Chapter 1.7.1

GENERAL THEORY OF ELASTIC WAVE SCATTERING

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PART 1 SCATTERING OF WAVES BY MACROSCOPIC TARGETS

Topic 1.7 Elastodynamic Wave (Elastic) Scattering: Theory

Contents

§ 1. Introduction

The theory of elastodynamic wave scattering is based on two foundations; continuum mechanics of elastic media and the general principles of scattering theory. In this chapter, the basic elements of the theory of elastodynamic scattering theory are presented. In later chapters this is applied to scattering of body waves (Chapter 1.7.2), scattering of surface waves (Chapter 1.7.3), seismic imaging (Chapter 1.8.1) and nondestructive testing (Chapter 1.8.2).

There is a wide body of literature on elastodynamic wave scattering. In their treatise of theoretical physics, Morse and Feshbach (1953a,b) include several aspects of continuum mechanics and elastic wave propagation. The foundations of continuum mechanics are described in great detail by Malvern (1969). A comprehensive overview of scattering of acoustic, electromagnetic and elastic waves is presented in a unified way by de Hoop (1995). The propagation of elastic waves forms the foundation of seismology. The required theory and application to seismological problems is presented in great detail by Aki and Richards (1980) Hudson (1980) and by Ben-Menahem and Singh (1981).

§ 2. Principles of Elasticity

In this section an outline is given of the principles of elasticity that are of relevance for elastic wave scattering. A more detailed description of the theory of elastic waves needed for the description of elastic wave scattering can be found in Aki and Richards (1981). In general the derivations are given in the frequency domain. The frequency-domain formulation and the time-domain formulation are related by the Fourier transform. For the temporal and spatial Fourier transforms the following convention is used:

$$f(t) = \frac{1}{2\pi}\int F(\omega)e^{-i\omega t}d\omega, \tag{1}$$

$$h(x) = \frac{1}{2\pi}\int H(k)e^{ikx}dk. \tag{2}$$

For the inverse transform the exponents have the opposite sign and the factor 2π is omitted.

A crucial element in the description of elastic waves is the stress tensor τ. Consider an infinitesimal surface with surface area dS with normal vector $\hat{\mathbf{n}}$. The force acting on this surface per unit surface area is called the *traction*; this quantity is given by

$$\mathbf{T} = \hat{\mathbf{n}} \cdot \tau. \tag{3}$$

The stress tensor is such a useful quantity because expression (3) makes it possible to compute the force acting on *any* surface once the stress tensor is known. The stress tensor is symmetric:

$$\tau_{ij} = \tau_{ji}. \tag{4}$$

This property follows from the requirement that angular momentum is conserved (Malvern, 1969; Goldstein, 1980).

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For a general medium the stress in the medium depends in a complicated way on the deformation of the medium. The deformation of the medium results from a displacement vector $\mathbf{u}(\mathbf{r})$ in the medium that varies with position, because a constant displacement vector $\mathbf{u}$ does not generate an internal deformation. For small elastic deformations the relation between the deformation and the resulting stress can be linearised; this linearisation is expressed by Hooke's law:

$$\tau_{ij} = c_{ijkl} e_{kl}. \tag{5}$$

Throughout this chapter the summation convention is used where one sums over repeated indices. In expression (5) the infinitesimal strain tensor e_{ij} is defined by

$$e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{6}$$

where the notation ∂_i stands for the derivative with respect to the x_i coordinate: $\partial_i f \equiv \partial f/\partial x_i$. The fourth-order tensor c_{ijkl}, called the *elasticity tensor*, generalises the concept of the spring constant of a simple spring to three-dimensional elastic media. One should keep in mind that Hooke's law (5) is a linearisation that breaks down in situations when the response of the medium is nonlinear. This is for example the case at fractures or in the earthquake source region, where nonlinear fracture behaviour governs the response of the medium.

It follows from the definition (6) that the strain tensor is symmetric:

$$e_{ij} = e_{ji}. \tag{7}$$

Because of this property, the elasticity tensor is symmetric when the last indices are exchanged: $c_{ijkl} = c_{ijlk}$. Since according to (4) the stress tensor is also symmetric, the elasticity tensor is also symmetric in the first two indices: $c_{ijkl} = c_{jikl}$. Energy considerations (Aki and Richards, 1980) imply that the elasticity tensor is also symmetric for exchange of the first and last pair of indices: $c_{ijkl} = c_{klij}$. The elasticity tensor therefore has the symmetry properties

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{8}$$

Inserting (6) in (5) and using the symmetry of the elasticity tensor in the first pair of indices, the relation between the stress and the displacement can be written as

$$\tau_{ij} = c_{ijkl} \partial_k u_l. \tag{9}$$

In general a tensor of rank 4 in three dimensions has 81 components. The symmetry relations (8) reduce the number of independent components of the elasticity tensor to the number of independent elements of a symmetric 6×6 matrix (Backus, 1970; Helbig, 1994), which means that the elasticity tensor has in three dimensions 21 independent elements. Of special importance is the case of an *isotropic* elastic medium. In such a medium the elasticity tensor does not depend on some preferred direction, but only on the Lamé parameters λ and μ

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{10}$$

where δ_{ij} is the Kronecker delta defined by $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. For the moment we consider a general elasticity tensor.

The force associated with the stress acting on a volume is given by $\oint \mathbf{T}\, dS$, where the surface integral is over the surface that bounds the volume. By the theorem of Gauss and expression (3), this force is given by $\int \nabla \cdot \tau \, dV$. This implies that the force generated by the internal stress per unit volume is given by the divergence of the stress tensor $(\nabla \cdot \tau)$. In addition to this force, forces such as gravity or a seismic source that are not resulting from the deformation may be operative. These forces per unit volume are denoted by the force vector $\mathbf{f}$. Since the mass per unit volume is given by the density ρ, Newton's law can be written as

$$\rho \ddot{\mathbf{u}} = \nabla \cdot \tau + \mathbf{f}. \tag{11}$$

It should be noted that this is the linearised form of Newton's law. Since we are dealing with a continuous medium the acceleration contains advective terms as well; hence, the left-hand side should be written as $\rho \partial \mathbf{v}/\partial t + \rho \mathbf{v} \cdot \nabla \mathbf{v}$, with $\mathbf{v}$ being the velocity vector. However, when the material velocity $\mathbf{v}$ is much smaller than the wave velocity, the advective terms can be ignored (Snieder, 2001). By inserting (9) in Newton's law, one obtains the wave equation for the displacement vector

$$\rho \ddot{u}_i = \partial_j \left(c_{ijkl} \partial_k u_l \right) + f_i. \tag{12}$$

In the frequency domain this corresponds to

$$\rho \omega^2 u_i + \partial_j (c_{ijkl} \partial_k u_l) = -f_i. \tag{13}$$

In a shorthand notation we will write this expression also as

$$\mathbf{L}\mathbf{u} = -\mathbf{f}, \tag{14}$$

where the operator $\mathbf{L}$ is defined by

$$L_{ij} = \rho \omega^2 \delta_{ij} + \partial_k c_{iklj} \partial_l. \tag{15}$$

The wave equation is a second-order differential equation for the displacement vector $\mathbf{u}$ that needs to be supplemented with boundary conditions at the surface of the medium. At a stress-free surface S, the tractions acting on the surface vanish:

$$\mathbf{T} = \hat{\mathbf{n}} \cdot \tau = 0 \quad \text{at } S. \tag{16}$$

This boundary condition is appropriate when the surface S forms the outer boundary of the elastic body that is surrounded by empty space. Ignoring the stress imposed on an elastic body imposed by the gas or fluid in which the body is possibly embedded, the tractions vanish at the surface. Using (3) and (9) this boundary condition can be expressed as a condition for the displacement at the free surface:

$$n_j c_{ijkl} \partial_k u_l = 0 \quad \text{at } S. \tag{17}$$

For a number of applications it is useful to define the power flux in an elastic medium. When a force **F** acts on a particle moving with velocity **v** (given by $\dot{\mathbf{u}}$), the power delivered by the force is given by $(\mathbf{F}\cdot\mathbf{v})$. Consider a surface element $d\mathbf{S}$, according to (3) the force acting on that surface element is given by $\mathbf{T}dS = \tau\cdot\hat{\mathbf{n}}dS = \tau\cdot d\mathbf{S}$. The power delivered at the surface is therefore given by $\mathbf{v}\cdot\tau\cdot d\mathbf{S}$. The vector $d\mathbf{S}$ points out of a volume; to obtain the power delivered to a volume we should consider the energy flow *into* the volume by adding a minus sign. The power flux $\mathbf{J}_P$ is therefore given by

$$\mathbf{J}_P = -(\tau\cdot\dot{\mathbf{u}}). \tag{18}$$

Expression (18) is the elastodynamic equivalent of the Pointing vector in electromagnetism. The power that flows through a surface is given by

$$P = \int\int \mathbf{J}_P\cdot\hat{\mathbf{n}}\, dS. \tag{19}$$

It can be shown (Ben-Menahem and Singh, 1981) that the energy density satisfies the conservation law

$$\frac{\partial E}{\partial t} + (\nabla\cdot\mathbf{J}_P) = \rho(\mathbf{f}\cdot\dot{\mathbf{u}}). \tag{20}$$

The power flux and energy density depend quadratically on the displacement because expression (18) contains the product of the displacement and the stress (which is a linear combination of partial derivatives of the displacement). One should therefore not simply replace the real time-domain quantities by the complex counterpart in the frequency domain. With the Fourier convention (1) in the frequency domain, the power flux averaged over one period is given by

$$\mathbf{J}_P = \frac{\omega}{2}\Im m(\tau\cdot\mathbf{u}^*), \tag{21}$$

where the asterisk denotes the complex conjugate.

