pg. 253 – 254

Chapter 6 Review : #11, 15

11. $(x-1)y'' + 3y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$xy'' - y'' + 3y = 0$$

$$x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 3c_n x^n = 0$$

Reindex: $$\sum_{n=3}^{\infty} (n-1)(n-2)c_{n-1} x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} 3c_{n-2} x^{n-2} = 0$$

$$\sum_{n=3}^{\infty} (n-1)(n-2)c_{n-1} x^{n-2} - 2(1)c_2 x^0 - \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}$$

$$+ 3c_0 x^0 + \sum_{n=3}^{\infty} 3c_{n-2} x^{n-2} = 0$$

$$(-2c_2 + 3c_0) + \sum_{n=3}^{\infty}\left[(n-1)(n-2)c_{n-1} - n(n-1)c_n + 3c_{n-2}\right]x^{n-2} = 0$$

$-2c_2 + 3c_0 = 0$ $\qquad$ $(n-1)(n-2)c_{n-1} - n(n-1)c_n + 3c_{n-2} = 0$

$\underline{c_2 = \frac{3c_0}{2}}$ $\qquad$ $\boxed{c_n = \dfrac{(n-1)(n-2)c_{n-1} + 3c_{n-2}}{n(n-1)}, \quad n \geq 3}$

Recurrence Relation

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$n=0$: $c_0$, $n=1$: $c_1$

$n=2$: $c_2 = \frac{3c_0}{2}$

$n=3$: $c_3 = \frac{(2)(1)c_2 + 3c_1}{(3)(2)} = \frac{3c_0 + 3c_1}{(3)(2)} = \frac{c_0}{2} + \frac{c_1}{2}$

$n=4$: $c_4 = \frac{(3)(2)c_3 + 3c_2}{4(3)} = \frac{3c_0 + 3c_1 + \frac{9}{2}c_0}{(4)(3)} = \frac{15}{24}c_0 + \frac{c_1}{4} = \frac{5}{8}c_0 + \frac{c_1}{4}$

$$y = c_0 + c_1 x + \frac{3c_0}{2}x^2 + \left(\frac{c_0}{2} + \frac{c_1}{2}\right)x^3 + \left(\frac{5c_0}{8} + \frac{c_1}{4}\right)x^4 + \ldots$$

11. (cont.)

$$\boxed{\begin{aligned} y = {} & c_0\left[1 + \tfrac{3}{2}x^2 + \tfrac{1}{2}x^3 + \tfrac{5}{8}x^4 + \ldots\right] \\ & + c_1\left[x + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^4 + \ldots\right] \end{aligned}}$$

$$\boxed{\begin{aligned} y_1 &= c_0\left[1 + \tfrac{3}{2}x^2 + \tfrac{1}{2}x^3 + \tfrac{5}{8}x^4 + \ldots\right] \\ y_2 &= c_1\left[x + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^4 + \ldots\right] \end{aligned}}$$

15: $y'' + xy' + 2y = 0$, $y(0) = 3$, $y'(0) = -2$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

Reindex $$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} (n-2)c_{n-2} x^{n-2} + \sum_{n=2}^{\infty} 2c_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left[n(n-1)c_n + (n-2)c_{n-2} + 2c_{n-2}\right] x^{n-2} = 0$$

$$n(n-1)c_n + (n-2)c_{n-2} + 2c_{n-2} = 0$$

$$n(n-1)c_n + n c_{n-2} = 0$$

$$c_n = \frac{-\cancel{n}c_{n-2}}{\cancel{n}(n-1)}$$

$$\boxed{c_n = \frac{-c_{n-2}}{n-1}, \quad n \geq 2}$$

Recurrence Relation.

$$y=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

$n=0$: $c_0$ , $n=1$: $c_1$

$n=2$: $c_2 = \frac{-c_0}{1}$

$n=3$: $c_3 = -\frac{c_1}{2}$

$n=4$: $c_4 = -\frac{c_2}{3} = \frac{c_0}{3}$

$n=5$: $c_5 = \frac{-c_3}{4} = \frac{c_1}{8}$

$$y = c_0 + c_1 x - c_0 x^2 - \frac{c_1}{2} x^3 + \frac{c_0}{3} x^4 + \frac{c_1}{8} x^5 + \ldots$$

$$y = c_0\left[1 - x^2 + \frac{x^4}{3} + \ldots\right] + c_1\left[x - \frac{x^3}{2} + \frac{x^5}{8} + \ldots\right]$$

$$y' = c_1 - 2c_0 x - \frac{3c_1}{2}x^2 + \frac{4c_0}{3}x^3 + \frac{5c_1}{8}x^4 + \ldots$$

