MATH225, Fall 2011
Worksheet 12 (10.1, 10.2, 10.3, 10.4)

Name: Solutions
Section:

For full credit, you must show all work and box answers.

1. Find all the critical (equilibrium) points of the given systems.

(a) $$\frac{dx}{dt} = x - y$$ $$\frac{dy}{dt} = 5x + 2y$$

$x - y = 0$ & $5x + 2y = 0$

$\underline{y = x}$

$\underline{y = x}$: $5x + 2x = 0$

$7x = 0$

$\underline{\underline{x = 0}}$

$\downarrow$

$\underline{\underline{y = x = 0}}$

$\boxed{\text{EP}: (0,0)}$

(b) $$x' = x^2 + xy$$ $$y' = x^3 + 3y + 2x$$

$x(x+y) = 0$ & $x^3 + 3y + 2x = 0$

$\underline{x = 0}$ $\quad x + y = 0$

$\underline{y = -x}$

$\underline{x = 0}$: $3y = 0$

$\underline{\underline{y = 0}}$

$\underline{y = -x}$: $x^3 - 3x + 2x = 0$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$\underline{\underline{x = 0}}$ $\quad x^2 - 1 = 0$

$\downarrow$ $\quad x^2 = 1$

$\underline{\underline{y = 0}}$ $\quad \underline{\underline{x = 1}}, \underline{\underline{x = -1}}$

$\downarrow \quad \downarrow$

$\underline{\underline{y = -1}} \quad \underline{\underline{y = 1}}$

$\boxed{\text{EP}: (0,0), (1,-1), (-1,1)}$

(c) $$\frac{dx}{dt} = (x+1)e^y$$ $$\frac{dy}{dt} = 5y(e^x - 1)$$

$(x+1)e^y = 0$ & $5y(e^x - 1) = 0$

$x + 1 = 0$, $e^y \neq 0$

$\underline{x = -1}$

$\underline{x = -1}$: $5y(e^{-1} - 1) = 0$

$5y = 0$

$\underline{\underline{y = 0}}$

$\boxed{\text{EP}: (-1,0)}$

2. Given the system

$$\frac{dX}{dt} = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} X \quad A = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det \begin{pmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{pmatrix} = 0$$

$$(-1-\lambda)(1-\lambda) - (5)(-1) = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\lambda = \pm 2i$$

$$\lambda_1 = 2i : (A - \lambda_1 I)\vec{V_1} = \vec{O}$$

$$\begin{pmatrix} -1-2i & 5 \\ -1 & 1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-1-2i)v_1 + 5v_2 = 0$$

$$v_2 = \frac{(1+2i)v_1}{5}$$

$$\begin{pmatrix} v_1 \\ \frac{(1+2i)v_1}{5} \end{pmatrix}, \quad v_1 = 5, \quad \vec{V_1} = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix}$$

$$e^{\lambda_1 t}\vec{V_1} = e^{2it}\begin{pmatrix} 5 \\ 1+2i \end{pmatrix} = (\cos(2t) + i\sin(2t))\begin{pmatrix} 5 \\ 1+2i \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos(2t) + 5i\sin(2t) \\ \cos(2t) + 2i\cos(2t) + i\sin(2t) - 2\sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i\begin{pmatrix} 5\sin(2t) \\ 2\cos(2t) + \sin(2t) \end{pmatrix}$$

$$\boxed{\vec{X}(t) = C_1\begin{pmatrix} 5\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + C_2\begin{pmatrix} 5\sin(2t) \\ 2\cos(2t) + \sin(2t) \end{pmatrix}}$$

(b) Classify the critical (equilibrium) point $(0, 0)$ of the system.

$\lambda = \pm 2i$, Pure Imaginary, $\boxed{\text{Center}}$

$a=0, \ b=2$

(c) Discuss the nature of solutions in a neighborhood of $(0, 0)$.

$$\text{Period} = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

Solutions are periodic. Solutions are ellipses that circle the origin every π units of time.

(d) Find the solution that satisfies the initial condition $X(0) = (-2, 2)$. Report your solution as one real vector.

$$\vec{X}(0) = C_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$5C_1 = -2 \qquad C_1 + 2C_2 = 2$$

$$C_1 = -\frac{2}{5} \qquad C_2 = \frac{2 - C_1}{2} = \frac{12}{10} = \frac{6}{5}$$

$$\boxed{\vec{X}(t) = \begin{pmatrix} -2\cos(2t) + 6\sin(2t) \\ 2\cos(2t) + 2\sin(2t) \end{pmatrix}}$$

3. Classify the critical (equilibrium) point $(0,0)$ of the linear system by computing the trace τ and the determinant Δ. Use this information to match the given linear systems to their phase portraits.