§3. The Anatomy of a Seismogram

As an introduction to the different types of elastic wave propagation, we consider Fig. 1 in which the vertical component of the ground motion is shown after an earthquake at Jan-Mayen Island in the North Atlantic. The ground motion was recorded by one of the stations of the Network of Autonomously Recording Seismographs (NARS) that was placed in Naroch, Belarus. The earthquake takes place at time $t = 0$. In the seismogram impulsive waves arrive around 300 and 550 s. After this, a strong extended wave train arrives between 620 and 900 s. The impulsive arrivals are waves that have propagated deep in the Earth; these waves are called *body waves*. In contrast to these body waves, the wave train that arrives around 700 s corresponds to waves that are guided along the Earth's surface; for this reason such waves are called *surface waves*. The paths of propagation of these types of waves are sketched in Fig. 2. The surface wave propagates effectively in only two dimensions whereas the body waves propagate in three dimensions. For this reason the surface wave suffers less from geometrical spreading during the propagation and therefore has in general a larger amplitude than the body waves; this can be seen clearly in Fig. 1.

Elastodynamic waves are vector waves, with no restriction on the polarisation. Because there are three spatial dimensions, elastodynamic waves can have three different polarisations. In an isotropic medium, the body waves separate into a longitudinal wave and

Figure 1 Vertical component of the ground motion at a seismic station in Naroch (Belarus) after an earthquake at Jan-Mayen Island. This station, part of the Network of Autonomously Recording Seismographs (NARS), is operated by Utrecht University.

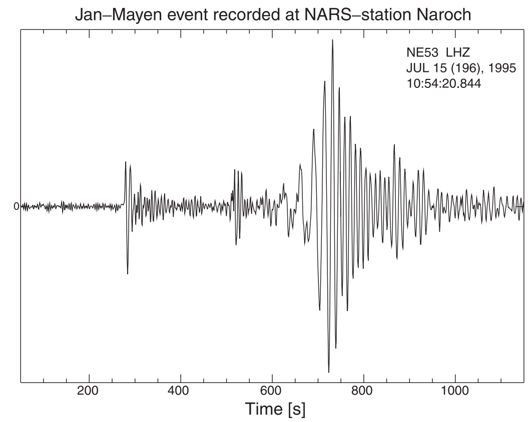
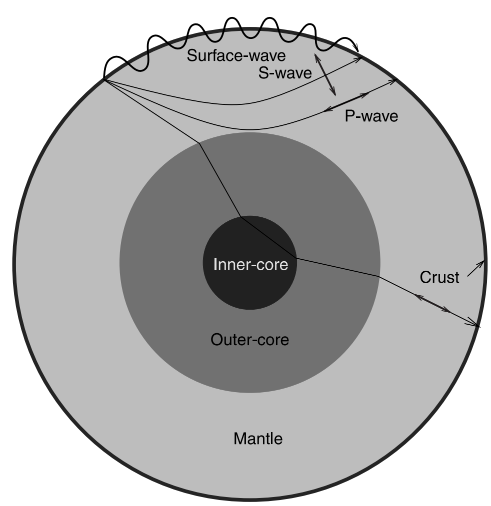


Figure 2 Sketch of the paths of propagation and polarisation of P-waves, S-waves and surface waves in the Earth.



transverse waves. The polarisation of these waves is indicated by the arrows in Fig. 2. Longitudinal waves propagate at a higher velocity than do the transverse waves. The nomenclature “P wave” and “S waves” historically denotes the first arriving (primary) and later arriving (secondary) body waves. The wave arriving around 300 s therefore is the P wave while the wave arriving around 550 s is the S wave. There are two directions of polarisation perpendicular to the path of propagation; for this reason there are two S waves in an isotropic medium that for reason of symmetry propagate at the same velocity. When the medium is anisotropic, there are still three polarisations of the body waves, but they do not correspond to longitudinal and transverse directions of polarisations. This is treated in more detail under Plane Wave Solutions.

The surface wave is guided along the Earth’s surface. For an isotropic Earth model where the elastic parameters and the density depend only on depth there are two types of surface wave: *Rayleigh waves* and *Love waves*. The Love wave is linearly polarised in the horizontal direction perpendicular to the propagation path, while Rayleigh waves have an ellipsoidal polarisation in the vertical plane through the path of propagation (Aki and Richards, 1980). Love and Rayleigh waves propagate in different modes, the number of modes depending in a nontrivial way on the Earth model and on frequency.

The velocity of wave propagation changes with depth in the Earth. Since the surface waves penetrate to different depths at different frequencies, the phase velocity of each surface wave mode in general depends on frequency. This implies that surface waves are in general dispersive. This can clearly be seen in the surface wave that arrives in Fig. 1; the early part of the wave train around 650 s has a frequency much lower than that of the later part of the wave train, which arrives around 800 s.

§ 4. The Representation Theorem

Green’s functions are very important in scattering problems. In elasticity theory the Green’s function gives the displacement generated by a point force in a certain direction. Since both the point force and the displacement are vectors with three components, the Green’s function is a 3×3 tensor. The ith component of a unit force in the n direction at location $\mathbf{r}'$ is given by $\delta_{in}\delta(\mathbf{r}-\mathbf{r}')$. The Green’s tensor $G_{in}(\mathbf{r},\mathbf{r}')$ is the displacement at location $\mathbf{r}$ in the i direction due to this point force

$$\rho\omega^2 G_{in}(\mathbf{r},\mathbf{r}') + \partial_j(c_{ijkl}\partial_k G_{ln}(\mathbf{r},\mathbf{r}')) = -\delta_{in}\delta(\mathbf{r}-\mathbf{r}'), \quad (22)$$

or, using the shorthand notation of Eq. (15),

$$L_{ij}G_{jn}(\mathbf{r},\mathbf{r}') = -\delta_{in}\delta(\mathbf{r}-\mathbf{r}'). \quad (23)$$

In order to derive the representation theorem for elastic waves, let us consider a displacement $\mathbf{u}^{(1)}$ that is generated by an excitation $\mathbf{f}^{(1)}$, and a displacement $\mathbf{u}^{(2)}$ that is caused by an excitation $\mathbf{f}^{(2)}$:

$$\begin{aligned}\rho\omega^2 u_i^{(1)} + \partial_j\left(c_{ijkl}\partial_k u_l^{(1)}\right) &= -f_i^{(1)},\\ \rho\omega^2 u_i^{(2)} + \partial_j\left(c_{ijkl}\partial_k u_l^{(2)}\right) &= -f_i^{(2)}.\end{aligned} \quad (24)$$

Multiply the first equation by $u_i^{(2)}$ and the second by $u_i^{(1)}$, subtract the two expressions and integrate over a volume V:

$$\begin{aligned}&\int \left\{u_i^{(2)}\partial_j\left(c_{ijkl}\partial_k u_l^{(1)}\right) - u_i^{(1)}\partial_j\left(c_{ijkl}\partial_k u_l^{(2)}\right)\right\}dV\\ &= -\int \left\{f_i^{(1)}u_i^{(2)} - f_i^{(2)}u_i^{(1)}\right\}dV.\end{aligned} \quad (25)$$

The volume integrals on the left-hand side can be converted into surface integrals using the identities $\int v_i\partial_j(c_{ijkl}\partial_k u_l)dV = \int \partial_j(v_i c_{ijkl}\partial_k u_l)dV - \int(\partial_j v_i)c_{ijkl}(\partial_k u_l)dV = \oint n_j v_i c_{ijkl}\partial_k u_l \; dS - \int(\partial_j v_i)c_{ijkl}(\partial_k u_l)dV$, where Gauss’s theorem is used in the last equality. Applying this to both terms on the left-hand side of (25) gives

$$\begin{aligned}&\oint \left\{n_j u_i^{(2)} c_{ijkl}\partial_k u_l^{(1)} - n_j u_i^{(1)} c_{ijkl}\partial_k u_l^{(2)}\right\} dS\\ &-\int \left\{\left(\partial_j u_i^{(2)}\right) c_{ijkl}\left(\partial_k u_l^{(1)}\right) - \left(\partial_j u_i^{(1)}\right) c_{ijkl}\left(\partial_k u_l^{(2)}\right)\right\}dV\\ &= -\int \left\{f_i^{(1)}u_i^{(2)} - f_i^{(2)}u_i^{(1)}\right\}dV.\end{aligned} \quad (26)$$

Because of the symmetry properties of the elasticity tensor as shown in the last identity of (8), the terms in the second line vanish so that this expression reduces to

$$\begin{aligned}&\oint \left\{n_j u_i^{(2)} c_{ijkl}\partial_k u_l^{(1)} - n_j u_i^{(1)} c_{ijkl}\partial_k u_l^{(2)}\right\} dS\\ &= -\int \left\{f_i^{(1)}u_i^{(2)} - f_i^{(2)}u_i^{(1)}\right\}dV.\end{aligned} \quad (27)$$

This general expression holds for any volume V, which is not necessarily the complete extent of the elastic body through which the waves propagate.