$y(0) = c_0 = 3$ , $y'(0) = c_1 = -2$

$$\boxed{y = 3\left[1 - x^2 + \frac{x^4}{3} + \ldots\right] - 2\left[x - \frac{x^3}{2} + \frac{x^5}{8} + \ldots\right]}$$

# Chapter 7 Review, pg. 300–302 

\# 2, 6, 7, 9, 11, 12, 13, 15, 17, 19, 29, 33, 35, 36, 37

2. $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_2^4 e^{-st}(1)\,dt$$
$$= -\frac{1}{s} e^{-st}\Big|_2^4$$
$$= \boxed{-\frac{1}{s}e^{-4s} + \frac{1}{s}e^{-2s}}$$

6. $\boxed{\text{False}}$

$F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$

$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$ #16

$\ne fg$

7. $\boxed{\mathcal{L}\{e^{-7t}\} = \frac{1}{s+7}}$, #3, $a = -7$

9. $\boxed{\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}}$, #5, $b = 2$

11. $\boxed{\mathcal{L}\{t\sin(2t)\} = \frac{2(2)s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}}$, #14, $b = 2$

12. $\mathcal{L}\{\sin(2t)\, \mathcal{U}(t-\pi)\}$

#19, $a = \pi$, $g(t) = \sin(2t)$, $g(t+\pi) = \sin(2(t+\pi))$

$= \sin(2t + 2\pi)$

$= \sin(2t)\cos(2\pi) + \cos(2t)\sin(2\pi)$ Trig. Iden.

$= \sin(2t)$

$$\mathcal{L}\{\sin(2t)\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(2t)\}$$
$$= e^{-\pi s}\left(\frac{2}{s^2+4}\right)$$
$$= \boxed{\frac{2e^{-\pi s}}{s^2+4}}$$

13. $\mathcal{L}^{-1}\left\{\frac{20}{s^6}\right\} = \frac{20}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\}$

$= \boxed{\frac{1}{6}t^5}$, #4, $n=5$

15. $\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s-5)^3}\right\}$

$= \boxed{\frac{1}{2}t^2 e^{5t}}$, #11, $n=2$, $a=5$

17. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-10s+29}\right\}$

$s^2-10s+29$: $b^2-4ac = 100-4(29) < 0$, No real factors

$\underline{CS}$: $s^2-10s+29 = s^2-10s+\left(\frac{-10}{2}\right)^2+29-\left(\frac{-10}{2}\right)^2$

$= (s-5)^2+4$

$\frac{s}{s^2-10s+29} = \frac{s}{(s-5)^2+4} = \frac{s-5}{(s-5)^2+4} + \frac{5}{2}\left(\frac{2}{(s-5)^2+4}\right)$

#13, $a=5$, $b=2$ #12

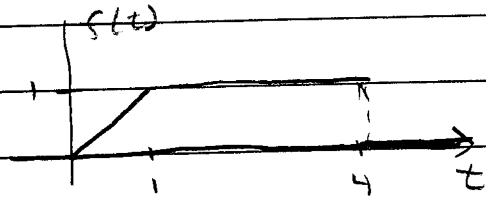
$$\boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2-10s+29}\right\} = e^{5t}\cos(2t) + \frac{5}{2}e^{5t}\sin(2t)}$$

19. $\mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{s}{s^2+\pi^2}+\frac{\pi}{s^2+\pi^2}\right)\right\}$, #18, $a=1$

#6, $b=\pi$ #5

$= \boxed{\mathcal{U}(t-1)\left(\cos(\pi(t-1))+\sin(\pi(t-1))\right)}$

29.




$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 4 \\ 0, & t \ge 4 \end{cases}$

$f(t) = t + \mathcal{U}(t-1)(1-t) + \mathcal{U}(t-4)(0-1)$

$f(t) = t + \mathcal{U}(t-1)(1-t) - \mathcal{U}(t-4)$

#4, $n=1$ #19, $a=1$, $g(t)=1-t$ #17, $a=4$

$g(t+1) = -t$

$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{-t\} - \frac{e^{-4s}}{s}$

$\boxed{F(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2}\right) - \frac{e^{-4s}}{s}}$

29. (cont.)

