(A-.05503)^* u = 1445.71 \text{ MeV} \]

Example 3: binding energy of Vanadium-50 (A=50, Z=23):

\[ \text{In[9]:=} \]
\[ \text{Print}[\text{N[BE[50,23]/.\text{const}]\text{ MeV}}] \]
\[ 437.735 \text{ MeV} \]

Using appendix C of Meyerhof:
\[ \text{BE} = (Z \text{ Mhyd} + N \text{ Mneut})-(A-.0^*) u = .434.62 \text{ MeV} \]
Use the semiempirical mass formula to estimate the binding energy of $^{179}$W.

$$A = 2 = 74, \quad N = 105$$

\[ \begin{align*}
W^{179}W &= \text{BE}^{179}W = 15.67 \cdot 1.79 - 17.23 \cdot 1.79^{1\frac{3}{2}} - 23.288 \cdot \frac{(105-74)^2}{179} \\
& \quad - 0.714 \cdot \frac{74^2}{179^{1\frac{1}{2}}} + \Delta = 0 \quad \text{(even-odd)}
\end{align*} \]

\[\begin{align*}
= 2.894.93 - 547.25 - 125.0266 - 693.7622 \\
= 1435.69 \text{ MeV}
\end{align*}\]

\[\boxed{\Delta = 93.02 \text{ MeV}}\]

\[\Rightarrow M(179W) = 179m + \Delta = 166.66 \text{ MeV} = 17.8.9.7.9.2\]

\[\begin{align*}
74 \cdot 1.007625 + 105 \cdot 1.008664 - 178.9.7.9.0.7 = 1.51472 \\
= 14.36 \text{ MeV}
\end{align*}\]
Using the semiempirical mass formula

(caution: not good for detail work)

Average Binding Energy per nucleon, BE/A

\[
BE/A = a_v - a_s A^{-1/3} - a_{sym} (N-Z)^2 A^{-2} - a_c Z(Z-1) A^{-4/3}
\]

\[
\frac{d}{dZ} (BE/A) = 0 \quad \implies \quad Z \text{ stable}
\]

\(A\)-fixed

\[
Q = \Delta A
\]

\[
Q = A - A_f \quad \text{fission is exothermic for } A > A_f
\]

For \(A < 60\)

For \(A < 60\)
Radioactivity

I. Conservation laws

II. Characterization of common processes \( \alpha, \beta, \gamma \)

III. Energetics and Stability

IV. Law of Radioactive Decay
Conservation Laws

I. Energy and momentum:

\[ E_i = E_f \]

\[ \vec{p}_i = \vec{p}_f \]

II. Charge:

\[ \sum q_i = \sum q_f \]

III. Nucleon number:

\[ \sum A_i = \sum A_f \]

IV. Angular momentum:

\[ \sum j_i = \sum j_f \]

(Note: in β-decay, both N and Z change, but the SUM is conserved.)

\[ n \rightarrow \bar{p} + e^- + \bar{\nu}_e \]

Barion (38)

\[ l \rightarrow l \]

Lepton (0 → +1 + -1 = 0)
Example 1: alpha-decay of radium

\[ ^{226}\text{Ra} \rightarrow ^{222}\text{Rn} + ^{4}\text{He} + Q \]  (\(\alpha\))

Energy conservation -->

\[ M(^{226}\text{Ra}) = M(^{222}\text{Rn}) + M(^{4}\text{He}) + Q/c^2 \]  \((t_{\nu} \approx 1600\text{y})\)

Charge conservation -->

\[ Z(\text{Ra}) = Z(\text{Rn}) + Z(\text{He}) \checkmark \]

Nucleon number conservation -->

\[ A(\text{Ra}) = A(\text{Rn}) + A(\text{He}) \checkmark \]

\[ ^{226}\text{Ra} \rightarrow \alpha + ? \]

\[ A = 4 \quad A_f = 226 - 4 \]
\[ Z = 2 \quad Z_f = 88 - 2 \]
Example 2: beta-decay of carbon-14

\[ ^{14}\text{C} \quad \longrightarrow \quad ^{14}\text{N} + \quad e^- + \quad \bar{\nu} \]

Energy conservation -->

\[ M(^{14}\text{C}) = M(^{14}\text{N}) + M(e^-) + Q/c^2 \]

\( Q > 0 \)

Charge conservation -->

\[ Z(\text{C}) = Z(\text{N}) + Z(e^-) \]

Nucleon number conservation -->

\[ A(\text{C}) = A(\text{N}) \]
Alternative Forms of β-decay

\[ \beta^+ \]