We now specialise momentarily for the case wherein the Green’s function satisfies homogeneous boundary conditions on the bounding surface S. This means that either the Green’s function or its associated traction vanish on the boundary: $G_{in}(\mathbf{r},\mathbf{r}') = 0$ or $n_j c_{ijkl}\partial_k(G_{ln}) = 0$ when $\mathbf{r}$ is located on S. When point forces $f_i^{(1)}(\mathbf{r}) = \delta_{in}(\mathbf{r}-\mathbf{r}_1)$ and $f_i^{(2)}(\mathbf{r}) = \delta_{im}(\mathbf{r}-\mathbf{r}_2)$ are used to excite the wave fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, these wave fields are by definition equal to the Green’s tensors: $u_i^{(1)}(\mathbf{r}) = G_{in}(\mathbf{r},\mathbf{r}_1)$ and $u_i^{(2)}(\mathbf{r}) = G_{im}(\mathbf{r},\mathbf{r}_2)$. For this special excitation the following identity holds: $\int f_i^{(1)}u_i^{(2)}dV = \int \delta_{in}(\mathbf{r}-\mathbf{r}_1)G_{im}(\mathbf{r},\mathbf{r}_2)dV = G_{nm}(\mathbf{r}_1,\mathbf{r}_2)$. Inserting these results in (27) then gives the **reciprocity theorem:**

$$G_{nm}(\mathbf{r}_1,\mathbf{r}_2) = G_{mn}(\mathbf{r}_2,\mathbf{r}_1). \quad (28)$$

This theorem states that n component of the displacement at $\mathbf{r}_1$ due to a point source excitation in $\mathbf{r}_2$ in the m direction is identical to the m component of the displacement at $\mathbf{r}_2$ due to a point source excitation in $\mathbf{r}_1$ in the n direction. In other words, when the roles of source and receiver are exchanged, one obtains exactly the same wave field.

Note that in deriving the representation theorem the volume V can be any volume, and the wavefield solution can have any value at the surface S that bounds this volume. Let us now leave out the superscripts (1) in $\mathbf{u}^{(1)}$ and $\mathbf{f}^{(1)}$ and let the (2) solution be the Green's function for a unit point force at $\mathbf{r}'$ in the n direction. This means that $f_i^{(2)}(\mathbf{r}) = \delta_{in}(\mathbf{r}-\mathbf{r}')$, and the associated response is by definition given by the Green's tensor: $u_i^{(2)}(\mathbf{r}) = G_{in}(\mathbf{r},\mathbf{r}')$. Using this in expression (27) yields

$$u_n(\mathbf{r}') = \int G_{in}(\mathbf{r},\mathbf{r}')f_i(\mathbf{r})dV + \oint \{G_{in}(\mathbf{r},\mathbf{r}')n_j c_{ijkl}\partial_k u_l(\mathbf{r}) - u_i(\mathbf{r})n_j c_{ijkl}\partial_k G_{ln}(\mathbf{r},\mathbf{r}')\}dS. \tag{29}$$

The reciprocity theorem (28) can be applied to the Green's tensors on the right-hand side. Exchanging the coordinates $\mathbf{r} \leftrightarrow \mathbf{r}'$ in the resulting expression and interchanging the indices $i \leftrightarrow n$ then gives the **representation theorem:**

$$u_i(\mathbf{r}) = \int G_{in}(\mathbf{r},\mathbf{r}')f_n(\mathbf{r}')dV' + \oint \{G_{in}(\mathbf{r},\mathbf{r}')n_j c_{njkl}\partial'_k u_l(\mathbf{r}') - u_n(\mathbf{r}')n_j c_{njkl}\partial'_k G_{il}(\mathbf{r},\mathbf{r}')\}dS'. \tag{30}$$

Note that the integration and differentiation are now over the primed coordinates. When the exciting force $f_n(\mathbf{r}')$ is known within the volume and when the wave field $u_n(\mathbf{r}')$ and the associated traction $n_j c_{ijkl}\partial_k u_l(\mathbf{r}')$ are known on the surface S, one can compute the wave field everywhere within the volume V. This expression forms the basis of the elastic equivalent of the Kirchhoff integral.

§ 5. The Lippman–Schwinger Equation for Elastic Waves

In order to describe scattering it is necessary to define a reference medium in which an unperturbed wave propagates and a perturbation of the medium that acts as a secondary source that generates scattered waves. For this reason we divide the elasticity tensor $\mathbf{c}$ into a reference tensor $\mathbf{c}^{(0)}$ and a perturbation $\mathbf{c}^{(1)}$, and do the same for the density:

$$\begin{aligned} \mathbf{c}(\mathbf{r}) &= \mathbf{c}^{(0)}(\mathbf{r}) + \mathbf{c}^{(1)}(\mathbf{r}), \\ \rho(\mathbf{r}) &= \rho^{(0)}(\mathbf{r}) + \rho^{(1)}(\mathbf{r}). \end{aligned} \tag{31}$$

With this decomposition one can also separate the operator $\mathbf{L}$ defined in expression (15) into an operator $\mathbf{L}^{(0)}$ for the reference medium and an operator $\mathbf{L}^{(1)}$ associated with the perturbation of the medium. It should be noted that the reference medium is not necessarily a homogeneous medium. For the Earth, for example, an Earth model in which the properties depend only on depth is a natural reference model. Let $\mathbf{u}^{(0)}$ be the solution for the reference medium, and denote the Green's function for the reference medium by $G^{(0)}$; this means that in operator notation

$$L_{ij}^{(0)} G_{jn}^{(0)}(\mathbf{r},\mathbf{r}') = -\delta_{in}\delta(\mathbf{r}-\mathbf{r}'). \tag{32}$$

Let us now assume that at the surface of the medium the tractions vanish. This boundary condition holds both for the reference medium and for the perturbed medium.

The total wave field in the perturbed medium that is excited by a force $\mathbf{f}(\mathbf{r})$ is given by

$$\left(\mathbf{L}^{(0)} + \mathbf{L}^{(1)}\right)\mathbf{u}(\mathbf{r}) = -\mathbf{f}(\mathbf{r}), \tag{33}$$

or

$$\mathbf{L}^{(0)}\mathbf{u}(\mathbf{r}) = -\left(\mathbf{f}(\mathbf{r}) + \mathbf{L}^{(1)}\mathbf{u}(\mathbf{r})\right). \tag{34}$$

According to this expression the total wave field in the perturbed medium is identical to the wave field in the unperturbed medium that is generated by an effective force given by $\mathbf{f}(\mathbf{r})+\mathbf{L}^{(1)}\mathbf{u}(\mathbf{r})$. The representation theorem (30) for the reference medium can be applied to this result. Using that the effective force is given by $\mathbf{f}(\mathbf{r})+\mathbf{L}^{(1)}\mathbf{u}(\mathbf{r})$, the wave field is given by

$$u_i(\mathbf{r}) = \int G_{in}^{(0)}(\mathbf{r},\mathbf{r}')f_n(\mathbf{r}')\,dV' + \int G_{in}^{(0)}(\mathbf{r},\mathbf{r}')L_{nj}^{(1)}(\mathbf{r}')u_j(\mathbf{r}')\,dV' + \oint \left\{G_{in}^{(0)}(\mathbf{r},\mathbf{r}')n_j c_{njkl}^{(0)}\partial'_k u_l(\mathbf{r}') - u_n(\mathbf{r}')n_j c_{njkl}^{(0)}\partial'_k G_{il}^{(0)}(\mathbf{r},\mathbf{r}')\right\}dS'. \tag{35}$$

Note that the Green's tensor for the reference medium is used and that in the surface integral the elasticity tensor $\mathbf{c}^{(0)}$ of the reference medium is used. The first term on the right-hand side is the unperturbed wave:

$$u_i^{(0)}(\mathbf{r}) = \int G_{in}^{(0)}(\mathbf{r},\mathbf{r}')f_n(\mathbf{r}')dV'. \tag{36}$$

At this point we assume that the reference medium has zero tractions at the surface:

$$n_j c_{njkl}^{(0)}\partial'_k G_{il}^{(0)}(\mathbf{r},\mathbf{r}') = 0. \tag{37}$$

Because of this boundary condition the last term in (35) vanishes. The perturbed medium also has zero tractions at the surface S, but for this boundary condition we must use the perturbed elasticity tensor and the total wave field:

$$n_j\left(c_{ijkl}^{(0)} + c_{ijkl}^{(1)}\right)\partial_k u_l(\mathbf{r}) = 0. \tag{38}$$

