Name: Solutions
Section:

MATH 225 Fall 2011
Exam II

In order to receive full credit, **SHOW ALL YOUR WORK.** Full credit will be given only if all reasoning and work is provided and final answers are simplified. All solutions must be reported in real form. Where appropriate, please enclose your final answers in boxes.

1. (10 points) Find the Laplace Transform of the following functions using the method indicated.

   (a) Use the definition of the Laplace transform for:

$$f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 4t, & 4 \le t \end{cases}$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st}\,dt + \int_4^\infty 4te^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^4 + \left(-\frac{4t}{s}e^{-st} - \frac{4}{s^2}e^{-st}\right)\Big|_4^\infty, \quad s>0$$

$$= -\frac{1}{s}e^{-4s} + \frac{1}{s} + 0 + 0 + \frac{16}{s}e^{-4s} + \frac{4}{s^2}e^{-4s}$$

$$= \boxed{\frac{15}{s}e^{-4s} + \frac{4}{s^2}e^{-4s} + \frac{1}{s}}$$

$u \quad dv$
$4t \quad e^{-st}$
$4 \quad -\frac{1}{s}e^{-st}$
$0 \quad \frac{1}{s^2}e^{-st}$

   (b) Use the table provided to find the Laplace transform for:

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 \le t \end{cases}$$

$$g(t) = 0 + \mathcal{U}(t-1)(t-1) = \mathcal{U}(t-1)(t-1) = (t-1)\,\mathcal{U}(t-1)$$

#19, $a=1$, $g(t) = t-1$
$g(t+1) = t+1-1 = t$

$$G(s) = \mathcal{L}\{g(t)\} = e^{-s}\underbrace{\mathcal{L}\{t\}}_{\#4,\ n=1} = e^{-s}\left(\frac{1}{s^2}\right) = \boxed{\frac{e^{-s}}{s^2}}$$

2. (10 points) Find the inverse Laplace transform of the following.

   (a) $F(s) = \frac{(s+2)^2}{s^3} = \frac{s^2+4s+4}{s^3} = \frac{1}{s} + 4\left(\frac{1}{s^2}\right) + \frac{4}{2}\left(\frac{2!}{s^3}\right)$

#1   #4, $n=1$ #4, $n=2$

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\} = 1 + 4t + 2t^2}$$

   (b) $G(s) = \frac{e^{-4s}}{s^2+2s+2} = e^{-4s}\left(\frac{1}{s^2+2s+2}\right) = e^{-4s}\left(\frac{1}{(s+1)^2+1}\right)$, #19, $a=4$

$F(s) = \frac{1}{(s+1)^2+1}$
#12, $a=-1$, $b=$

$b^2-4ac = 2^2-4(1)(2) < 0$, No Real Factors

CS: $s^2+2s+2 = \underbrace{s^2+2s+\left(\tfrac{2}{2}\right)^2}+2-\left(\tfrac{2}{2}\right)^2$

$= (s+1)^2+1$

$$\boxed{\begin{aligned} g(t) &= \mathcal{L}^{-1}\{G(s)\} \\ &= \mathcal{U}(t-4)e^{-(t-4)}\sin(t-4) \end{aligned}}$$

3. (25 points) Solve the following initial-value problems with Laplace transforms.

(a) $\frac{dy}{dt} + 3y = 13\sin(2t), \quad y(0) = 0$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 13\,\mathcal{L}\{\sin(2t)\}$

#1(a) # 5, b=2

$sY(s) - 0 + 3Y(s) = 13\left(\frac{2}{s^2+4}\right)$

$Y(s) = \frac{26}{(s+3)(s^2+4)}$

PF: $\frac{26}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$

$A(s^2+4) + (Bs+C)(s+3) = 26$

$As^2 + 4A + Bs^2 + 3Bs + Cs + 3C = 26$

$A+B=0 \quad 3B+C=0 \quad 4A+3C=26$

$A=-B \quad C=-3B \quad -4B-9B=26$

$A=2 \quad C=6 \quad B=-2$

$Y(s) = 2\left(\frac{1}{s+3}\right) - 2\left(\frac{s}{s^2+4}\right) + \frac{6}{2}\left(\frac{1(2)}{s^2+4}\right)$

