1. Introduction. We wish to develop a vector and tensor analysis which holds in more general coordinate systems. It is assumed that the reader has already an understanding of vectors and tensors in the Cartesian sense and is familiar with curvilinear coordinate systems such as cylindrical and spherical polar. The analysis we develop is similar to that as introduced by Cartesian tensors, however there are some distinctions which need to be made for transformations in different system and, also, partial differentiation requires the development of some new concepts to enable us to preserve the rules we learned in the Cartesian sense. Another consideration that we must include in our analysis is the means for everything to reduce in the Cartesian sense to the rules we are already familiar with.

2. Notation. In the generalized tensor sense, it is necessary to distinguish between two types of transformation. When discussing Cartesian tensors, this distinction did not exist and allowed for a simplified notation when representing tensors. As a result of this added complexity, we will represent the coordinates of a point in a general system with a superscript rather than a subscript as we have done in the Cartesian system. That is, \( x^i \), \( i = 1, 2, 3 \), will stand for the coordinates of a point \( P \) in place of the \( x_i \) we have used to date. For example, in cylindrical polar coordinates, we would have \( x^1 = r \), \( x^2 = \theta \) and \( x^3 = z \).

If we transform to another system, the bar will continue to be used to denote the coordinates in the transformed system. For example, if the transformed system is cylindrical polar coordinates, we would have the equations defining the transformation as

\[
\bar{x}^1 = \left( (x^1)^2 + (x^2)^2 \right)^{\frac{1}{2}}, \quad \bar{x}^2 = \tan^{-1}(x^2/x^1), \quad \bar{x}^3 = x^3. \tag{2.1}
\]

We will now use the subscript associated with an element to imply partial differentiation with respect to a coordinate. So,

\[
v_i = \frac{\partial v}{\partial x^i}.
\]

**Summation Convention:** Any index which occurs in both the subscript and superscript positions in a product of terms is held to be summed over its values. Because of this, if an index is repeated in just superscripts, then it is not assumed that there exists a summation. A formula such as

\[
A^{ij}_k = E^{i}F^{j}_{kp}G^p
\]

would imply a summation over the index \( p \), \( p = 1, 2, 3 \).

3. Contravariant Vectors. When dealing with Cartesian coordinates, the transformation between coordinates was linear. In general, we cannot expect that the transformation between two coordinates is linear; as is exhibited by the transformation between Cartesian and cylindrical
polar coordinates. However, differentials of coordinates are connected by the rules of partial differentiation, as
\[ dx^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \] (summation on j). For example, consider the transformation between Cartesian and cylindrical polars (2.1). We have,
\[ \begin{align*}
dx^1 &= \frac{x^1 dx^1}{((x^1)^2 + (x^2)^2)^{\frac{1}{2}}} + \frac{x^2 dx^2}{((x^1)^2 + (x^2)^2)^{\frac{1}{2}}} \\
- dx^2 &= \frac{-x^2 dx^1}{((x^1)^2 + (x^2)^2)^{\frac{1}{2}}} + \frac{x^1 dx^2}{((x^1)^2 + (x^2)^2)^{\frac{1}{2}}} \\
dx^3 &= dx^3.
\end{align*} \]
This type of transformation is called a contravariant transformation and, as we did in the Cartesian system, we define a contravariant vector to be anything which transforms in this manner.

**Definition 3.1.** We say that \( a^i \) are the components of a contravariant vector at a point in the coordinate system \( O123 \) if under a transformation of coordinates to the system \( \bar{O}1\bar{2}\bar{3} \) the components of the vector become
\[ \bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^j} a^j. \]
As an example of a contravariant vector, suppose that \( x^i(t) \) is the position of a particle at time \( t \), then
\[ v^i = \frac{dx^i}{dt} \]
is its velocity. Using the chain rule, we have
\[ \bar{v}^i = \frac{dx^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} \frac{dx^j}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} v^j, \]
which shows that the velocity is a contravariant vector. Similarly, the acceleration and other derivatives are contravariant vectors.