From this it follows that at the surface $n_j c^{(0)}_{ijkl}\partial_k u_l(\mathbf{r}) = -n_j c^{(1)}_{ijkl}\partial_k u_l(\mathbf{r})$. This expression can be used in the third term of the right-hand side of (35). Using these results and inserting also definition (15) for the operator $L^{(1)}$ one obtains that

$$\begin{aligned} u_i(\mathbf{r}) = u_i^{(0)}(\mathbf{r}) &+ \omega^2 \int G^{(0)}_{ij}(\mathbf{r},\mathbf{r}')\rho^{(1)}(\mathbf{r}')u_j(\mathbf{r}')\,dV' \\ &+ \int G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\partial'_k\left(c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\right) dV' \\ &- \oint G^{(0)}_{in}(\mathbf{r},\mathbf{r}')n_j c^{(1)}_{njkl}\partial'_k u_l(\mathbf{r}')\,dS'. \end{aligned} \tag{39}$$

Note that the perturbed elasticity tensor $\mathbf{c}^{(1)}$ is present in the surface integral as well as in the volume integral in the second line. The theorem of Gauss can be used to convert this volume integral in a surface integral:

$$\begin{aligned} &\int G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\partial'_k\left(c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\right) dV' \\ &= \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\right) dV' \\ &\quad - \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\,dV' \\ &= \oint n_k G^{(0)}_{in}(\mathbf{r},\mathbf{r}')c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')dS' \\ &\quad - \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\,dV'. \end{aligned} \tag{40}$$

The surface integral in this expression is the same as the surface integral in expression (39) but it has the opposite sign. When (40) is inserted in (39) these terms cancel so that finally

$$\begin{aligned} u_i(\mathbf{r}) = u_i^{(0)}(\mathbf{r}) &+ \omega^2 \int G^{(0)}_{ij}(\mathbf{r},\mathbf{r}')\rho^{(1)}(\mathbf{r}')u_j(\mathbf{r}')\,dV' \\ &- \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j(\mathbf{r}')\,dV'. \end{aligned} \tag{41}$$

This is the Lippman–Schwinger equation for elastic wave scattering. In this expression the total wave field is decomposed in the unperturbed wave plus scattered waves that are generated by the perturbations of the medium. The strength of these scattered waves depends on the total wave field at the location of the scatterers as well as on the size of the perturbations from the reference medium. Conveniently, expression (41) is simpler than the original expression (35) because the surface integral has disappeared. In addition, the Green's function and the wave field appear in a more symmetric way in the last term of (41) because the gradient of both terms is taken.

§ 6. The Born Approximation of Elastic Waves

Up to this point no approximations have been made. The simplicity of (41) is elusive because one needs to know the total wave field at the scatterers in order to compute the integrals on the right-hand side. For many problems the Born approximation is a useful tool for computing the scattered waves. This approximation is obtained by replacing the total wave field on the right-hand side of (41) with the unperturbed wave. This means that in the Born approximation the *scattered waves* u_i^{B} are given by

$$\begin{aligned} u_i^{\mathrm{B}}(\mathbf{r}) = \omega^2 &\int G^{(0)}_{ij}(\mathbf{r},\mathbf{r}')\rho^{(1)}(\mathbf{r}')u_j^{(0)}(\mathbf{r}')\,dV' \\ &- \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j^{(0)}(\mathbf{r}')\,dV'. \end{aligned} \tag{42}$$

An alternative way to obtain the Born approximation is to iterate the integral equation (41). This resulting series is nothing but the Neumann series solution for elastodynamic wave scattering where the wave field is written as a sum of the unperturbed wave, the single-scattered waves, the double-scattered waves, etc. Truncating this series after the second term gives (42) for the scattered waves. This implies that the Born approximation (42) retains the single-scattered waves in an elastic medium. Analytic estimates of the domain of applicability for the Born approximation for elastic waves are given by Hudson and Heritage (1982).

So far we have considered volumetric perturbations of the density and elasticity tensor. In many applications, the perturbations of the medium are best described by perturbations in the position of interfaces within the medium. Because the properties of the reference medium are discontinuous at interfaces, we consider perturbations of height h away from their position in the reference medium. For perturbations of the boundary height small compared to the wavelength of elastic waves, the associated perturbation of the density is equivalent to a volumetric perturbation $\left[\rho^{(0)}\right]$ over a thickness h across the interface (Hudson, 1977). The straight brackets denote the contrast in the density across the interface. Similarly the associated perturbation of the elasticity tensor is given a volumetric perturbation $\left[\mathbf{c}^{(1)}\right]$ over a thickness h across the interface. Since h is assumed to be small compared to a wavelength, the resulting volume integrals in (42) reduce to surface integrals over the perturbed interface multiplied with the height of the perturbation. This then generalises the previous expression to include perturbations of the interfaces in the reference medium:

$$\begin{aligned} u_i^{\mathrm{B}}(\mathbf{r}) = \omega^2 &\int G^{(0)}_{ij}(\mathbf{r},\mathbf{r}')\rho^{(1)}(\mathbf{r}')u_j^{(0)}(\mathbf{r}')\,dV' \\ &- \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)c^{(1)}_{nklj}(\mathbf{r}')\partial'_l u_j^{(0)}(\mathbf{r}')\,dV' \\ &+ \sum \omega^2 \int G^{(0)}_{ij}(\mathbf{r},\mathbf{r}')h(\mathbf{r}')\left[\rho^{(0)}(\mathbf{r}')\right]u_j^{(0)}(\mathbf{r}')\,dS' \\ &- \sum \int \partial'_k\left(G^{(0)}_{in}(\mathbf{r},\mathbf{r}')\right)h(\mathbf{r}')\left[c^{(0)}_{nklj}(\mathbf{r}')\right]\partial'_l u_j^{(0)}(\mathbf{r}')\,dS'. \end{aligned} \tag{43}$$

The summation is over the perturbed interfaces. The details of this derivation are given by Hudson (1977).

One must be careful in the application of this expression to the displacement of an interface that separates a fluid from a solid (Woodhouse, 1976). The reason is that at such a boundary the component of the displacement parallel to the interface is not continuous, and the equivalence of a displacement of such an interface with a volumetric perturbation is not obvious. This is relevant for a variety of situations such as the scattering of elastic waves at the core–mantle boundary in the Earth, the scattering of waves by irregularities in the ocean floor and engineering applications where an elastic body is surrounded by a fluid.

§7. The Green's Tensor in Dyadic Form

In this section examples are given of the Green's tensor for different media. In all these examples the Green's tensor $\mathbf{G}(\mathbf{r},\mathbf{r}')$ can be written as a sum of dyads of the form $\mathbf{p}(\mathbf{r})g(\mathbf{r},\mathbf{r}')\mathbf{p}^{\dagger}(\mathbf{r}')$. (The dagger denotes the Hermitian conjugate defined as the complex conjugate of the transpose: $\mathbf{p}^{\dagger} = \left(\mathbf{p}^{T}\right)^{*}$.) In this expression, the vectors $\mathbf{p}$ are polarisation vectors and the scalar function $g(\mathbf{r},\mathbf{r}')$ accounts for propagation. The fact that the polarisation vectors at the source $\mathbf{r}$ and the receiver $\mathbf{r}'$ enter this expression in a symmetric way is due to the reciprocity property (28) of the Green's tensor.

A Homogeneous Infinite Isotropic Medium

As a first example, consider the Green's tensor for a homogeneous, infinite isotropic medium. This Green's tensor is given in expression (4.12) of Ben-Menahem and Singh (1981)

$$\mathbf{G}(\mathbf{r}) = \frac{ik_\alpha}{12\pi(\lambda+2\mu)}\left(\mathbf{I}\,h_0^{(1)}(k_\alpha r) + (\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^T)\,h_2^{(1)}(k_\alpha r)\right) - \frac{ik_\beta}{12\pi\mu}\left(-2\mathbf{I}\,h_0^{(1)}(k_\beta r) + (\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^T)\,h_2^{(1)}(k_\beta r)\right), \quad (44)$$

where $\mathbf{r}$ is the *relative* location of the observation point relative to the source. (In this expression $\mathbf{I}$ denotes the identity matrix with elements $\mathbf{I}_{ij} = \delta_{ij}$.) The caret above a vector denotes the unit vector pointing in the same direction, hence $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, where r is the distance between the source and observation points. In expression (44) the wave numbers k_α and k_β are given by

$$k_\alpha = \omega/\alpha \quad \text{and} \quad k_\beta = \omega/\beta. \quad (45)$$

where α is the wave velocity for compressive waves and β is the wave velocity for shear waves:

$$\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} \quad \text{and} \quad \beta = \sqrt{\frac{\mu}{\rho}}. \quad (46)$$