(a) $\dfrac{dx}{dt} = 4x + y$, $\dfrac{dy}{dt} = x + 2y$ (b) $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = 2x - y$ (c) $\dfrac{dx}{dt} = -x + y$, $\dfrac{dy}{dt} = -2x - y$ (d) $\dfrac{dx}{dt} = x + 5y$, $\dfrac{dy}{dt} = 5x + 2y$

(a) $A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

$\tau = 4 + 2 = 6 > 0$

$\Delta = (4)(2) - (1)(1) = 7 > 0$

$\tau^2 - 4\Delta = 36 - 28 > 0$

$\boxed{\text{Unstable Node}}$

(b) $A = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$\tau = -1 - 1 = -2 < 0$

$\Delta = (-1)(-1) - 0 = 1 > 0$

$\tau^2 - 4\Delta = 4 - 4 = 0$

$\boxed{\text{Degenerate Stable Node}}$

(c) $A = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}$

$\tau = -1 - 1 = -2 < 0$

$\Delta = (-1)(-1) - (1)(-2) = 3 > 0$

$\tau^2 - 4\Delta = 4 - 12 < 0$

$\boxed{\text{Stable Spiral Point}}$

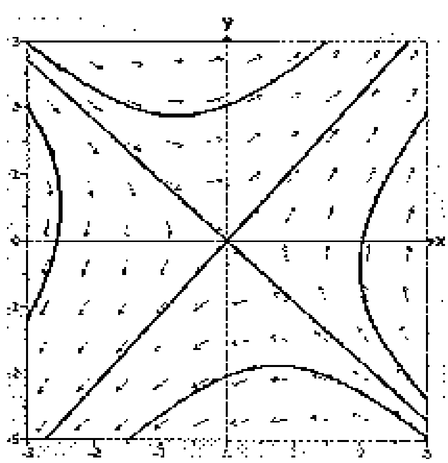
(d) $A = \begin{pmatrix} 1 & 5 \\ 5 & 2 \end{pmatrix}$

$\tau = 1 + 2 = 3$

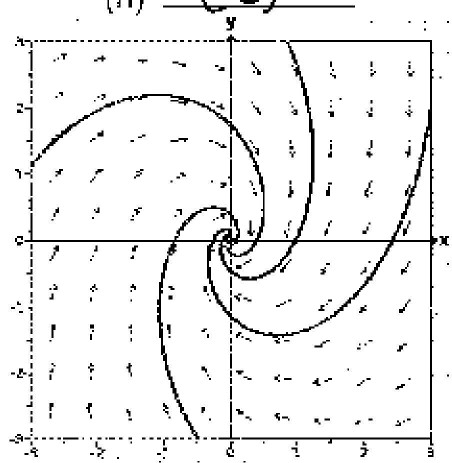
$\Delta = (1)(2) - (5)(5) = -23 < 0$

$\boxed{\text{Saddle Point}}$

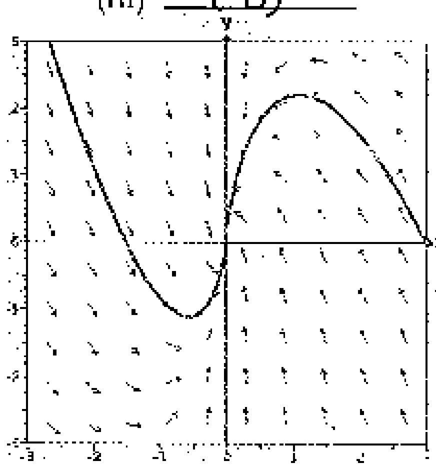
(I) (d)



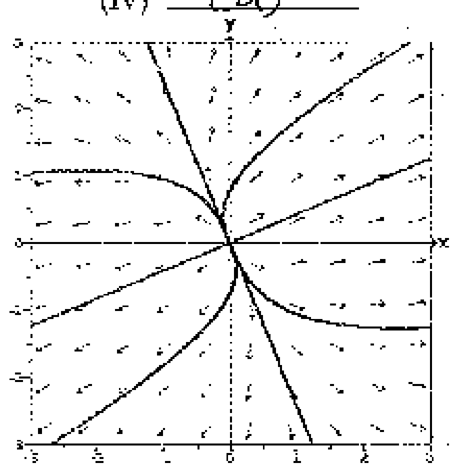
(II) (c)



(III) (b)



(IV) (a)



4. Consider the system:

$$\frac{dx}{dt} = x(1-3x+y) = x(1-3x)+xy = x-3x^2+xy$$
$$\frac{dy}{dt} = y(4-y-2x) = 4y(1-\tfrac{y}{4})-2xy = 4y-y^2-2xy$$

(a) Are these species predator-prey or competing?

$\boxed{\text{Predator - Prey}}$, x is the predator since its interaction term (+xy) is positive, y is the prey, interaction term (-2xy) negative.

(b) What type of growth does species y exhibit in absence of species x?

$\underline{x=0}$: $\frac{dy}{dt} = 4y(1-\frac{y}{4})$, $\boxed{\text{Logistic Growth}}$

(c) Find all critical (equilibrium) points.