**4. Covariant Vectors.** Suppose that \( f(x^1, x^2, x^3) \) is a scalar function and that we transform it into a function of the variables \( \bar{x}^1, \bar{x}^2, \bar{x}^3 \). Then, its derivatives will transform according to the equation
\[ \frac{\partial f}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial f}{\partial x^j} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial f}{\partial x^j}. \] (4.1)
We could write,
\[ a_j = \frac{\partial f}{\partial x^j} \]
which enables us to rewrite (4.1) to get
\[ \bar{a}_i = \frac{\partial x^j}{\partial \bar{x}^i} a_j. \] (4.2)
We call this type of transformation a covariant transformation and anything which transforms in this manner is a covariant vector.

**Definition 4.1.** We say that $a_i$ are the components of a covariant vector at a point in the coordinate system $O_{123}$ if under a transformation of coordinates to the system $O_{\bar{1}\bar{2}\bar{3}}$ the components of the vector transform according to (4.2).

Recall, from the study of Cartesian transformations, the rotation transformation, $\bar{x}^i = \ell_{ij} x^j$. We write,

$$\bar{x}^i = \ell_{ij} x^j,$$

and

$$x^j = \ell_{ij} \bar{x}^i,$$

which implies $\partial \bar{x}^i / \partial x^j = \ell_{ij}$ and $\partial x^j / \partial \bar{x}^i = \ell_{ij}^{-1}$, and notice that $\ell_{ij}$ and $\ell_{ij}^{-1}$ are both exactly the same as $\ell_{ij}$ for Cartesian tensors. This yields, from the definitions of contravariant and covariant vectors, that there is no distinction between contravariant and covariant transformations in the Cartesian frame of reference.

As an example of a covariant vector, let $y = (y^1, y^2, y^3)$ be the Cartesian coordinates of a point and $x^1, x^2, x^3$ be its coordinates in another coordinate system. For $i = 1, 2, 3$, $y^i = y^i(x^1, x^2, x^3)$ is a scalar function and so, from the definition of a covariant vector, the quantities

$$\frac{\partial y^i}{\partial x^j}, \quad j = 1, 2, 3,$$

are the components of a covariant vector.

**5. Contravariant and Covariant Tensors.** Just as in the Cartesian case, the definitions for vectors can be expanded into dimensions of two or greater and a vector can be thought of as a tensor of order one.

If we form the quantities, $A^{ij} = B^i C^j$, where $B^i$ and $C^j$ are the components of two contravariant vectors, then it follows that the $A^{ij}$ transform according to

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} A^{kl}, \quad (5.1)$$

In general, if we have the set of quantities $A^{ij}$ which transform according to (5.1), then we call $A^{ij}$ a **contravariant tensor** of second order.

Similarly, if we have the set of quantities $A_{ij}$ which transform according to

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl},$$

then we call $A_{ij}$ a **covariant tensor** of second order.

Moreover, if we have the set of quantities $A_i^j$ whose transformation law is

$$\bar{A}_i^j = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^j} A_i^k,$$

then we call $A_i^j$ a **mixed tensor** of second order.

For dimensions higher than two, the definition of a tensor is analogous to that of two dimensions and is not presented here in an effort to not complicate this discussion.
As an example, consider the permutation symbol $\varepsilon^{ijk}$. We have already defined this symbol when considering Cartesian tensors and would like to verify that it is still a third order tensor. From the definition of the determinant,

$$\frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} \varepsilon^{ijk} = J_{\bar{x}} \varepsilon^{pqr}.$$

Which implies,

$$\bar{\varepsilon}^{pqr} = J^{-1} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} \varepsilon^{ijk} = \varepsilon^{pqr},$$

and evidently $\varepsilon^{ijk}$ is a third order covariant tensor. Similarly, if $\varepsilon^{ijk}$ is defined the same way then it is also a third order contravariant tensor.

6. Addition, Subtraction and Multiplication of Tensors. Clearly, $A^{ij} + B^i$ is not a tensor because it does not transform. However, if we consider tensors of the same type, then the properties of a group hold. For example, for the two tensors $A^i_{jk}$ and $B^i_{jk}$, we can form a third tensor $\alpha A^i_{jk} + \beta B^i_{jk}$ which obeys the transformation law.