These velocities are derived under An Isotropic Medium (§8). In expression (44) the $h_m^{(1)}$ are spherical Bessel functions of the first kind. It should be noted that Ben-Menahem and Singh (1981) use a Fourier transform with the opposite sign in the exponent to the convention (1) that is used here. For this reason, the complex conjugate is taken, which implies that the spherical Bessel function of the second kind $h_m^{(2)}$ in their expression is replaced by the spherical Bessel function of the first kind $h_m^{(1)}$. These functions are given by

$$h_0^{(1)}(x) = -i\,\frac{e^{ix}}{x}, \qquad h_2^{(1)}(x) = \left(\frac{i}{x} - \frac{3}{x^2} - \frac{3i}{x^3}\right)e^{ix}. \quad (47)$$

Let us first analyse the Green's tensor in the far field. The far field is defined as the region of space that is far away from the source compared to a wavelength; hence it is defined by the condition $kr \gg 1$. In the far field only terms $1/kr$ are retained, but terms of higher powers in $1/kr$ are ignored. Using this in (47) and (44) gives the Green's tensor in the far field:

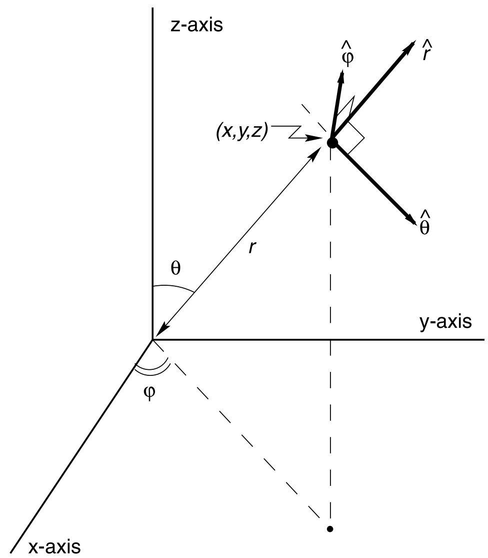
$$\mathbf{G}^{FF}(\mathbf{r}) = \frac{1}{4\pi(\lambda+2\mu)}\frac{e^{ik_\alpha r}}{r}\hat{\mathbf{r}}\hat{\mathbf{r}}^T + \frac{1}{4\pi\mu}\frac{e^{ik_\beta r}}{r}\left(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}^T\right). \quad (48)$$

Note that this expression is not yet in the form of a superposition of dyads $\mathbf{p}g(\mathbf{r},\mathbf{r}')\mathbf{p}^T$. However, (48) can be written in such form by using the closure relation for the identity matrix: $\mathbf{I} = \hat{\mathbf{r}}\hat{\mathbf{r}}^T + \hat{\theta}\hat{\theta}^T + \hat{\varphi}\hat{\varphi}^T$, where $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\varphi}$ are the unit vectors in the direction of increasing values of the variables r, θ and φ used in spherical coordinates. The directions of these unit vectors are shown in Fig. 3. Using this result and using the relations $\lambda + 2\mu = \rho\alpha^2 = \rho\omega^2/k_\alpha^2$ and $\mu = \rho\beta^2 = \rho\omega^2/k_\beta^2$, the far field Green's function can be written as

$$\mathbf{G}^{FF}(\mathbf{r}) = \frac{k_\alpha^2}{4\pi\rho\omega^2}\frac{e^{ik_\alpha r}}{r}\hat{\mathbf{r}}\hat{\mathbf{r}}^T + \frac{k_\beta^2}{4\pi\rho\omega^2}\frac{e^{ik_\beta r}}{r}\left(\hat{\theta}\hat{\theta}^T + \hat{\varphi}\hat{\varphi}^T\right). \quad (49)$$

Let us consider the first term. For a general excitation $\mathbf{f}$ at the origin, the inner product of the force with the Green's tensor should be taken. This means that the first term gives a contribution proportional to $\hat{\mathbf{r}}\left(e^{ik_\alpha r}/r\right)(\hat{\mathbf{r}}\cdot\mathbf{f})$, and the displacement of this wave is in the radial direction. This term therefore corresponds to a *longitudinal* wave. The wave propagates with a wave number k_α through space, and the corresponding wave velocity is given by α defined in (46). It follows from (46) and the fact that the Lamé constants positive that α is larger than β. This means that the P wave arrives first in the time domain. During the propagation the wave decays because of geometrical spreading with a factor $1/r$. The excitation of this wave is given by the component $(\hat{\mathbf{r}}\cdot\mathbf{f})$ of the force in the direction of propagation. Since the unit vector

Figure 3 Definition of the unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\varphi}$ that are used in a system of spherical coordinates.



$\hat{\mathbf{r}}$ gives the polarisation of this wave, it will be called the *polarisation vector*.

In the same way, the last terms in (49) describe the propagation of S waves. These waves have a wave number k_β and propagate away from the source with a velocity β defined in (46). For these waves, oscillation is in the direction $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\varphi}}$. As shown in Fig. 3, both directions are orthogonal to the direction of propagation $\hat{\mathbf{r}}$. This implies that the waves described by the last terms in (49) are transverse waves. The excitation of each of these waves depends on the components of the exciting force $(\hat{\boldsymbol{\theta}}\cdot\mathbf{f})$ or $(\hat{\boldsymbol{\varphi}}\cdot\mathbf{f})$ along the direction of polarisation. Note that there are two directions of oscillation for the S wave because in three dimensions two vectors are orthogonal to a given direction of propagation.

These results imply that both the P wave and the two S waves can be written in the same form. For each of these waves a polarisation vector can be defined; for the P wave the polarisation vector is given by $\hat{\mathbf{r}}$, while for the S waves the polarisation vector is given by $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ respectively. The far field Green's tensor can be written as a sum over these polarisation vectors. Let the summation over the polarisation vectors be denoted by a Greek subscript, then the far field Green's tensor can be written as

$$\mathbf{G}^{\mathrm{FF}}(\mathbf{r}) = \sum_\nu \frac{k_\nu^2}{4\pi\rho\omega^2}\hat{\mathbf{p}}_\nu \frac{e^{ik_\nu r}}{r}\hat{\mathbf{p}}_\nu^\dagger, \qquad (50)$$

where $k_\nu = k_\alpha$ for the P wave and $k_\nu = k_\beta$ for the S wave. Since the polarisation vectors are real, the asterisk has no effect here but it is used because, as shown in §1 of Chapter 1.7.3 the Green's tensor for Rayleigh waves has polarisation vectors that are complex.

The terms in the first line of (44) depend only on the P-wave velocity α, while the terms in the second line depend only on the S-wave velocity β. It is therefore tempting to identify the first line with the P-wave motion and the second line with the S-wave motion. It follows from (47), however, that each of these terms has a $1/r^3$ singularity in the near field ($kr < 1$). A $1/r$ or $1/r^2$ singularity of the Green's tensor poses no problem because these singularities are integrable in three dimensions. (They are absorbed by a contribution r^2 from the Jacobian in spherical coordinates.) This means that the P-wave terms and the S- wave terms separately have a nonintegrable singularity at the origin. As shown by Wu and Ben-Menahen (1985), the *sum* of these terms is integrable in the near field:

$$\mathbf{G}^{\mathrm{NF}}(\mathbf{r}) = -\frac{1}{8\pi\mu}\left(1 - \frac{\beta^2}{\alpha^2}\right)\frac{1}{r}(\mathbf{I} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}). \qquad (51)$$

The lesson we learn from this is that the notion of P and S waves is meaningful only in the far field; in the near field the compressive and shear motions combine in an intricate way. Similarly, the distinction between body waves and surface waves is physically meaningless in the near field.

The Ray-Geometric Green's Tensor for Body Waves

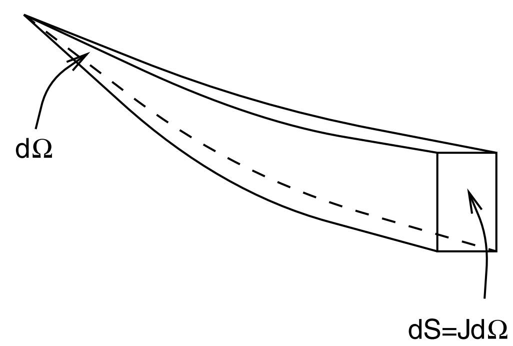
As a second example of the Green's function we consider the ray-geometric propagation of body waves in elastic media. Because the algebraic complexity of ray theory for elastic media hides the physical simplicity a heuristic derivation is given here. Details are given by Červený and Ravindra (1971), Červený and Hron (1980) and Aki and Richards (1980). Here we consider the wave field at location $\mathbf{r}$ due to a point force $\mathbf{f}$ at source location $\mathbf{r}_s$.