$x(1-3x+y)=0$ & $y(4-y-2x)=0$

$\underline{x=0}$ $\quad 1-3x+y=0$, $x=\frac{1+y}{3}$

$\underline{x=0}$: $y(4-y)=0$, $y=0$, $4-y=0$, $y=4$

$\underline{x=\frac{1}{3}+\frac{y}{3}}$: $y(4-y-\frac{2}{3}-\frac{2}{3}y)$, $y(\frac{10}{3}-\frac{5}{3}y)=0$

$y=0 \downarrow x=\frac{1}{3}$ $\qquad \frac{10}{3}-\frac{5}{3}y=0$, $y=2 \downarrow x=\frac{1}{3}+\frac{2}{3}$, $x=1$

$\boxed{\text{EP}: (0,0), (0,4), (\tfrac{1}{3},0), (1,2)}$

(d) Using the Jacobian matrix, classify (if possible) each critical (equilibrium) point as a stable node, a stable spiral point, an unstable node, an unstable spiral point, or a saddle point.

$$J(x,y) = \begin{pmatrix} 1-6x+y & x \\ -2y & 4-2y-2x \end{pmatrix}$$

$\underline{(0,0)}$: $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$\tau = 1+4 = 5>0$

$\Delta = (1)(4)-0 = 4>0$

$\tau^2-4\Delta = 25-16>0$

$\boxed{\text{Unstable Node}}$

$\underline{(0,4)}$: $J(0,4) = \begin{pmatrix} 5 & 0 \\ -10 & -4 \end{pmatrix}$

$\tau = 5-4 = 1>0$

$\Delta = (5)(-4)-0 = -20<0$

$\boxed{\text{Saddle Point}}$

$(\frac{1}{3},0)$: $J(\frac{1}{3},0) = \begin{pmatrix} -1 & \frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$

$\tau = -1+\frac{10}{3} = \frac{7}{3}>0$

$\Delta = (-1)(\frac{10}{3})-0 = -\frac{10}{3}<0$

$\boxed{\text{Saddle point}}$

$\underline{(1,2)}$: $J(1,2) = \begin{pmatrix} -3 & 1 \\ -4 & -2 \end{pmatrix}$

$\tau = -3-2 = -5<0$

$\Delta = (-3)(-2)-(1)(-4) = 10>0$

$\tau^2-4\Delta = 25-40<0$

$\boxed{\text{Stable Spiral Point}}$

5. Consider a pendulum made of a rigid rod with a ball at one end. The second-order differential equation which models the damped pendulum is:

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0$$

where $\theta(t)$ is the angle at time, t, measured in a counterclockwise direction from the downward vertical. The parameter g is gravity, l is the length of the pendulum, b is the damping coefficient, and m is the mass of the ball (we neglect the mass of the rod). For simplicity, let $l = m = 1$, $b = 2$, and $g \approx 9.8$, thus the equation becomes:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 9.8\sin(\theta) = 0$$

(a) Let $v = \frac{d\theta}{dt}$ and convert the second-order differential equation to a first-order system.

$V = \frac{d\theta}{dt}$

$\frac{dv}{dt} = \frac{d^2\theta}{dt^2} = -9.8\sin\theta - 2\frac{d\theta}{dt}$

$= -9.8\sin\theta - 2V$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= V \\ \frac{dv}{dt} &= -9.8\sin\theta - 2V \end{aligned}}$$

(b) Find the equilibrium points of the system for $0 \le \theta < 2\pi$.

$\underline{V=0}$ + $-9.8\sin\theta - 2V = 0$

$\underline{V=0}$; $-9.8\sin\theta = 0$

$\sin\theta = 0$

$\underline{\theta = 0, \pm\pi, \pm 2\pi, \ldots}$

$\boxed{\underline{EP}: (0,0), (\pi,0)}$

(c) Using the Jacobian matrix, classify (if possible) each critical (equilibrium) point for $0 \le \theta < 2\pi$ as a stable node, a stable spiral point, an unstable node, an unstable spiral point, or a saddle point.

$J(\theta, V) = \begin{pmatrix} 0 & 1 \\ -9.8\cos\theta & -2 \end{pmatrix}$

$\underline{(0,0)}$: $J(0,0) = \begin{pmatrix} 0 & 1 \\ -9.8 & -2 \end{pmatrix}$

$\tau = 0 - 2 = -2 < 0$

$\Delta = (0)(-2) - (1)(-9.8) = 9.8 > 0$

$\tau^2 - 4\Delta = 4 - 4(9.8) < 0$

$\boxed{\text{Stable spiral point}}$

$\underline{(\pi,0)}$: $J(\pi,0) = \begin{pmatrix} 0 & 1 \\ 9.8 & -2 \end{pmatrix}$

$\tau = 0 - 2 = -2 < 0$

$\Delta = (0)(-2) - (1)(9.8)$

$= -9.8 < 0$

$\boxed{\text{Saddle Point}}$