In particular, we may add and subtract two tensors, $A^i_{jk} + B^i_{jk}$, $A^i_{jk} - B^i_{jk}$ and produce a tensor. Rules exactly like contraction and the quotient rule in the Cartesian sense apply and are not discussed again here.

7. The Metric Tensor. In this section we derive the definition for a tensor of fundamental importance. We start the derivation in the Cartesian frame of reference, however there is no such restriction placed on our definition – it is valid in any frame of reference.

Consider a point $P$ in the Cartesian system with coordinates $y^i$ and another point $Q$ with coordinates $y^i + dy^i$. The distance between $P$ and $Q$, $ds$, is given by

$$ds^2 = \sum_{k=1}^{3} dy^k dy^k$$

(summation notation does not apply since both scripts are in the super position). If $x^i, i = 1, 2, 3$ are the coordinates of the point $P$ in another system, then

$$dy^k = \frac{\partial y^k}{\partial x^i} dx^i,$$

and,

$$ds^2 = \sum_{k=1}^{3} \left( \frac{\partial y^k}{\partial x^i} dx^i \right) \left( \frac{\partial y^k}{\partial x^j} dx^j \right)$$

$$= g_{ij} dx^i dx^j$$

where

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \quad (7.1)$$

defines the metric tensor $g_{ij}$. Notice that the definition (7.1) relates distance between coordinate systems and is independent of the Cartesian system in which we started. Also, it can be shown that the metric tensor obeys the transformation laws of a second order covariant tensor.
For an example, again using cylindrical polar coordinates (2.1), we have

\[ g_{11} = 1, \quad g_{22} = (x^1)^2, \quad g_{33} = 1, \]

and all \( g_{ij} = 0, i \neq j \). When it is the case that only the diagonal elements of the metric tensor are nonzero, then the coordinate system in which we are dealing is said to be orthogonal. We do not consider non-orthogonal systems in this presentation, however allowing for them does not add additional special cases to this analysis. For reasons that become apparent later, we write

\[ g_{ii} = h^2_i, \]

where the \( h_i \) are called scale factors.

If we write \( g_{ij} \) as a matrix and let \( g \) denote its determinant, then \( g \) is nonzero and hence \( g_{ij} \) is invertible. If we write its inverse as \( g^{ij} \) then we have

\[ g^{ij} g_{jk} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \] (7.2)

which is precisely the definition of the Kronecker delta which we now will write \( \delta^i_j \).

8. Physical Components. When dealing with Cartesian coordinates, all have the physical dimension of length, but in general, we cannot expect this of curvilinear coordinate system. Consider, for example, cylindrical polar coordinates, \( x^1 = r, x^2 = \theta \) and \( x^3 = z \) – the first and third components have the dimension of length, however the second has no such dimension associated with it. The leads to the contravariant velocity components, \( v^i = dx^i/dt \) not all having the same physical dimensions. This differs from what we would expect to be the physical components of velocity. A similar discrepancy arises when we consider the covariant components of velocity.

Assuming that we are dealing with an orthogonal coordinate system, consider the contravariant vectors tangent to the the coordinate lines,

\[ e^i_{(1)} = \delta^i_1 / h_1, \quad e^i_{(2)} = \delta^i_2 / h_2, \quad e^i_{(3)} = \delta^i_3 / h_3. \]

If the contravariant vector \( A^i \) is represented as a linear combination of these basis vectors, then we can write

\[ A^i = A(1)e^i_{(1)} + A(2)e^i_{(2)} + A(3)e^i_{(3)}, \]

which implies

\[ A(1) = h_1 A^1, \quad A(2) = h_2 A^2, \quad A(3) = h_3 A^3. \]

The \( A(i) \) are called the physical components of the contravariant vector \( A^i \).