When the variations of the properties of the medium and of the amplitude of the waves is small over a wavelength, the equations of elastodynamic wave propagation are approximated well by the eikonal equation for the phase and the transport equation for the amplitude. Ray theory thus constitutes a high-frequency approximation. The eikonal and transport equations should be applied to the P wave and the two S waves separately. Here we consider a mode of propagation ν that may be either the P wave or an S wave. The corresponding velocity is denoted by c_ν. The eikonal equation states that the gradient of the phase in the ray direction is given by ω/c_ν. The arrival time of waves is determined by the phase in the frequency domain. Integrating along the ray, the phase of mode ν is given by

$$\varphi_\nu = \omega\int \frac{1}{c_\nu}ds = \int k_\nu ds. \qquad (52)$$

The transport equation ensures that energy is conserved by requiring that $\rho c J A^2$ is conserved, where A

Figure 4 Definition of the geometrical spreading factor J.



is the amplitude of the wave field and J is the geometrical spreading factor. As shown in Fig. 4, $Jd\Omega$ is the surface area spanned by a bundle of rays with a solid angle $d\Omega$ at the source. This result implies that the amplitude of the wave is given by $A = C/(\sqrt{\rho c J})$, where the constant C needs to be determined. The polarisation vector $\hat{\mathbf{p}}_\nu$ is parallel to the ray direction for the P wave and is orthogonal to the ray direction for the S waves. Using these results one finds that the displacement for mode ν is given by

$$\mathbf{u}_\nu(\mathbf{r},\mathbf{r}_s) = \frac{C}{\sqrt{\rho_\nu(\mathbf{r})c_\nu(\mathbf{r})J_\nu(\mathbf{r},\mathbf{r}_s)}}\hat{\mathbf{p}}_\nu(\mathbf{r})e^{i\varphi_\nu}. \tag{53}$$

The value of the constant C follows from the fact that close to the source, the medium can be treated as if it is homogeneous with the properties of the source region. The corresponding wave field in the vicinity of the source is given by (50) contracted with the excitation $\mathbf{f}$. (The contraction means that the quantities are combined using the rules for matrix multiplication, i.e., $(\mathbf{G}\cdot\mathbf{f})_i$ stands for $\sum_j G_{ij}f_j$.) For a homogeneous medium, the rays are straight lines. It follows from Fig. 4 that, near the source, the geometrical spreading is given by $J(\mathbf{r},\mathbf{r}_s) = |\mathbf{r}-\mathbf{r}_s|^2$. These results lead to the following constraint on the constant C:

$$\frac{1}{4\pi\rho(\mathbf{r}_s)c_\nu^2(\mathbf{r}_s)}\frac{\left(\hat{\mathbf{p}}_\nu^\dagger(\mathbf{r}_s)\cdot\mathbf{f}\right)}{|\mathbf{r}-\mathbf{r}_s|} = \frac{C}{\sqrt{\rho(\mathbf{r}_s)c_\nu(\mathbf{r}_s)|\mathbf{r}-\mathbf{r}_s|^2}}. \tag{54}$$

Solving this expression for C leads to the displacement of mode ν

$$\mathbf{u}_\nu(\mathbf{r},\mathbf{r}_s) = \frac{1}{4\pi\sqrt{\rho(\mathbf{r})\rho(\mathbf{r}_s)c_\nu(\mathbf{r})c_\nu^3(\mathbf{r}_s)}}\hat{\mathbf{p}}_\nu(\mathbf{r}) \times \frac{e^{i\varphi_\nu}}{\sqrt{J_\nu(\mathbf{r},\mathbf{r}_s)}}\left(\hat{\mathbf{p}}_\nu^\dagger(\mathbf{r}_s)\cdot\mathbf{f}\right). \tag{55}$$

Summing over the polarisations ν the complete ray-geometric Green's tensor can then be written as

$$\mathbf{G}^{FF}(\mathbf{r},\mathbf{r}_s) = \sum_\nu \frac{1}{4\pi\sqrt{\rho(\mathbf{r})\rho(\mathbf{r}_s)c_\nu(\mathbf{r})c_\nu^3(\mathbf{r}_s)}}\hat{\mathbf{p}}_\nu(\mathbf{r}) \times \frac{\exp\left(i\int k_\nu ds\right)}{\sqrt{J_\nu(\mathbf{r},\mathbf{r}_s)}}\hat{\mathbf{p}}_\nu^\dagger(\mathbf{r}_s). \tag{56}$$

Note the similarity between this expression and Eq. (50) for the Green's tensor in a homogeneous medium. The polarisation vectors $\hat{\mathbf{p}}_\nu(\mathbf{r})$ and $\hat{\mathbf{p}}_\nu(\mathbf{r}_s)$ are defined by the ray direction at the point $\mathbf{r}$ and the source point $\mathbf{r}_s$ respectively. In (50) these vectors are identical because the reference medium is homogeneous in that development. Since the rays in general are curved, the polarisation vectors $\hat{\mathbf{p}}_\nu(\mathbf{r})$ and $\hat{\mathbf{p}}_\nu(\mathbf{r}_s)$ usually have a different direction. The integral of the phase is computed along the ray that joins the points $\mathbf{r}$ and $\mathbf{r}_s$. In the ray-geometric approximation the waves with different polarisations are not coupled; this is reflected by the fact that expression (56) contains a *single* sum over the different modes of polarisations.

The reader may be puzzled by the fact that the Green's tensor (56) appears to violate the principle of reciprocity (28) because the velocity at the source enters through a factor $c_\nu^3(\mathbf{r}_s)$, whereas the velocity at location $\mathbf{r}$ is present with a factor $c_\nu(\mathbf{r})$. Snieder and Chapman (1998) show that for a point source in three dimensions the geometrical spreading factor satisfies the reciprocal rule $J(\mathbf{r},\mathbf{r}_s)c^2(\mathbf{r}_s) = J(\mathbf{r}_s,\mathbf{r})c^2(\mathbf{r})$. The velocity term in this expression ensures that the Green's tensor (56) indeed satisfies the reciprocity property (28).

Other Examples of Dyadic Green's Tensors

It is shown in §1 of Chapter 1.7.3 that the Green's tensor for surface waves in an isotropic half-space that varies only with depth can be written as

$$\mathbf{G}(\mathbf{r},\mathbf{r}_s) = \sum_\nu \mathbf{p}_\nu(z,\varphi)\frac{e^{i(k_\nu X+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\mathbf{p}_\nu^\dagger(z_s,\varphi). \tag{57}$$

In this expression the $\mathbf{p}_\nu$ are appropriate polarisation vectors that describe the particle motion of Love and Rayleigh waves, and X is the horizontal distance between $\mathbf{r}$ and $\mathbf{r}_s$. Again, the Green's tensor is given by a sum over polarisation vectors that correspond to the (surface wave) modes ν of the system. The term $\exp(ik_\nu X)/\sqrt{k_\nu X}$ accounts for the phase shift and geometrical spreading in the horizontal propagation from the source at $\mathbf{r}_s$ to the point $\mathbf{r}$ and φ denotes the azimuth of the source–receiver direction.

The Green's tensor of the Earth can be expressed as a sum over normal modes of the Earth. These normal modes also incorporate self-gravitation in the Earth. The theory of the Earth's normal modes is described in detail by Dahlen and Tromp (1998). Let the displacement of a normal mode ν be denoted by $\mathbf{p}_\nu(\mathbf{r})$ and let the mode have eigenfrequency ω_ν. According to Section 6.3.3.2 of Ben-Menahem and Singh (1981), the modes are orthogonal and can be normalised to form an orthonormal set with respect to the inner product

$$\langle \mathbf{p}_\nu | \mathbf{p}_\mu \rangle = \delta_{\nu\mu}, \tag{58}$$

where the inner product is defined by

$$\langle \mathbf{u}|\mathbf{v}\rangle \equiv \sum_i \int \rho(\mathbf{r}) u_i^*(\mathbf{r}) v_i(\mathbf{r}) dV = \int \rho(\mathbf{r}) \mathbf{u}^\dagger(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) dV. \quad (59)$$

Note the presence of the mass density in this inner product. As shown in Ben-Menahem and Singh (1981) the displacement for a point source $\mathbf{f}$ at location $\mathbf{r}_s$ is given by

$$\mathbf{u}(\mathbf{r}) = \sum_\nu \frac{1}{\omega_\nu^2 - \omega^2} \mathbf{p}_\nu(\mathbf{r}) \langle \mathbf{p}_\nu(\mathbf{r}_s)|\mathbf{f}\rangle, \quad (60)$$

where ω is the angular frequency of the excitation. This implies that the Green's tensor is given by

$$\mathbf{G}(\mathbf{r}, \mathbf{r}_s) = \sum_\nu \mathbf{p}_\nu(\mathbf{r}) \frac{1}{\omega_\nu^2 - \omega^2} \mathbf{p}_\nu^\dagger(\mathbf{r}_s). \quad (61)$$

Again the Green's tensor is written as a sum over dyads composed of the polarisation vector multiplied with a function that measures the strength of the response. Note that the response is strongest when the excitation has a frequency close to a resonance ($\omega \approx \omega_\nu$). At a resonance ($\omega = \omega_\nu$) expression (61) for the Green's tensor is infinite, but anelastic damping prevents a singularity in the solution. In the presence of anelastic damping, however, the eigenfunctions are not orthogonal so one of the polarisation vectors in each term of (61) must be replaced by the polarisation vector of the dual eigenfunctions. The reader is referred to Dahlen and Tromp (1998) for details.