#2, a=-3 #6, b=2 #5, b=2

$\boxed{y(t) = 2e^{-3t} - 2\cos(2t) + 3\sin(2t)}$

(b) $y'' + 3y' + 2y = \delta(t-1) + 3\mathcal{U}(t-2), \quad y(0) = 0, \quad y'(0) = 1$

$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\} + 3\mathcal{L}\{\mathcal{U}(t-2)\}$

#1(b) #1(a) #20, a=1 #17, a=2

$s^2Y(s) - 0 - 1 + 3(sY(s) - 0) + 2Y(s) = e^{-s} + \frac{3e^{-2s}}{s}$

$(s^2+3s+2)Y(s) = 1 + e^{-s} + e^{-2s}\left(\frac{3}{s}\right)$

$Y(s) = \frac{1}{(s+2)(s+1)} + e^{-s}\left(\frac{1}{(s+2)(s+1)}\right) + e^{-2s}\left(\frac{3}{s(s+2)(s+1)}\right)$

PF: $\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$

$A(s+1) + B(s+2) = 1$

$s=-1$: $B=1$

$s=-2$: $-A=1$, $A=-1$

$\frac{3}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$

$A(s+2)(s+1) + Bs(s+1) + Cs(s+2) = 3$

$s=-2$: $B(-2)(-1) = 3$, $B = \frac{3}{2}$

$s=-1$: $C(-1)(1) = 3$, $C = -3$

$s=0$: $A(2)(1) = 3$, $A = \frac{3}{2}$

$Y(s) = -\frac{1}{s+2} + \frac{1}{s+1} + e^{-s}\left[-\frac{1}{s+2} + \frac{1}{s+1}\right] + e^{-2s}\left[\frac{3}{2}\left(\frac{1}{s}\right) + \frac{3}{2}\left(\frac{1}{s+2}\right) - 3\left(\frac{1}{s+1}\right)\right]$

3, a=-2 #3, a=-1 #18, a=1 #18, a=2

$\boxed{y(t) = -e^{-2t} + e^{-t} + \mathcal{U}(t-1)\left[-e^{-2(t-1)} + e^{-(t-1)}\right] + \mathcal{U}(t-2)\left[\frac{3}{2} + \frac{3}{2}e^{-2(t-2)} - 3e^{-(t-2)}\right]}$

4. (12 points) Solve the following integral equation with Laplace transforms.

$$y(t) + \int_0^t \underbrace{y(\tau)}_{f(\tau)}\,d\tau = 1 \qquad g(t-\tau)=1$$

$$f * g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$y(t) + (y * 1) = 1$$

$$\underset{\#16}{\mathcal{L}\{y\} + \mathcal{L}\{y*1\}} = \underset{\#2}{\mathcal{L}\{1\}}$$

$$Y(s) + Y(s)\left(\frac{1}{s}\right) = \frac{1}{s}$$

$$Y(s)\left(1+\frac{1}{s}\right) = \frac{1}{s}$$

$$Y(s)\left(\frac{s+1}{s}\right) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s+1}, \quad \#3, a=-1$$

$$\boxed{y(t) = e^{-t}}$$

5. (12 points) Given the system of differential equations

$$\frac{dx}{dt} = 3x + 9y$$
$$\frac{dy}{dt} = x + 3y$$

$$A = \begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & 9 \\ 1 & 3-\lambda \end{pmatrix}$$

Find the general solution of the system.

$$\det\begin{pmatrix} 3-\lambda & 9 \\ 1 & 3-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(3-\lambda) - (1)(9) = 0$$

$$\lambda^2 - 6\lambda = 0$$

$$\lambda(\lambda - 6) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 6$$

$\underline{\lambda_1 = 0}$: $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3v_1 + 9v_2 = 0$$

$$v_2 = \frac{-v_1}{3}$$

$$\begin{pmatrix} v_1 \\ -\frac{v_1}{3} \end{pmatrix}, \quad v_1 = 3, \quad \vec{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 6}$: $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3v_1 + 9v_2 = 0$$

$$v_2 = \frac{v_1}{3}$$

$$\begin{pmatrix} v_1 \\ \frac{v_1}{3} \end{pmatrix}, \quad v_1 = 3$$

$$\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{X}(t) = c_1 e^{0t}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}} \quad \underline{\text{or}} \quad \boxed{\begin{aligned} x(t) &= 3c_1 + 3c_2 e^{6t} \\ y(t) &= -c_1 + c_2 e^{6t} \end{aligned}}$$

