Math 514: Homework 7

(Due: 9 December, 2011)

1. **Elastic equations of motion.**
    a. Write the elastic equations of motion, presented in class, in vector form. Hint: You may require the use of a matrix or two.
    b. Given the elastic equations of motion and a time-harmonic source, show that by taking the shear modulus $\mu = 0$, the scalar Helmholtz equation can be derived.

2. The Helmholtz equation can be factored into a product of incoming and outgoing wave operators so that only one-half of the wave equation can be solved. In 2-D polar, the elliptic Helmholtz equation can be written in the form

$$\left(\frac{\partial}{\partial r} - ik_0\sqrt{1+X}\right)\left(\frac{\partial}{\partial r} + ik_0\sqrt{1+X}\right)p + \left[\frac{\partial}{\partial r}, ik_0\sqrt{1+X}\right]p = 0, \quad X = \frac{1}{k_0^2}\left(\frac{\partial^2}{\partial z^2} + k^2 - k_0^2\right)$$

where $k_0 = \omega / c_0$ is a reference wave number, c_0 is a representative sound speed, and

$$\left[L, M\right] = LMp - MLp$$

is the commutator of the operators L and M. For range-independent media, the two operators commute and the commutator is exactly zero. Retaining only the outgoing wave, the resulting fluid medium parabolic wave equation is:

$$\frac{\partial p}{\partial r} = ik_0\sqrt{1 + X}\,p.$$

This factoring technique is known as the parabolic approximation because it reduces the elliptic wave equation into a product of parabolic equations. Consider the elastic equations of motion in 2-D, ignoring the azimuthal coordinate (=> $v = 0$), and write the elastic equations of motion in the form,

$$L\frac{\partial^2}{\partial r^2}\overline{u} + M\overline{u} = N\frac{\partial}{\partial r}\overline{u}.$$

where the matrices L, M, and N contain only depth operators.

   a. Write the solution to this system using the quadratic equation.
   b. It would be pretty, nice, and easier to factor if $N = 0$. Show that by choosing the dependent variable $\overline{u} = (u, w)$, that this does not make $N = 0$.
   c. Consider the dependent variable $\overline{u} = (u_r, w)$, where $u_r = \partial u / \partial r$. By taking the derivative with respect to r of the first equation of motion, along with the second equation of motion, making use of the u_r definition, derive an elastic parabolic equation:

$$\frac{\partial}{\partial r}\begin{pmatrix} u_r \\ w \end{pmatrix} = ik_0\sqrt{I + X}\begin{pmatrix} u_r \\ w \end{pmatrix}, \quad X = \frac{1}{k_0^2}\left(L^{-1}M - k_0^2 I\right).$$