Physical Interpretation of Dyadic Green's Tensor

In Eqs. (50), (56), (57) and (61) the far field Green's function is written as a sum of dyads:

$$\mathbf{G}(\mathbf{r}, \mathbf{r}_s) = \sum_\nu \mathbf{p}_\nu(\mathbf{r}) R(\mathbf{r}, \mathbf{r}_s) \mathbf{p}_\nu^\dagger(\mathbf{r}_s). \quad (62)$$

In this expression $R(\mathbf{r}, \mathbf{r}_s)$ is a response function that measures the strength of the response. The displacement generated by a force $\mathbf{f}$ is given by

$$\mathbf{u}(\mathbf{r}) = \sum_\nu \mathbf{p}_\nu(\mathbf{r}) R(\mathbf{r}, \mathbf{r}_s) \langle \mathbf{p}_\nu(\mathbf{r}_s)|\mathbf{f}\rangle. \quad (63)$$

Reading this expression from right to left one can follow the life history of each mode ν. A mode can refer to either a normal mode of the Earth or a surface wave mode, but it may also refer to the three polarisations of body wave propagation. At the source the mode is excited, the excitation given by the projection of the force on the polarisation vector: $\langle \mathbf{p}_\nu(\mathbf{r}_s)|\mathbf{f}\rangle$. The wave then travels from the source to the receiver; this is accounted for by the response function $R(\mathbf{r}, \mathbf{r}_s)$. Finally, at the receiver location $\mathbf{r}$ the mode oscillates in the direction given by the polarisation vector $\mathbf{p}_\nu(\mathbf{r})$. The general expression (62) therefore not only unifies the various Green's tensors from a mathematical point of view, it also provides a physical description of the role played by the Green's tensor in the life history of the waves.

An Example of a Complicated Green's Tensor

The Green's tensors given in the previous sections are for simple models of elastic media. These Green's tensors could be obtained because of the symmetries of the models considered and the simplifying assumptions that have been made (such as smoothness of the model). In practice the response of an arbitrary elastic medium can be very complex. The Green's function gives the impulse response of the medium. This means that for a localised earthquake, the recorded ground motion gives a direct measurement of the Green's function in the Earth.

The top panel of Fig. 5 shows the vertical component of the ground motion recorded with the Parkfield array in California (Fletcher *et al.*, 1992) after a local earthquake. The signals have a noisy appearance. At almost the same location a second earthquake occurred on the same day; the resulting ground motion measured by the same array is shown in the bottom panel of Fig. 5. The ground motion recorded after these two earthquakes is virtually identical, each trace in the top panel matches the corresponding trace in the bottom panel "wiggle by wiggle." In fact, the discrepancies between the corresponding traces are on the same order of magnitude as the fluctuations in the traces before the first impulsive arrival.

This similarity implies that the "noisy" signals in Fig. 5 are not noise but deterministic Earth response. At these frequencies the Earth is a strongly scattering medium, and the resulting ground motion is a complex interference pattern of multiply scattered and mode-converted elastic waves. The complexity of the waves that propagate through the Earth is a strong function of frequency. At present it is not known to what extent these complex waves can be used for imaging purposes. This issue and the possible relationship between classical chaos and wave chaos are discussed further in Scales and Snieder (1997), Snieder (1999) and Snieder and Scales (1998).

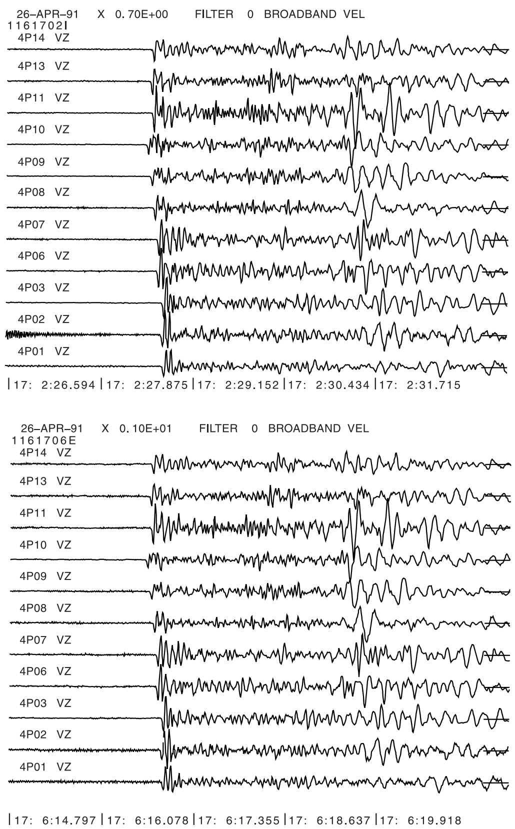
§ 8. Anisotropic Media

Plane-Wave Solutions

When the elastic medium is anisotropic, interesting wave phenomena that have no counterpart in isotropic media occur. The phenomena are introduced here by considering a plane wave that propagates in a homogeneous elastic medium

$$\mathbf{u}(\mathbf{r}) = \hat{\mathbf{p}}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (64)$$

Figure 5 Wave field recorded at the Parkfield array (Fletcher *et al.*, 1992) in California after two nearby earthquakes (top and bottom, respectively). Note the extreme resemblance of each trace in the top panel with the corresponding trace in the bottom panel. (Courtesy of Peggy Hellweg of the USGS).



where $\hat{\mathbf{p}}$ is the polarisation vector and $\mathbf{k}$ the wave vector. As illustrated in Fig. 6, the wave fronts of this plane wave are perpendicular to a unit vector $\hat{\mathbf{n}}$. With the phase velocity denoted by v the wave vector can be written as

$$\mathbf{k} = \frac{\omega}{v}\hat{\mathbf{n}}. \tag{65}$$

Inserting the solution (64) into the equation of motion (13) and using that ρ and **c** do not depend on the position one finds that

$$\rho\omega^2 p_i - c_{ijkl}k_j k_k p_l = 0. \tag{66}$$

With expression (65) for the wave vector this result can also be written as

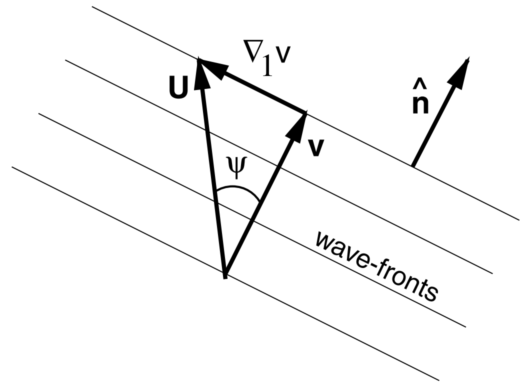
$$\mathbf{\Gamma}\hat{\mathbf{p}} = \rho v^2 \hat{\mathbf{p}}, \tag{67}$$

with $\mathbf{\Gamma}$ defined by

$$\Gamma_{ij} = c_{iklj} n_k n_l. \tag{68}$$

Let us consider the direction of wave propagation $\hat{\mathbf{n}}$ to be specified. This defines the matrix $\mathbf{\Gamma}$ through expression (68). Equation (67) then constitutes an eigenvalue problem; the eigenvectors of $\mathbf{\Gamma}$ define the polarisation vectors $\hat{\mathbf{p}}$ and the eigenvalues are given by

Figure 6 The phase velocity $\mathbf{v} = v\hat{\mathbf{n}}$ and the group velocity $\mathbf{U}$ in relation to the orientation of the wave fronts in an anisotropic medium. The transverse gradient ∇_1 is defined in expression (76).



ρv^2, so that the eigenvalues provide the phase velocity of the plane wave. Three important issues should be noted: (i) $\mathbf{\Gamma}$ is a real symmetric 3×3 matrix; hence there are three real eigenvalues. (ii) Each eigenvalue gives a velocity v of the plane wave with polarisation vector $\hat{\mathbf{p}}$ that corresponds to that mode of propagation. The eigenvalues of $\mathbf{\Gamma}$ are equal to ρv^2; these eigenvalues are positive.[1] This means that when v is a solution, then another solution is $-v$; this reflects the reciprocity of waves in an elastic medium. (iii) The polarisation vectors and the phase velocity do not depend on frequency, this means that there is no dispersion for the homogeneous medium.

The divergence and the curl of the displacement (64) are given by

$$\begin{aligned}(\nabla\cdot\mathbf{u}) &= \frac{i\omega}{v}\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{p}}\right) e^{i\mathbf{k}\cdot\mathbf{r}},\\ \nabla\times\mathbf{u} &= \frac{i\omega}{v}\left(\hat{\mathbf{n}}\times\hat{\mathbf{p}}\right) e^{i\mathbf{k}\cdot\mathbf{r}}.\end{aligned} \tag{69}$$

For an anisotropic medium one cannot state that one solution is polarised in the direction of propagation ($\hat{\mathbf{p}}//\hat{\mathbf{n}}$) and that two solutions are polarised in the transverse directions ($\hat{\mathbf{p}} \perp \hat{\mathbf{n}}$). It follows from (69) that the wave is curl-free when the polarisation is longitudinal ($\hat{\mathbf{p}}//\hat{\mathbf{n}}$) and that the wave is divergence-free when the wave is transversely polarised ($\hat{\mathbf{p}} \perp \hat{\mathbf{n}}$). Therefore, in a general anisotropic medium the plane wave solutions are neither divergence-free nor curl-free.

