MATH 225 Fall 2011
Exam II, Make-up

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Section:

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided and final answers are simplified. All solutions must be reported in real form. Where appropriate, please enclose your final answers in boxes.

1. (10 points) Find the Laplace Transform of the following functions using the method indicated.

   (a) Use the definition of the Laplace transform for:

$$f(t) = \begin{cases} 2t, & 0 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

$u = t \quad dv = e^{-st}$

$du = dt \quad v = -\frac{1}{s}e^{-st}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 2te^{-st}\,dt$$

$$= -\frac{t}{s}e^{-st}\Big|_0^4 + \frac{1}{s}\int_0^4 e^{-st}dt = -\frac{t}{s}e^{-st}\Big|_0^4 - \frac{1}{s^2}e^{-st}\Big|_0^4$$

$$= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2}$$

   (b) Use the table provided to find the Laplace Transform for:

$$f(t) = \begin{cases} 1, & 0 \le t < 3 \\ 2(t-3), & 3 \le t \end{cases} = 1 - U(t-3) + 2(t-3)U(t-3)$$

$$= \frac{1}{s} - \frac{e^{-3s}}{s} + 2e^{-3s}\left(\frac{1}{s^2}\right)$$

2. (10 points) Find the inverse Laplace transform of the following.

   (a) $F(s) = \frac{(2s+3)^2}{s^3} = \frac{4s^2 + 12s + 9}{s^3} = \frac{4}{s} + \frac{12}{s^2} + \frac{9}{s^3}$

$$\mathcal{L}^{-1}(F(s)) = 4 + 12t + \mathcal{L}^{-1}\left(\frac{9}{s^3}\right) = 4 + 12t + \frac{9}{2}\mathcal{L}^{-1}\left(\frac{2}{s^3}\right)$$

$$= 4 + 12t + \frac{9}{2}t^2$$

   (b) $G(s) = \frac{e^{-2s}}{s^2+s+2} = \frac{e^{-2s}}{(s+\frac{1}{2})^2 + \frac{7}{4}} = U(t-2)\left[e^{-\frac{1}{2}(t+2)}\frac{2}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}(t+2)\right)\right]$

3. (25 points) Solve the following initial-value problems with Laplace transforms.

(a) $\frac{d^2y}{dt^2} + 3y = 13\cos(2t), \quad y(0) = y'(0) = 0$

$$s^2Y + 3Y = \frac{13s}{s^2+4} \Rightarrow Y(s^2+3) = \frac{13s}{(s^2+4)(s^2+3)}$$

$$13s = (As+B)(s^2+3) + (Cs+D)(s^2+4) = \frac{-13s}{s^2+4} + \frac{13s}{s^2+3}$$

$$= As^3 + Bs^2 + 3As + 3B + Cs^3 + Ds^2 + 4Cs + 4D$$

$$= (A+C)s^3 + (B+D)s^2 + (3A+4C)s + 3B + 4D$$

$\Rightarrow \quad A+C=0 \quad B+D=0, \quad 3A+4C=13, \quad 3B+4D=0$

$A=-C \qquad \Rightarrow B=-D \quad \Rightarrow D=B=0$

$\Rightarrow C=13 \Rightarrow A=-13$

$$\Rightarrow y(t) = -13\cos(2t) + 13\cos(\sqrt{3}t)$$

(b) $y' + 2y = \delta(t-1) + e^{2t}, \quad y(0) = 1$

$$sY - 1 + 2Y = e^{-s} + \frac{1}{s-2} \Rightarrow (s+2)Y = 1 + e^{-s} + 1/_{s-2}$$

$$\Rightarrow Y(s) = \frac{1}{s+2} + \frac{e^{-s}}{(s+2)} + \frac{1}{(s+2)(s-2)} = \frac{1}{s^2-4}$$

$$= e^{-2t} + U(t-1)e^{-2(t+1)} + \tfrac{1}{2}\sinh(2t)$$

4. (12 points) Solve the following equation with Laplace transforms.

$$y'(t) + \int_0^t y(\tau)\,d\tau = 3, \quad y(0) = 0$$

$\Rightarrow sY + 1 * y = 3 \Rightarrow 3Y + \frac{1}{s}Y = \frac{3}{s}$

$\Rightarrow \left(\frac{\cancel{3s}s^2+1}{s}\right)Y = \frac{3}{s} \Rightarrow Y = \frac{3}{s^2+1}$

$\Rightarrow y(t) = 3\sin(t)$

5. (12 points) Given the system of differential equations

$$\begin{pmatrix} -\lambda & -2 \\ -5 & 3-\lambda \end{pmatrix} \qquad \begin{aligned} \frac{dx}{dt} &= -2y \\ \frac{dy}{dt} &= -5x + 3y \end{aligned} \quad = \underbrace{\begin{pmatrix} 0 & -2 \\ -5 & 3 \end{pmatrix}}_{=A} \bar{X}$$

Find the general solution of the system.