$$\mathcal{L}\{e^t f(t)\} = F(s-1) \quad \#9, a=1$$

$$= \boxed{\frac{1}{(s-1)^2} - e^{-(s-1)}\left(\frac{1}{(s-1)^2}\right) - e^{-4(s-1)}\left(\frac{1}{s-1}\right)}$$

33. $y'' - 2y' + y = e^t, \quad y(0) = 0, \quad y'(0) = 5$

$$\underset{\#1(b)}{\mathcal{L}\{y''\}} - 2\underset{\#1(a)}{\mathcal{L}\{y'\}} + \mathcal{L}\{y\} = \underset{\#3,\, a=1}{\mathcal{L}\{e^t\}}$$

$$s^2 Y(s) - s(0) - 5 - 2(sY(s) - 0) + Y(s) = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)Y(s) = 5 + \frac{1}{(s-1)}$$

$$Y(s) = \frac{5}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$Y(s) = 5\underset{\#11,\, n=1}{\left(\frac{1}{(s-1)^2}\right)} + \frac{1}{2}\underset{n=2}{\left(\frac{2}{(s-1)^3}\right)}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = 5e^t t + \frac{1}{2}e^t t^2}$$

35. $y'' + 6y' + 5y = t - t\,\mathcal{U}(t-2), \quad y(0) = 1, \quad y'(0) = 0$

$$\underset{\#1(b)}{\mathcal{L}\{y''\}} + 6\underset{\#1(a)}{\mathcal{L}\{y'\}} + 5\mathcal{L}\{y\} = \underset{\#4,\, n=1}{\mathcal{L}\{t\}} - \underset{\#19,\, a=2}{\mathcal{L}\{t\,\mathcal{U}(t-2)\}}$$

$g(t) = t$

$g(t+2) = t+2$

$$s^2 Y(s) - s(1) - 0 + 6(sY(s) - 1) + 5Y(s) = \frac{1}{s^2} - e^{-2s}\mathcal{L}\{t+2\}$$

$$(s^2 + 6s + 5)Y(s) = s + 6 + \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

$$Y(s) = \frac{s+6}{(s+1)(s+5)} + \frac{1}{s^2(s+1)(s+5)} - e^{-2s}\left(\frac{1}{s^2(s+1)(s+5)} + \frac{2}{s(s+1)(s+5)}\right)$$

PF: $\dfrac{s+6}{(s+1)(s+5)} = \dfrac{A}{s+1} + \dfrac{B}{s+5}$

$$A(s+5) + B(s+1) = s+6$$

$s = -5: \quad -4B = 1, \quad B = -\frac{1}{4}$

$s = -1: \quad 4A = 5, \quad A = \frac{5}{4}$

35. (cont.)

PF: $\frac{1}{s^2(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+5}$

$As(s+1)(s+5) + B(s+1)(s+5) + Cs^2(s+5) + Ds^2(s+1) = 1$

$A(s^3+6s^2+5s) + B(s^2+6s+5) + C(s^3+5s^2) + D(s^3+s^2) = 1$

$(A+C+D)s^3 + (6A+B+5C+D)s^2 + (5A+6B)s$

$+5B = 1$

$A+C+D=0 \qquad 6A+B+5C+D=0 \qquad 5A+6B=0 \qquad 5B=1$

$C=-D-A \qquad \frac{-36}{25}+\frac{5}{25}-5D+\frac{30}{25}+D=0 \qquad A=\frac{-6B}{5} \qquad B=\frac{1}{5}$

$C=-D+\frac{6}{25} \qquad -4D=\frac{1}{25} \qquad A=\frac{-6}{25}$

$C=\frac{1}{100}+\frac{24}{100} \qquad D=\frac{-1}{100}$

$C=\frac{25}{100}=\frac{1}{4}$

PF: $\frac{2}{s(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+5}$

$A(s+1)(s+5) + Bs(s+5) + Cs(s+1) = 2$

$A(s^2+6s+5) + B(s^2+5s) + C(s^2+s) = 2$

$(A+B+C)s^2 + (6A+5B+C)s + 5A = 2$

$A+B+C=0 \qquad 6A+5B+C=0 \qquad 5A=2$

$C=-A-B \qquad \frac{12}{5}+5B-\frac{2}{5}-B=0 \qquad A=\frac{2}{5}$

$C=-\frac{2}{5}-B \qquad 4B=-2$

$C=\frac{-4}{10}+\frac{5}{10} \qquad B=-\frac{1}{2}$

$C=\frac{1}{10}$

$\#3, a=-1 \qquad \#3, a=-5 \qquad \#2 \qquad \#4, n=1$

$Y(s) = \frac{5}{4}\left(\frac{1}{s+1}\right) - \frac{1}{4}\left(\frac{1}{s+5}\right) + \frac{-6}{25}\left(\frac{1}{s}\right) + \frac{1}{5}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s+1}\right)$