As an example, consider the magnitude of the vector \( A \) for a general orthogonal coordinate system,

\[ |A|^2 = (h_1 A^1)^2 + (h_2 A^2)^2 + (h_3 A^3)^2 \]

where the scale factors cause both sides of the equation to have the same physical dimensions. Specifically, if we consider cylindrical polar coordinates, we have

\[ ds^2 = (dx^1)^2 + (x^1 dx^2)^2 + (dx^3)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2 \]
The physical components of a covariant vector can be constructed in the same way as we did for the contravariant vector, however it turns out that the physical components are the same in both cases, which is reassuring as it is desirable to have consistent physical components.

9. Differentiation. We observed in §4 that the set of partial derivatives \( \frac{\partial f}{\partial y^i} \) of a scalar function \( f(x^1, x^2, x^3) \) represents a covariant vector. If we form the set of second partial derivatives \( \frac{\partial}{\partial y^j} \left( \frac{\partial f}{\partial y^i} \right) \) of the covariant vector \( \frac{\partial f}{\partial y^i} \), we get

\[
\frac{\partial^2 f}{\partial y^j \partial y^i} = \frac{\partial}{\partial y^j} \left( \frac{\partial f}{\partial x^r} \frac{\partial x^r}{\partial y^i} \right) = \frac{\partial^2 f}{\partial x^r \partial y^j} \frac{\partial x^r}{\partial y^i} + \frac{\partial f}{\partial x^r} \frac{\partial^2 x^r}{\partial y^j \partial y^i},
\]

which is not a second order tensor unless the term \( \frac{\partial f}{\partial x^r} \frac{\partial^2 x^r}{\partial y^j \partial y^i} = 0 \), which is in general not true. Therefore, the set of partial derivatives of a covariant vector does not, in general, transform as a second order tensor. This differs from the Cartesian system where the second partial derivatives formed a second order tensor. To this end, we would like to define a derivative on tensors which preserves the tensor character, obeys the standard rules of differentiation such as the product rule and reduces to regular partial differentiation when dealing with Cartesian vectors and tensors.

9.1. Christoffel Symbols. Our goal is to build expressions involving the derivatives of a tensor, which are themselves a tensor. We do this through the use of two function which are based on the metric tensor \( g_{ij} \). The functions are called the Christoffel symbols of the first and second kind defined respectively by

\[
[ij, k] = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \quad (9.1)
\]

and

\[
\left\{ \begin{array}{c} l \\ ij \end{array} \right\} = g^{lk} [ij, k]. \quad (9.2)
\]

A useful formula which we can get derive (9.1) is that the derivative of the metric tensor,

\[
\frac{\partial g_{ik}}{\partial x^j} = [ij, k] + [kj, i]. \quad (9.3)
\]

Given that the fundamental tensor \( g_{ij} \), is second order and covariant, it transforms according to

\[
\bar{g}_{lm} = \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} g_{ij}.
\]

If we differentiate this equation with respect to \( \bar{x}^n \) and use (9.3), we get

\[
\frac{\partial \bar{g}_{lm}}{\partial \bar{x}^n} = \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^n} + \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^m} \frac{\partial x^j}{\partial \bar{x}^n} g_{ij} + \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial^2 x^k}{\partial \bar{x}^n \partial \bar{x}^k} g_{ij}.
\]

By cyclicly interchanging the indices \( l, m \) and \( n \) we subtract from this equation two similar equations. Dividing by two, and with appropriate changes in the dummy indices, we have

\[
[lm, n] = [ij, k] \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} + g_{ij} \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^m} \quad (9.4)
\]
where the bar bar over the Christoffel symbol indicates that it was calculated in the \( \bar{x}^i \) coordinate system with respect to the metric tensor \( \bar{g}_{ij} \).

