Math 514: Homework 1

Due, September 9, 2011

1. Find the dispersion relation for the propagation of plane harmonic waves in the systems

   (a) $i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=0$
   (the free space Schrodinger equation, which arises in quantum mechanics),

   (b) $\frac{\partial \phi}{\partial t}+\alpha\frac{\partial \phi}{\partial x}=\beta\frac{\partial^3 \phi}{\partial x^3}-\gamma\frac{\partial^5 \phi}{\partial x^5}$
   (the linearized fifth order KdV equation).

   In each case, determines both the phase and group velocities. For each equation, determine $\phi(x,t)$ if $\phi(x,0)=\delta(x)$. To impose the initial condition, you will need to 'sum' over k, using

   $$\phi(x,t)=\int_{-\infty}^{\infty}A(k)e^{i(kx-\omega t)}dk.$$

   For part (b) only, use the method of stationary phase to find an approximation to the solution for $t>>1$ with $v=x/t=O(1)$, v constant

2. Consider the equation

   $$\psi_{xxtt}=-g\alpha\psi_{xx},$$

   where g and α are positive constants. This equation arises as a model for internal waves in a fluid. Calculate the dispersion relation, and hence determine the solution when

   $$\psi(x,0)=\begin{cases}0, & |x|>1\\ 1, & |x|\le 1\end{cases},$$

   $$\psi_t(x,0)=0.$$

3. An exercise in separation of variables. Consider the finite string that we studies in class. When $t=0$ the displacement and velocity of the string are given by

   $$y(x,0)=x(L-x),$$

   (a) $\frac{\partial y}{\partial t}(x,0)=0,$

$$y(x,0) = 0,$$

(b) $$\frac{\partial y}{\partial t}(x,0) = 1, \quad |x - d| \le a,$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \quad 0 \le x \le d - a, \quad d + a \le x \le L.$$

In each case determine the displacement of the string for $t > 0$ and the amount of energy stored in the string. Note that case (b) corresponds to a string struck by a hammer with width $2a$ and unit velocity at the point $x = d$.

4. Consider the infinite string that we studied in class. When $t = 0$ the displacement and velocity of the string are

$$y(x,0) = \sin(x), \quad -\pi \le x \le \pi,$$

(a) $$y(x,0) = 0, \quad x > \pi, \quad x < -\pi,$$

$$\frac{\partial y}{\partial t}(x,0) = 0,$$

$$y(x,0) = 0,$$

(b) $$\frac{\partial y}{\partial t}(x,0) = 0, \quad |x| > a,$$

$$\frac{\partial y}{\partial t}(x,0) = 1, \quad |x| \le a.$$

In each case, use d'Alembert's solution to find the displacement for $t > 0$. Sketch the solutions at various times and describe what is happening to the string. The particular solution that satisfies the initial conditions

$$y(x,0) = y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = v_0(x)$$

is given by

$$y(x,t) = \frac{1}{2}\{y_0(x - ct) + y_0(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)ds.$$

5. Any infinitely long stretched string has density ρ for $x < 0$ and 4ρ for $x > 0$. When $t = 0$ the string is stationary and has displacement

$$y(x,0) = \begin{cases} 1, & -2 < x < -1, \\ 0, & otherwise. \end{cases}$$

Determine and sketch the displacement of the string for $t > 0$.

6. **Prove:** $$\frac{d}{dx}\int_0^{a(x)} f(x,y)dy = \int_0^a \frac{\partial f}{\partial x}dy + a'(x)f(x,a).$$