$-\frac{1}{100}\left(\frac{1}{s+5}\right) - e^{-2s}\left[-\frac{6}{25}\left(\frac{1}{s}\right) + \frac{1}{5}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s+1}\right) - \frac{1}{100}\left(\frac{1}{s+5}\right)\right.$

$\left. + \frac{2}{5}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{10}\left(\frac{1}{s+5}\right)\right]$

$\#18, a=2$

$$\boxed{\begin{aligned} y(t) &= \frac{3}{2}e^{-t} - \frac{13}{50}e^{-5t} - \frac{6}{25} + \frac{1}{5}t \\ &- u(t-2)\left[+\frac{4}{25} + \frac{1}{5}(t-2) - \frac{1}{4}e^{-(t-2)} + \frac{9}{100}e^{-5(t-2)}\right] \end{aligned}}$$

36. $y' - 5y = f(t)$, $y(0) = 1$

$$f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

$$f(t) = t^2 + U(t-1)(0 - t^2)$$

$$y' - 5y = t^2 - U(t-1)\,t^2$$

$$\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{t^2\} - \mathcal{L}\{U(t-1)t^2\}$$

#1(a) #4, $n=2$ #19, $a=1$, $g(t) = t^2$

$g(t+1) = (t+1)^2 = t^2 + 2t + 1$

$$sY(s) - 1 - 5Y(s) = \frac{2}{s^3} - e^{-s}\mathcal{L}\{t^2 + 2t + 1\}$$

#4, $n=2$, $n=1$ #2

$$(s-5)Y(s) = 1 + \frac{2}{s^3} - e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]$$

$$Y(s) = \frac{1}{s-5} + \frac{2}{s^3(s-5)} - e^{-s}\left[\frac{2}{s^3(s-5)} + \frac{2}{s^2(s-5)} + \frac{1}{s(s-5)}\right]$$

$$Y(s) = \frac{1}{s-5} + \frac{2}{s^3(s-5)} - e^{-s}\left[\frac{2 + 2s + s^2}{s^3(s-5)}\right]$$

PF: $\frac{2}{s^3(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-5}$

$As^2(s-5) + Bs(s-5) + C(s-5) + Ds^3 = 2$

$(A+D)s^3 + (-5A+B)s^2 + (-5B+C)s - 5C = 2$

$A + D = 0$ $-5A + B = 0$ $-5B + C = 0$ $-5C = 2$

$D = +\frac{2}{125}$ $A = \frac{-2}{125}$ $B = -\frac{2}{25}$ $C = \frac{-2}{5}$

PF: $\frac{2 + 2s + s^2}{s^3(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-5}$

$As^2(s-5) + Bs(s-5) + C(s-5) + Ds^3 = 2 + 2s + s^2$

$(A+D)s^3 + (-5A+B)s^2 + (-5B+C)s - 5C = 2$

$A + D = 0$ $-5A + B = 1$ $-5B + C = 2$ $-5C = 2$

$D = +\frac{37}{125}$ $A = \frac{1-B}{-5}$ $B = \frac{2-C}{-5}$ $C = \frac{-2}{5}$

$A = \frac{-37}{125}$ $B = \frac{-12}{25}$

$$Y(s) = \frac{1}{s+5} - \frac{2}{125}\left(\frac{1}{s}\right) - \frac{2}{25}\left(\frac{1}{s^2}\right) - \frac{2}{5(2)}\left(\frac{1(2)}{s^3}\right) + \frac{2}{125}\left(\frac{1}{s-5}\right)$$

$$-e^{-s}\left[\frac{-37}{125}\left(\frac{1}{s}\right) - \frac{12}{25}\left(\frac{1}{s^2}\right) - \frac{2}{5(2)}\left(\frac{1(2)}{s^3}\right) + \frac{37}{125}\left(\frac{1}{s-5}\right)\right]$$

#18, $a=1$ #2 #4, $n=1$ #4, $n=2$ #3, $a=+5$

36. (cont.)

$$\boxed{y(t) = \frac{127}{125}e^{5t} - \frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 - u(t-1)\left[\frac{-37}{125} - \frac{12}{25}(t-1) - \frac{1}{5}(t-1)^2 + \frac{37}{125}e^{5(t-1)}\right]}$$