Consider now the transformation law of the inverse of the metric tensor, \( g^{ij} \)

\[
g^{np} = g^{rs} \frac{\partial \bar{x}^n}{\partial x^r} \frac{\partial \bar{x}^p}{\partial x^s} \tag{9.5}\]

If we multiply (9.4) on the inside by the corresponding sides of (9.5) and reduce the result, we obtain the relation

\[
\left\{ \begin{array}{c} p \\ lm \end{array} \right\} = \left\{ \begin{array}{c} s \\ ij \end{array} \right\} \frac{\partial \bar{x}^n}{\partial x^r} \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} + \frac{\partial \bar{x}^n}{\partial x^r} \frac{\partial^2 \bar{x}^j}{\partial \bar{x}^l \partial \bar{x}^m}. \tag{9.6}\]

Incidentally, at this point, equations (9.4) and (9.6) show how Christoffel symbols transform and show, in general, that they are not tensors.

Multiplying (9.6), on the inside, with \( \frac{\partial x^r}{\partial \bar{x}^p} \) yields the main result of this section,

\[
\frac{\partial^2 x^r}{\partial \bar{x}^l \partial \bar{x}^m} = \left\{ \begin{array}{c} p \\ lm \end{array} \right\} \frac{\partial \bar{x}^n}{\partial x^r} \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} - \left\{ \begin{array}{c} r \\ ij \end{array} \right\} \frac{\partial \bar{x}^n}{\partial x^r} \frac{\partial^2 \bar{x}^j}{\partial \bar{x}^l \partial \bar{x}^m}. \tag{9.7}\]

This equation, which expresses second derivatives in terms of first derivatives and Christoffel symbols of the second kind, will be used repeatedly in subsequent sections to derive expressions for derivatives in the general sense.

### 9.2. Covariant Differentiation of Vectors.

Consider a contravariant vector \( A^k \) which transforms according to

\[
A^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i \tag{9.8}\]

and differentiate it with respect to \( x^i \), to obtain

\[
\frac{\partial A^k}{\partial x^i} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n} \frac{\partial x^n}{\partial x^r} \frac{\partial x^k}{\partial \bar{x}^i} + \bar{A}^i \frac{\partial^2 x^k}{\partial x^r \partial \bar{x}^n} \frac{\partial x^n}{\partial \bar{x}^i}. \]

Which we see is not, in general, a tensor. To obtain a tensor from this formula we make use of (9.7) to eliminate the partial derivatives of second order, and this gives us

\[
\frac{\partial A^k}{\partial x^i} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n} \frac{\partial x^n}{\partial \bar{x}^r} \frac{\partial x^k}{\partial \bar{x}^i} + \bar{A}^i \frac{\partial^2 x^k}{\partial \bar{x}^r \partial \bar{x}^n} \frac{\partial x^n}{\partial \bar{x}^i}. \]

By using (9.8) and by changing the appropriate dummy indices, this equation reduces to

\[
\frac{\partial A^k}{\partial x^i} + \left\{ \begin{array}{c} k \\ rj \end{array} \right\} A^r = \left[ \frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \left\{ \begin{array}{c} i \\ rn \end{array} \right\} \bar{A}^r \right] \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^k}{\partial \bar{x}^r}. \]

Introduce comma notation,

\[
A^k_{\bar{,}j} = \frac{\partial A^k}{\partial x^j} + \left\{ \begin{array}{c} k \\ rj \end{array} \right\} A^r, \]

and the above equation can be written

\[
A^k_{\bar{,}j} = \bar{A}^i_{\bar{,}n} \frac{\partial x^n}{\partial \bar{x}^j} \frac{\partial x^k}{\partial \bar{x}^i}. \]
It is clear from this formula that $A^k_j$ is a mixed tensor of second order and it is called the covariant derivative of $A^k$ with respect to $x^j$.

