MATH225, Fall 2011
Worksheet 1 (Review)

Name: Solutions
Section:

For full credit, you must show all work and box answers.

1. Find the roots of the following polynomials. (Solve for r.)

   (a) $2r - 10 = 0$

   $r-5=0$, $\boxed{r=5}$

   (b) $r^2 - 6r + 9 = 0$

   $(r-3)^2 = 0$

   $\boxed{r=3}$ Repeated Root

   (c) $r^2 + 5r + 4 = 0$

   $(r+4)(r+1)=0$

   $\boxed{r=-4,\ r=-1}$

   (d) $2r^2 + 5r + 2 = 0$

   $$r = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(2)}}{2(2)} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{4}$$

   $\boxed{r=-2,\ r=-\frac{1}{2}}$

   or $(2r+1)(r+2)=0$, $\boxed{r=-\frac{1}{2},\ r=-2}$

2. Use exponential and logarithmic rules to simplify the following.

   (a) $\dfrac{e^{4x}e^{5x}}{e^{9x}} = \dfrac{e^{4x+5x}}{e^{9x}} = e^{9x-9x} = e^{0} = \boxed{1}$

   (b) $e^{2\ln(3x)} = e^{\ln(3x)^2} = \boxed{9x^2}$, $x>0$

   (c) $\ln(2y) - \ln(t) + 2\ln(2t) = \ln\left(\frac{2y(2t)^2}{t}\right) = \boxed{\ln(8yt)}$, $y, t>0$

3. Verify the given function $y(t)$ satisfies the given differential equation.

   (a) $y = e^{-10t} + 10\cos(t) + \sin(t)$, $\quad y' + 10y = 101\cos(t)$

   $y' = -10e^{-10t} - 10\sin t + \cos t$

   lhs: $y' + 10y = \cancel{-10e^{-10t}} - \cancel{10\sin t} + \cos t + 10[\cancel{e^{-10t}} + 10\cos t + \cancel{\sin t}]$

   $= 101\cos t$: rhs, $y(t)$ is a soln

   (b) $y = 2\sin(t) + \cos(t) + 10$, $\quad y'' + y = 10$

   $y' = 2\cos t - \sin t$, $y'' = -2\sin t - \cos t$

   lhs: $y'' + y = -2\sin t - \cos t + 2\sin t + \cos t + 10$

   $= 10$: rhs, y is a soln

   (c) $y = e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right)$, $\quad 2y'' + 5y' + 4y = 0$

   $y' = -\frac{5}{4}e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right) - \frac{\sqrt{7}}{4}e^{-\frac{5}{4}t}\sin\left(\frac{\sqrt{7}}{4}t\right)$

   $y'' = \frac{25}{16}e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right) + \frac{5\sqrt{7}}{16}e^{-\frac{5}{4}t}\sin\left(\frac{\sqrt{7}}{4}t\right) + \frac{5\sqrt{7}}{16}e^{-\frac{5}{4}t}\sin\left(\frac{\sqrt{7}}{4}t\right) - \frac{7}{16}e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right)$

   lhs: $2y'' + 5y' + 4y = 2\left[\frac{9}{16}e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right) + \cancel{\frac{5\sqrt{7}}{8}e^{-\frac{5}{4}t}\sin\left(\frac{\sqrt{7}}{4}t\right)}\right] + 5\left[-\frac{5}{4}e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right) - \cancel{\frac{\sqrt{7}}{4}e^{-\frac{5}{4}t}\sin\left(\frac{\sqrt{7}}{4}t\right)}\right]$

   $+ 4\left[e^{-\frac{5}{4}t}\cos\left(\frac{\sqrt{7}}{4}t\right)\right] = 0$: rhs, y is a soln

4. Evaluate the following definite integrals.

(a) $\int_1^e \frac{e^{\ln(x)}}{2x}\,dx$ $= \int_0^1 \frac{e^u}{2}\,du = \frac{1}{2}e^u \Big|_0^1 = \boxed{\frac{1}{2}e - \frac{1}{2}}$ or $= \boxed{\frac{1}{2}(e-1)}$

u-sub.

$u = \ln x$, $du = \frac{1}{x}dx$

$x = 1$, $u = \ln(1) = 0$

$x = e$, $u = \ln(e) = 1$

or $\int_1^e \frac{e^{\ln x}}{2x}\,dx = \int_1^e \frac{x}{2x}\,dx = \int_1^e \frac{1}{2}\,dx = \frac{1}{2}x\Big|_1^e = \boxed{\frac{1}{2}(e-1)}$

(b) $\int_0^1 \frac{(2t+1)e^{2t}}{e^{3t}}\,dt$ $= \int_0^1 (2t+1)e^{-t}\,dt = \left(-(2t+1)e^{-t} - 2e^{-t}\right)\Big|_0^1$

Int. By Parts

$= -3e^{-1} - 2e^{-1} - [-1-2]$

$= \boxed{-\frac{5}{e} + 3}$

u		dv
$2t+1$	$+$	e^{-t}
2	$-$	$-e^{-t}$
0		e^{-t}

(c) $\int_0^{\frac{\pi}{4}} \tan(y)\,dy$ $= \int_0^{\frac{\pi}{4}} \frac{\sin y}{\cos y}\,dy = \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u}\,du$