37. $y'(t) = \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau \quad , y(0)=1$

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{y(t) * \cos t\}$$

#1(a) #5, b=1 #16

$$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\left(\frac{s}{s^2+1}\right)$$

$$\left(s - \frac{s}{s^2+1}\right)Y(s) = 1 + \frac{s}{s^2+1}$$

$$\left(\frac{s(s^2+1)-s}{s^2+1}\right)Y(s) = 1 + \frac{s}{s^2+1}$$

$$Y(s) = \left(1 + \frac{s}{s^2+1}\right)\left(\frac{s^2+1}{s^3}\right)$$

$$Y(s) = \frac{s^2+1}{s^3} + \frac{s}{s^3} = \frac{1}{s} + \frac{1}{2}\frac{1(2)}{s^3} + \frac{1}{s^2}$$

#2 #4, n=2, n=1

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 + \frac{1}{2}t^2 + t}$$

Chapter 8 Review, pg. 337-338

\# 1, 2, 4, 5, 6, 7, 8

1. $\vec{X} = k\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}4k\\5k\end{pmatrix}$

$\vec{X}' = \begin{pmatrix}1&4\\2&-1\end{pmatrix}\vec{X} - \begin{pmatrix}8\\1\end{pmatrix}$

$\vec{X}' = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}1&4\\2&-1\end{pmatrix}\vec{X} - \begin{pmatrix}8\\1\end{pmatrix} = \begin{pmatrix}1&4\\2&-1\end{pmatrix}\begin{pmatrix}4k\\5k\end{pmatrix} - \begin{pmatrix}8\\1\end{pmatrix}$

$= \begin{pmatrix}4k+20k\\8k-5k\end{pmatrix} - \begin{pmatrix}8\\1\end{pmatrix}$

$= \begin{pmatrix}24k-8\\3k-1\end{pmatrix}$

$\vec{X}' = \begin{pmatrix}1&4\\2&-1\end{pmatrix}\vec{X} - \begin{pmatrix}8\\1\end{pmatrix}$

$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}24k-8\\3k-1\end{pmatrix}$

$24k-8=0 \qquad 3k-1=0$

$k = \frac{8}{24} = \frac{1}{3} \qquad k = \frac{1}{3}$

$\boxed{k = \frac{1}{3}}$

2. $\vec{X} = c_1 e^{-9t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{7t}\begin{pmatrix}5\\3\end{pmatrix}$

$\vec{X}' = \begin{pmatrix}1&10\\6&-3\end{pmatrix}\vec{X}, \quad \vec{X}(0) = \begin{pmatrix}2\\0\end{pmatrix}$

$\vec{X}(0) = c_1\begin{pmatrix}-1\\1\end{pmatrix} + c_2\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$-c_1 + 5c_2 = 2 \qquad c_1 + 3c_2 = 0$

$3c_2 + 5c_2 = 2 \qquad c_1 = -3c_2$

$8c_2 = 2 \qquad \boxed{c_1 = -\frac{3}{4}}$

$\boxed{c_2 = \frac{1}{4}}$

$\vec{X}(t) = \begin{pmatrix}\frac{3}{4}e^{-9t} + \frac{5}{4}e^{7t}\\ -\frac{3}{4}e^{-9t} + \frac{3}{4}e^{7t}\end{pmatrix}$

4. $\vec{X}' = A\vec{X}$

$\lambda_1 = 1+2i, \quad \vec{K}_1 = \vec{V}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$e^{\lambda_1 t}\vec{V}_1 = e^{(1+2i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 1 \\ i \end{pmatrix}$

$= e^t \begin{pmatrix} \cos(2t) + i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{pmatrix}$

$= e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + i e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$

$$\boxed{\vec{X}(t) = C_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}}$$

5. $\frac{dx}{dt} = 2x + y$

$\frac{dy}{dt} = -x$

$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$

$(2-\lambda)(-\lambda) - (1)(-1) = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$, repeated

$\lambda_1 = 1 : (A - \lambda_1 I)\vec{V}_1 = \vec{0}$

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 + v_2 = 0$

$v_2 = -v_1$

$\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix}, \ v_1 = 1, \ \vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - \lambda_1 I)\vec{V}_2 = \vec{V}_1$