6. (16 points) Given the system of differential equations

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\mathbf{X} \quad A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}, \; A-\lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det\begin{pmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda)-(-2)(2)=0$$

$$\lambda^2+3=0$$

$$\lambda^2=-3$$

$$\lambda = \pm\sqrt{-3} = \pm\sqrt{3}\,i$$

$\underline{\lambda_1=\sqrt{3}i}$: $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} 1-\sqrt{3}i & -2 \\ 2 & -1-\sqrt{3}i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1-\sqrt{3}i)v_1 - 2v_2 = 0$$

$$v_2 = \left(\frac{1-\sqrt{3}i}{2}\right)v_1$$

$$\begin{pmatrix} v_1 \\ \frac{(1-\sqrt{3}i)v_1}{2} \end{pmatrix}, \; v_1=2, \; \vec{v}_1 = \begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix}$$

$$e^{\lambda_1 t}\vec{v}_1 = e^{\sqrt{3}it}\begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix} = (\cos(\sqrt{3}t)+i\sin(\sqrt{3}t))\begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos(\sqrt{3}t)+2i\sin(\sqrt{3}t) \\ \cos(\sqrt{3}t)-\sqrt{3}i\cos(\sqrt{3}t)+i\sin(\sqrt{3}t)+\sqrt{3}\sin(\sqrt{3}t) \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t)+\sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} + i\begin{pmatrix} 2\sin(\sqrt{3}t) \\ -\sqrt{3}\cos(\sqrt{3}t)+\sin(\sqrt{3}t) \end{pmatrix}$$

$$\boxed{\vec{X}(t) = C_1\begin{pmatrix} 2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t)+\sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} + C_2\begin{pmatrix} 2\sin(\sqrt{3}t) \\ -\sqrt{3}\cos(\sqrt{3}t)+\sin(\sqrt{3}t) \end{pmatrix}}$$

(b) Find the solution that satisfies the initial-condition $\mathbf{X}(0) = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$. Report your solution as one real vector.

$$\vec{X}(0) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

$$2C_1 = 0 \qquad C_1 - \sqrt{3}C_2 = \sqrt{3}$$

$$C_1 = 0 \qquad C_2 = -1$$

$$\boxed{\vec{X}(t) = \begin{pmatrix} -2\sin(\sqrt{3}t) \\ \sqrt{3}\cos(\sqrt{3}t) - \sin(\sqrt{3}t) \end{pmatrix}}$$

7. (15 points) Using the power series method (centered at $x = 0$) on $y'' + \omega^2 x^2 y = 0$, where ω is a constant,

(a) Find the recurrence relation.

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$y'' + \omega^2 x^2 y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \omega^2 x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} \omega^2 c_n x^{n+2} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=4}^{\infty} \omega^2 c_{n-4} x^{n-2} = 0$$

$$2(1) c_2 x^0 + (3)(2) c_3 x + \sum_{n=4}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=4}^{\infty} \omega^2 c_{n-4} x^{n-2} = 0$$

$$\underbrace{(2c_2)}_{=0} + \underbrace{(6c_3)}_{=0} x + \sum_{n=4}^{\infty} \underbrace{[n(n-1) c_n + \omega^2 c_{n-4}]}_{=0} x^{n-2} = 0$$

$2c_2 = 0$ $\quad$ $6c_3 = 0$ $\quad$ $n(n-1)c_n + \omega^2 c_{n-4} = 0$

$\underline{c_2 = 0}$ $\quad$ $\underline{c_3 = 0}$ $\quad$ $\boxed{c_n = \dfrac{-\omega^2 c_{n-4}}{n(n-1)}, \quad n \geq 4}$

(b) Find the general solution up to degree 5 (up to and including the x^5 term).

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

$n=0$: c_0, $n=1$: c_1

$n=2$: $c_2 = 0$

$n=3$: $c_3 = 0$

$n=4$: $c_4 = \dfrac{-\omega^2 c_0}{12}$

$n=5$: $c_5 = \dfrac{-\omega^2 c_1}{20}$

$$\boxed{y \approx c_0 + c_1 x - \frac{\omega^2 c_0}{12} x^4 - \frac{\omega^2 c_1}{20} x^5}$$