"proton-rich" side of stability line:

\[ A^{(Z+1)} \rightarrow A^Z + e^+ + \nu_e \]

\[ \text{electrons} \rightarrow \text{protons} \] (Heavier)

\[ p + e^- \rightarrow n + \nu_e \]

\[ A^{(Z+1)} + e^- \rightarrow A^Z + \nu_e \] (K-shell)
Gamma-decay (photon emission)

\[
\begin{align*}
\text{^{12}C}^* & \rightarrow \text{^{12}C} + \gamma \\
\end{align*}
\]

Recall Pauli exclusion principle:

Note: the photon can be absorbed on an inner shell electron in which case the electron is ejected, a process called "internal conversion"
Alternative form of $\beta$-decay:

*Internal Conversion* (IT)

$A^x \rightarrow A + e^- (high\ energy\ electron\ ejected)$

$e^- (atomic\ electron)$
Figure 2. Relative Locations of the Products of Various Nuclear Processes

n = neutron  \quad \alpha = \text{alpha particle}

p = proton   \quad \beta^- = \text{negative electron}

d = deuteron \quad \beta^+ = \text{positron}

t = triton (³H) \quad \epsilon = \text{electron capture}
Stability Energetics

\[ Q > 0 \implies \text{energetically unstable} \]

Example: Is \(^{232}\text{Th}\) stable against \(\alpha\)-decay?

\[
\begin{align*}
^{232}\text{Th} & \longrightarrow ^{228}\text{Ra} + \alpha + Q \\
& \quad ?
\end{align*}
\]

\[
\Delta(\text{Th}) = .038050 \text{ u}
\]

\[
\Delta(\text{Ra}) = .031139 \text{ u}
\]

\[
\Delta(\text{He}) = .002603 \text{ u}
\]

If it decays, what is the energy of the alpha?
Law of Radioactive Decay

Time dependence -- when does the decay occur?

Don't know --- quantum mechanics only tells us the probability per time.

Let \( N(t) \) be the number of a certain radio-nuclide at time \( t \). The rate this number is decreasing is proportional to \( N \), i.e.

\[
\frac{dN}{dt} = -\lambda N
\]

The solution is the classical exponential decay law:

\[
N(t) = N(0) e^{-\lambda t}
\]
Half-life = time for half of the radionuclides to decay.

\[ N(t_{\text{half}}) = N(0)/2 = N(0) e^{-\lambda t_{\text{half}}} = N(0) e^{-t/\gamma} \]

\[ \gamma \rightarrow \text{"mean life"} \]

\[ \Longrightarrow t_{\text{half}} = \ln(2)/\lambda \]
Multiple generations:

\[ A \rightarrow B + x \]

\[ \rightarrow C + y \]

If you start with only A, what is time behavior of B?

Coupled equations:

\[
\frac{dN_A}{dt} = -\lambda_A N_A
\]

\[
\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A
\]

\[
\frac{dN_C}{dt} = \lambda_B N_B
\]
Solution:

\[ N_A(t) = N_A(0) e^{-\lambda_A t} \]

\[ N_B(t) = \frac{N_A(0) \lambda_A}{\lambda_B - \lambda_A} \left( e^{-\lambda_A t} - e^{-\lambda_B t} \right) \]
Suppose a very well insulated house has 1 mg of 226 Ra in the subfloor soil. (Half life of 226 Ra is 1600 y).

a. What is the Radium activity?

$$1 \text{Ci} = \text{activity of 1 g } ^{226}\text{Ra}$$

Thus,

$$\frac{dN}{dt} = -1 \text{ mCi} = 3.7 \times 10^2 \text{ Bq}$$

b. What is the equilibrium level of 222Rn in the house (Half life of 222Rn is 3.8 days)?

$$\frac{dN_{Ra}}{dt} = N_{Ra} \lambda_{Ra} - N_{Ra} \lambda_{en} \rightarrow 0 \text{ in equilibrium}$$

$$\Rightarrow N_{en} = \frac{\lambda_{Ra} N_{Ra}}{\lambda_{en}}$$

c. Starting from a zero radon level, how long does it take for the radon level to reach 90% of its final value?

$$\lambda_{en} \ll \lambda_{Ra} \Rightarrow N_{en} = \frac{N_{Ra} \cdot \lambda_{Ra}}{\lambda_{en}} \left( e^{-\lambda_{en} t} - e^{-\lambda_{Ra} t} \right)$$

$$N_{en}(t) = N_{Ra} \cdot \lambda_{Ra} \left( 1 - e^{-\lambda_{en} t} \right)$$

\[ \text{At 90% final} \]

$$0.9 = (1 - e^{-\lambda_{en} t}) \Rightarrow t_f = 12.6 \text{ days}$$