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$v_1 + v_2 = 1$

$v_2 = 1 - v_1$

$\begin{pmatrix} v_1 \\ 1 - v_1 \end{pmatrix}, \ v_1 = 0, \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{V}_2$

$$\boxed{\vec{X}(t) = C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\left[te^t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}$$

6. $\frac{dx}{dt} = -4x + 2y$

$\frac{dy}{dt} = 2x - 4y$

$A = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{pmatrix}$

$\det \begin{pmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{pmatrix} = 0$

$(-4-\lambda)(-4-\lambda) - (2)(2) = 0$

$\lambda^2 + 8\lambda + 12 = 0$

$(\lambda+6)(\lambda+2) = 0$

$\lambda_1 = -6, \quad \lambda_2 = -2$

$\lambda_1 = -6 : (A - \lambda_1 I)\vec{v_1} = \vec{0}$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2v_1 + 2v_2 = 0$

$v_2 = -v_1$

$\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix}, \; v_1 = 1, \; \vec{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -2 : (A - \lambda_2 I)\vec{v_2} = \vec{0}$

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2v_1 + 2v_2 = 0$

$v_2 = v_1$

$\begin{pmatrix} v_1 \\ v_1 \end{pmatrix}, \; v_1 = 1, \; \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}(t) = c_1 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

7. $\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\vec{x}$

$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix}$

$\det\begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = 0$

$(1-\lambda)(1-\lambda) - (2)(-2) = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$

$\lambda = 1 \pm 2i$

$\lambda_1 = 1 + 2i : (A - \lambda_1 I)\vec{v_1} = \vec{0}$

$\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2i\,v_1 + 2v_2 = 0$

$v_2 = i v_1$

$\begin{pmatrix} v_1 \\ i v_1 \end{pmatrix}, \; v_1 = 1$

$\vec{v_1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

7. (cont.)

$$e^{\lambda_1 t}\vec{v}_1 = e^{(1+2i)t}\begin{pmatrix}1\\i\end{pmatrix}$$
$$= e^{t}\begin{pmatrix}\cos(2t)\\-\sin(2t)\end{pmatrix} + i\,e^{t}\begin{pmatrix}\sin(2t)\\\cos(2t)\end{pmatrix}, \text{ See \#4}$$

$$\boxed{\vec{X}(t) = C_1 e^{t}\begin{pmatrix}\cos(2t)\\-\sin(2t)\end{pmatrix} + C_2 e^{t}\begin{pmatrix}\sin(2t)\\\cos(2t)\end{pmatrix}}$$

8. $\vec{X}' = \begin{pmatrix}-2 & 5\\-2 & 4\end{pmatrix}\vec{X}$

$A = \begin{pmatrix}-2 & 5\\-2 & 4\end{pmatrix}$, $A-\lambda I = \begin{pmatrix}-2-\lambda & 5\\-2 & 4-\lambda\end{pmatrix}$

$$\det\begin{pmatrix}-2-\lambda & 5\\-2 & 4-\lambda\end{pmatrix} = 0$$
$$(-2-\lambda)(4-\lambda) - (-2)(5) = 0$$
$$\lambda^2 - 2\lambda + 2 = 0$$
$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$$

$\lambda_1 = 1+i$ : $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix}-3-i & 5\\-2 & 3-i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$
$$(-3-i)v_1 + 5v_2 = 0$$
$$v_2 = \frac{(3+i)v_1}{5}$$
$$\begin{pmatrix}v_1\\\frac{(3+i)v_1}{5}\end{pmatrix},\ v_1 = 5,\ \vec{v}_1 = \begin{pmatrix}5\\3+i\end{pmatrix}$$

$$e^{\lambda_1 t}\vec{v}_1 = e^{(1+i)t}\begin{pmatrix}5\\3+i\end{pmatrix} = e^{t}(\cos t + i\sin t)\begin{pmatrix}5\\3+i\end{pmatrix}$$
$$= e^{t}\begin{pmatrix}5\cos t + 5i\sin t\\3\cos t + i\cos t + 3i\sin t - \sin t\end{pmatrix}$$
$$= e^{t}\begin{pmatrix}5\cos t\\3\cos t - \sin t\end{pmatrix} + i\,e^{t}\begin{pmatrix}5\sin t\\\cos t + 3\sin t\end{pmatrix}$$

$$\boxed{\vec{X}(t) = C_1 e^{t}\begin{pmatrix}5\cos t\\3\cos t - \sin t\end{pmatrix} + C_2 e^{t}\begin{pmatrix}5\sin t\\\cos t + 3\sin t\end{pmatrix}}$$