Now, consider a covariant vector $A^j$ which transforms according to

$$\tilde{A}^i = A^j \frac{\partial x^j}{\partial \tilde{x}^i}. \quad (9.9)$$

Differentiate this expression with respect to $\tilde{x}^i$, yielding

$$\frac{\partial \tilde{A}^i}{\partial \tilde{x}^l} = \frac{\partial A^j}{\partial x^m} \frac{\partial x^m}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} + A^j \frac{\partial^2 x^j}{\partial \tilde{x}^i \partial \tilde{x}^l}.$$  

We have already shown that, in general, this is not a tensor. Again, making use of (9.7) to eliminate the second order partial derivatives and changing dummy indices as needed we have, by using (9.9), that

$$\frac{\partial \tilde{A}^i}{\partial \tilde{x}^l} - \{m_{il}\} \tilde{A}^m = \left[ \frac{\partial A^j}{\partial x^n} - \{r_{jn}\} A^r \right] \frac{\partial x^j}{\partial \tilde{x}^l} \frac{\partial x^n}{\partial \tilde{x}^l}.$$  

Introduce another comma notation,

$$A^l_{,n} \equiv \frac{\partial A^j}{\partial x^n} \frac{\partial x^n}{\partial \tilde{x}^l} - \{r_{jn}\} A^r,$$

and the above equation can be written

$$\tilde{A}^i_{,l} = A^l_{,n} \frac{\partial x^j}{\partial \tilde{x}^l} \frac{\partial x^n}{\partial \tilde{x}^i},$$

which shows that $A^l_{,n}$ is a covariant tensor of second order. It is called the covariant derivative of $A^j$ with respect to $x^n$.

### 9.3. Covariant Differentiation of Tensors

The process we introduced to derive the covariant derivative of vectors can be generalized to allow us to differentiate a tensor covariantly. If we consider a second order mixed tensor $A^i_{jk}$, there will be no loss of generality as it contains both contravariant and covariant indices. This tensor transforms according to

$$\tilde{A}^i_{jk} = \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial \tilde{x}^j} A^l_{ij}$$

and upon multiplication on the inside by $\partial x^m/\partial \tilde{x}^i$ yields

$$\tilde{A}^i_{jk} \frac{\partial x^m}{\partial \tilde{x}^i} = A^l_{mn} \frac{\partial x^l}{\partial \tilde{x}^j}. \quad (9.10)$$

Differentiate this equation with respect to $\tilde{x}^k$, then again eliminate the second order partial derivatives by using (9.7) to obtain

$$\frac{\partial \tilde{A}^i_{jk}}{\partial \tilde{x}^l} \frac{\partial x^m}{\partial \tilde{x}^i} + \tilde{A}^j \left[ \left\{r \right\} \frac{\partial x^l}{\partial \tilde{x}^k} \frac{\partial x^n}{\partial \tilde{x}^j} + \tilde{A}^i \left[ \left\{l \right\} \frac{\partial x^l}{\partial \tilde{x}^k} \frac{\partial x^n}{\partial \tilde{x}^i} \right\} \right] = \tilde{A}^m_{lk} \frac{\partial x^m}{\partial \tilde{x}^j} \frac{\partial x^l}{\partial \tilde{x}^k} + A^m_{nr} \left[ \left\{r \right\} \frac{\partial x^l}{\partial \tilde{x}^k} \frac{\partial x^n}{\partial \tilde{x}^i} \right].$$

Using (9.10) and changing appropriate dummy indices, this equation becomes

$$\left[ \frac{\partial \tilde{A}^i_{jk}}{\partial \tilde{x}^k} + \tilde{A}^j \left\{r \right\} \frac{\partial x^l}{\partial \tilde{x}^i} \right] \frac{\partial x^m}{\partial \tilde{x}^i} = \left[ \frac{\partial A^m_{ij}}{\partial x^l} + \tilde{A}^j \left\{r \right\} \frac{\partial x^l}{\partial \tilde{x}^i} \right] \frac{\partial x^m}{\partial \tilde{x}^i}. $$
Introduce comma notation,

\[ A_{lm}^m = \frac{\partial A_{lm}^m}{\partial x^t} + \left\{ \begin{array}{l} m \\ r \end{array} \right\} A_r^m, \]

and the above equation takes the form

\[ \bar{A}_{lm}^m = A_{lm}^m \frac{\partial x^m}{\partial \bar{x}^l}. \]

Multiply on the inside by \( \partial \bar{x}^r / \partial x^m \) and we have

\[ \bar{A}_{lm}^r = A_{lm}^r \frac{\partial x^m}{\partial \bar{x}^l} \frac{\partial \bar{x}^r}{\partial x^m}. \]

Therefore, \( A_{lm}^m \) is a third order tensor and is called the covariant derivative of \( A_{lm}^m \) with respect to \( x^t \).