$= -\ln|u|\Big|_1^{\frac{\sqrt{2}}{2}}$

$= -\ln\left(\frac{\sqrt{2}}{2}\right) + 0$

$= \boxed{-\ln\left(\frac{\sqrt{2}}{2}\right)}$

$= -\ln\left(\frac{1}{\sqrt{2}}\right) = -\ln(2^{-\frac{1}{2}})$

$= \frac{\ln 2}{2}$

u-sub

$u = \cos y$, $du = -\sin y\,dy$

$-du = \sin y\,dy$

$y = 0$, $u = \cos(0) = 1$

$y = \frac{\pi}{4}$, $u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

5. Evaluate the following indefinite integrals.

(a) $\int (4t+2)\sqrt{t^2+t}\,dt$ $= \int 2(2t+1)(t^2+t)^{\frac{1}{2}}\,dt = \int 2u^{\frac{1}{2}}\,du$

u-sub

$u = t^2+t \quad du = (2t+1)\,dt$

$= 2\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C$

$= \boxed{\frac{4}{3}(t^2+t)^{\frac{3}{2}} + C}$

How to
choose u

Logs
Inverse Trig
Algebraic
Trig
Exponentials

(b) $\int 3x^2\ln(x)\,dx$ $= x^3\ln x - \int \frac{x^3}{x}\,dx = x^3\ln x - \int x^2\,dx$

$= \boxed{x^3\ln x - \frac{x^3}{3} + C}$

Int. By Parts

$u = \ln x \quad dv = 3x^2\,dx$

$du = \frac{1}{x}\,dx \quad v = x^3$

(c) $\int \frac{10}{y(5-y)}\,dy$ $= \int\left(\frac{2}{y} + \frac{2}{5-y}\right)dy = \boxed{2\ln|y| - 2\ln|5-y| + C}$

or

$= \boxed{\ln\left(\left(\frac{y}{5-y}\right)^2\right) + C}$

Partial Fractions

$\frac{10}{y(5-y)} = \frac{A}{y} + \frac{B}{5-y}$

$A(5-y) + By = 10$

$(-A+B)y + 5A = 10$

$-A+B=0 \quad 5A=10$

$B=A \quad A=2$

$B=2$

(d) $\int \frac{1}{1+y^2}\,dy$ $= \tan^{-1}y + C$

or

$= \arctan y + C$

Algebra and Calculus Review

In MATH225 you will need to use algebra, trigonometry, and calculus as tools to solve ordinary differential equations. Based on prerequisites, it is expected that you are skilled in these areas of mathematics already. Small arithmetic and calculus errors may be graded more intensely than what you may have been accustomed to in your previous math classes. This paper is intended to help review these topics but it should NOT be viewed as an all-encompassing review.

Quadratic Formula:

If $ax^2+bx+c=0$, then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Exponential Properties:

$a^{x+y}=a^xa^y,\quad a^{x-y}=\dfrac{a^x}{a^y},\quad (a^x)^y=a^{xy},\quad a^{-y}=\dfrac{1}{a^y},\quad a^0=1$

Note: a can be any positive real number including e.

Natural Log Properties:

$\ln(x)+\ln(y)=\ln(xy),\quad \ln(x)-\ln(y)=\ln\left(\dfrac{x}{y}\right),\quad y\ln(x)=\ln(x^y),\quad e^{\ln(x)}=x,\quad \ln(1)=0$

Note: x,y can be any positive real numbers.

u-substitution:

Ex:

$$\begin{aligned}\int xe^{x^2}\,dx &= \int\frac{1}{2}e^u\,du=\frac{1}{2}e^u+C=\frac{1}{2}e^{x^2}+C\\ \text{Let } u &= x^2,\quad du=2x\,dx\end{aligned}$$

Integration by Parts: $\int u\,dv=uv-\int v\,du$

Ex:

$$\begin{aligned}\int te^t\,dt &= te^t-\int e^t\,dt=te^t-e^t+C\\ \text{Let } u &= t,\quad dv=e^t\,dt\\ du &= dt,\quad v=e^t\end{aligned}$$

Partial Fraction Decomposition:

Ex:

$$\begin{aligned}\frac{s^2+s+4}{s(s-1)(s+2)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}\\ \frac{s^2+s+4}{s(s-1)(s+2)^2} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}+\frac{D}{(s+2)^2}\\ \frac{s^2+s+4}{s(s-1)(s^2+2)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+2}\end{aligned}$$

Completing the Square:

Ex:

$$\begin{aligned}s^2-2s+5 &= s^2-2s+\left(\frac{-2}{2}\right)^2+5-\left(\frac{-2}{2}\right)^2=(s-1)^2+4\\ s^2+s+5 &= s^2+s+\left(\frac{1}{2}\right)^2+5-\left(\frac{1}{2}\right)^2=\left(s+\frac{1}{2}\right)^2+\frac{19}{4}\end{aligned}$$