$\det(A - \lambda I) = \cancel{t} -\lambda(3-\lambda) - 10 = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$

$\Rightarrow \lambda = 5, -2$

$\lambda = 5$: $\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \quad -5v_1 - 2v_2 = 0 \Rightarrow v_2 = -\frac{5}{2}v_1$

$\Rightarrow \bar{v}_1 = \begin{pmatrix} v_1 \\ -\frac{5}{2}v_1 \end{pmatrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix}$

$\lambda = -2$ $\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix}(\bar{v}_2 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \bar{x}(t) = c_1 e^{5t}\begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

6. (16 points) Given the system of differential equations

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix} \mathbf{X} \qquad (A - \lambda I) = \begin{pmatrix} 2-\lambda & -8 \\ 1 & -2-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det(A-\lambda I) = (2-\lambda)(-2-\lambda)+8 = \lambda^2+4 \Rightarrow \lambda = \pm 2i$$

$$\lambda = 2i: \begin{pmatrix} 2-2i & -8 \\ 1 & -2-2i \end{pmatrix} \bar{v} = \quad (2-2i)v_1 - 8v_2 = 0$$

$$\Rightarrow v_2 = \left(\tfrac{1}{4} - \tfrac{1}{4}i\right)v_1 \Rightarrow \bar{v} = \begin{pmatrix} v_1 \\ \left(\tfrac{1}{4}-\tfrac{1}{4}i\right)v_1 \end{pmatrix}$$

$$\Rightarrow \bar{v} = \begin{pmatrix} 1 \\ \tfrac{1}{4}-\tfrac{1}{4}i \end{pmatrix}$$

$$\Rightarrow \bar{X}(t) = e^{2it}\begin{pmatrix} 1 \\ \tfrac{1}{4}-\tfrac{1}{4}i \end{pmatrix} = \cos(2t)+i\sin(2t)\begin{pmatrix} 1 \\ \tfrac{1}{4}-\tfrac{1}{4}i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2t) + i\sin(2t) \\ \tfrac{1}{4}\cos(2t) + \tfrac{1}{4}i\sin(2t) - \tfrac{1}{4}i\cos(2t) + \tfrac{1}{4}\sin(2t) \end{pmatrix}$$

$$\Rightarrow \bar{X}(t) = c_1\begin{pmatrix} \cos(2t) \\ \tfrac{1}{4}\cos(2t) + \tfrac{1}{4}\sin(2t) \end{pmatrix} + \begin{pmatrix} \sin(2t) \\ \tfrac{1}{4}\sin(2t) - \tfrac{1}{4}\cos(2t) \end{pmatrix} c_2$$

(b) Find the solution that satisfies the initial-condition $\mathbf{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Report your solution as one real vector.

$$\bar{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ \tfrac{1}{4} \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -\tfrac{1}{4} \end{pmatrix}$$

$$\Rightarrow c_1 = 2 \qquad 1 = \tfrac{1}{2} - \tfrac{c_2}{4} \Rightarrow \tfrac{1}{2} = -\tfrac{c_2}{4}$$

$$\Rightarrow c_2 = -2$$

$$\Rightarrow \bar{X}(t) = \begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ \cancel{2}\tfrac{1}{2}\cos(2t) + \cancel{2}\tfrac{1}{2}\sin(2t) - \cancel{2}\tfrac{1}{2}\sin(2t) + \tfrac{1}{2}\cos(2t) \end{pmatrix} = \begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

7. (15 points) Using the power series method (centered at $x = 0$) on $y'' - xy = 0$,

(a) Find the recurrence relation.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \implies y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0$$

$$\implies 2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - a_{n-3} x^{n-2} = 0$$

$$\implies a_2 = 0 \qquad a_n = \frac{a_{n-3}}{n(n-1)}, \ n \geq 3$$

(b) Find the general solution up to degree 5 (up to and including the x^5 term).

$n=3$: $a_3 = \frac{a_0}{3 \cdot 2}$

$$\implies y(x) \simeq a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \cdots$$

$n=4$ $a_4 = \frac{a_1}{4 \cdot 3}$

$n=5$ $a_5 = \frac{a_2}{5 \cdot 4} = 0$