It should be noted that the method we used to find the derivative of the mixed second order tensor can be used in the more general sense to show that the covariant derivative of higher order tensors are in fact tensors themselves, however the proof is tedious and is not of interest beyond the above derivation and we will not do it in this analysis.

10. Gradient, Laplacian, Divergence and Curl. The covariant derivative of a scalar function \( \varphi \) is a covariant vector \( \varphi_{,i} \) which is called the gradient of \( \varphi \). The contravariant form of the gradient is \( g^{ij} \varphi_{,i} \). When in the Cartesian system, both of these reduce to the same familiar form of \( \nabla \varphi \).

If we take the covariant derivative of the gradient with respect to \( x^j \) and sum on \( j \), then we get a scalar function which is called the Laplacian of \( \varphi \)

\[ \nabla^2 \varphi = g^{ij} \varphi_{,ij}. \]

If we write this in full and shuffle dummy indices, we have

\[ \nabla^2 \varphi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x^1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \varphi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left( \frac{h_1 h_3}{h_2} \frac{\partial \varphi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \varphi}{\partial x^3} \right) \right]. \]

The divergence of a contravariant vector, \( A^i \) is defined as

\[ \text{div} A = A^i_{,i}, \]

while the divergence of a covariant vector \( A_i \) is defined as

\[ \text{div} A = g^{ij} A_{,j}. \]

Writing this out in full in terms of the physical components of \( A \) we get

\[ \text{div} A = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x^1} (A(1)h_2 h_3) + \frac{\partial}{\partial x^2} (h_4 A(2) h_3) + \frac{\partial}{\partial x^3} (h_1 A(2) h_3) \right]. \]

The curl is defined for contravariant and covariant vectors respectively as

\[ \text{curl} A = \epsilon^{ijk} A_{k,j} \quad \text{and} \quad \text{curl} A = \epsilon^{ijk} g_{kp} A^p_{,j}. \]

In full, the \( i \)th physical component of the curl is given by

\[ \left\{ \begin{array}{c} \frac{1}{h_j} \frac{\partial A(k)}{\partial x^j} - \frac{1}{h_k} \frac{\partial A(j)}{\partial x^k} \\ \frac{1}{h_j} \frac{\partial h_k}{\partial x^j} - \frac{1}{h_k} \frac{\partial h_j}{\partial x^k} \end{array} \right\} A^p_{,j} g_{kp} A^p_{,j} \]

from which it can be shown that this definition is exactly the Cartesian definition, when in that coordinate system.
11. Spaces Other Than Euclidean. We have this far dealt with a Euclidean space. By definition, this is one in which a Cartesian coordinate system can be set up.

We defined the metric tensor from the Cartesian definition of the length element and transformed it into general coordinates.

It is possible to proceed more abstractly and say that a Riemannian space is one in which there exists a metric

\[ ds^2 = g_{ij}dx^i dx^j \]

where the \( g_{ij} \) is a covariant second order tensor field and develop a vector and tensor analysis from this starting point. As Euclidean space is a Riemannian space, then this would encompass the concepts we already have.

The concept of more abstract spaces can be useful when studying two-dimensional surface flows and is used when studying space-time relativity, however this is beyond the scope of this analysis and would not be useful in depicting the concepts in this paper.

12. Conclusions. Generalized tensors allow us to transform between coordinate systems which are not necessarily linear and spaces which are not necessarily Euclidean.

The ideas of what we have learned about Cartesian tensors still apply, however considerations need to be made when in other coordinate systems.

The concept of generalized tensors do provide a good theoretical background for Cartesian tensors, Also, it is reassuring that we can do things with tensors in the general sense without major reformulation of the underlying mathematics.

In conclusion, generalized tensors are interesting from a theoretical standpoint, however they are not commonly used as Cartesian tensors are much more convenient.

REFERENCES