Wave propagation in smoothly varying elastic media can be described by a formulation of geometric ray theory that is suitable for such media (Vlaar, 1968; Červený, 1972). From the analysis of their work the local polarisation vectors and the local phase velocity again follow from the eigenvalue problem (67), where the density and elasticity tensor are now evaluated at the position of the ray for a given ray direction $\hat{\mathbf{n}}$.

An Isotropic Medium

Let us now for the moment consider the special case of an isotropic elastic medium for which the elasticity tensor is given by (10). Using that $n_j n_j = (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = 1$ one finds that in this special case the matrix $\mathbf{\Gamma}$ from expression (68) is given by $\Gamma_{ij} = (\lambda + \mu) n_i n_j + \mu \delta_{ij}$, so that in vector notation

$$\mathbf{\Gamma} = (\lambda + \mu) \hat{\mathbf{n}} \hat{\mathbf{n}}^{\mathrm{T}} + \mu \mathbf{I}. \tag{70}$$

One eigenvector is given by $\hat{\mathbf{p}} = \hat{\mathbf{n}}$,

$$\mathbf{\Gamma} \hat{\mathbf{p}} = (\lambda + 2\mu) \hat{\mathbf{p}}. \tag{71}$$

Hence the associated wave velocity is given by $\rho v^2 = \lambda + 2\mu$. This velocity is the **P** velocity, which is denoted by α,

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \text{(46) again.}$$

Note that since $\hat{\mathbf{n}} = \hat{\mathbf{p}}$, this means that the waves are longitudinally polarised. For this type of wave $\hat{\mathbf{n}} \times \hat{\mathbf{p}} = 0$; with (69) this implies that the *P* waves are curl-free. The *P* waves thus involve only compressive motion, but no shear motion.

Another eigenvector is such that the polarisation is orthogonal to the direction of propagation: $\hat{\mathbf{p}} \perp \hat{\mathbf{n}}$, which means we now consider transverse waves. It follows from (70) that

$$\mathbf{\Gamma} \hat{\mathbf{p}} = \mu \hat{\mathbf{p}}. \tag{72}$$

According to (67) this corresponds to a wave velocity for which $\rho v^2 = \mu$. This is called the *S* velocity, which is denoted by β:

$$\beta = \sqrt{\frac{\mu}{\rho}} \qquad \text{(46) again.}$$

For these waves $\hat{\mathbf{p}} \perp \hat{\mathbf{n}}$ hence $(\hat{\mathbf{p}} \cdot \hat{\mathbf{n}}) = 0$. With (69) this implies that the *S* waves are divergence-free. *S* waves do not lead to compressive motion; they entail shear motion in the medium.

Distinction between Phase and Group Velocity

Let us now return to a general anisotropic medium. For a given direction of propagation $\hat{\mathbf{n}}$, the polarisation vectors follow from the eigenvalue problem (67). In general the polarisation vectors $\hat{\mathbf{p}}$ are no longer related in a simple way to the direction of propagation; they are not necessarily parallel or orthogonal to the direction of propagation, but they are mutually orthogonal. With (69) this implies that the waves in an anisotropic medium are neither curl-free nor divergence-free. In a general anisotropic medium one cannot therefore speak of the *P* wave and of *S* waves. However, when the anisotropy is weak, one of the waves is polarised almost in the longitudinal direction and two waves are polarised almost orthogonally. This means that in a weakly anisotropic medium one speaks of "quasi *P* waves" and "quasi *S* waves"(Coates and Chapman, 1990). The quasi *P* waves are not quite curl-free and the quasi *S* waves are not completely divergence-free.

For surface waves the situation is similar; in an anisotropic medium one cannot speak of "Rayleigh waves" and "Love waves", although for weak anisotropy one can distinguish "quasi Rayleigh waves" and "quasi Love waves" (Maupin, 1989). When the surface wave modes are nearly degenerate (in the sense that their phase velocities are nearly equal) these concepts must be used with care.

In an anisotropic medium the relation between the phase velocity and the group velocity is not trivial. This leads to observable phenomena. A plane wave has in the time domain the form $\exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$. The phase slowness $\mathbf{s}$ (defined as the reciprocal of the phase velocity) is given by $\mathbf{s} = \mathbf{k}/\omega$. The phase slowness is used rather than the phase velocity because this is a tensor of rank 1, while the phase velocity does not transform as a tensor. With (65) the phase slowness can be related to the direction of propagation $\hat{\mathbf{n}}$ and the magnitude of the phase velocity v as

$$\mathbf{s} = \frac{1}{v} \hat{\mathbf{n}}. \tag{73}$$

The phase slowness is shown in Fig. 7 for a number of points in the (k_x, k_z) plane by the dashed arrows. The phase slowness is parallel to the wave vector $\mathbf{k}$.

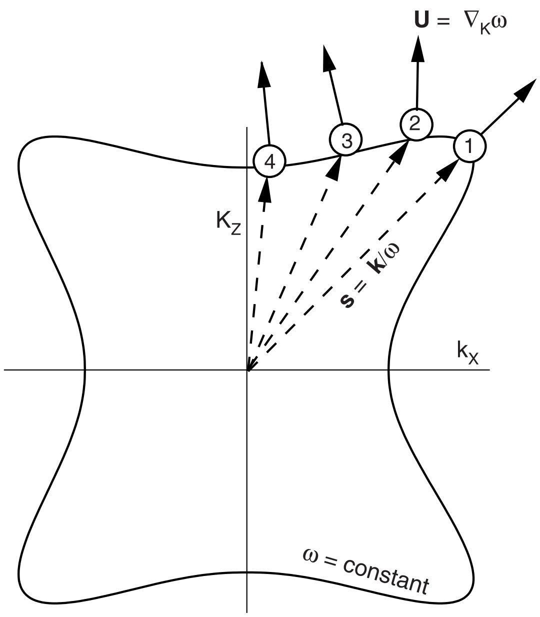
The group velocity of the plane wave is in contrast given by

$$\mathbf{U} = \mathbf{\nabla}_k \omega. \tag{74}$$

In this expression, $\mathbf{\nabla}_k$ is the gradient operator in wave-number space. The group velocity describes the energy flow of the waves in the medium (Babuska and Cara, 1991; Wolfe, 1998). Shown in Fig. 7 is the contour ω = constant in the (k_x, k_z) plane. Since the gradient of $\omega(\mathbf{k})$ is orthogonal to the contour lines ω = constant, the group velocity is orthogonal to the surface of constant values of ω. The direction of group velocity vectors at points 1, 2, 3 and 4 in the wave-number plane is indicated by the solid arrows in Fig. 7. It follows from the geometry of this figure that the group velocity and the phase slowness at any point in an anisotropic medium in general have a different direction. This is possible because the directionality contained in the elasticity tensor of an anisotropic medium breaks the symmetry of the system so that the direction propagation of the wave fronts and the energy flow are not necessarily aligned.

The difference in direction of propagation between the wave front and the wave group can be quantified by decomposing the gradient $\mathbf{\nabla}_k$ into a component parallel to the wave vector and an orthogonal component

Figure 7 Surface ω = const. in the wave-number plane. The phase slowness $\mathbf{s} = \mathbf{k}/\omega$ for the points 1, 2, 3 and 4 is indicated by arrows. The direction of the group velocity $\mathbf{U} = \nabla_k \omega$ is given by the solid arrows.



$$\nabla_k = \hat{\mathbf{n}}\frac{\partial}{\partial k} + \frac{1}{k}\nabla_1 , \tag{75}$$

where ∇_1 is the component of the gradient perpendicular to the direction of propagation. Using spherical coordinates in wave-number space, the orthogonal component can be written as

$$\nabla_1 = \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\varphi}}}{\sin\theta}\frac{\partial}{\partial\varphi}. \tag{76}$$

Inserting $\omega = vk$ in (74) and using (75), the group velocity is given by

$$\mathbf{U}(\hat{\mathbf{n}}) = \mathbf{v}(\hat{\mathbf{n}}) + \nabla_1 v(\hat{\mathbf{n}}). \tag{77}$$

With (76) it then follows that

$$\mathbf{U}(\theta, \varphi) = v(\theta, \varphi)\hat{\mathbf{n}} + \frac{\partial v}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}\tilde{\boldsymbol{\varphi}}. \tag{78}$$

Thus, in general the phase velocity $\mathbf{v} = v\hat{\mathbf{n}}$ and group velocity $\mathbf{U}$ have different directions. The difference in direction is proportional to the change in the phase velocity to the direction of propagation as described by the derivatives $\partial v/\partial\theta$ and $\partial v/\partial\varphi$.

Using that $\nabla_1 \perp \hat{\mathbf{n}}$, one finds from (77) and from $\mathbf{v} = v\hat{\mathbf{n}}$ that the components of the phase velocity and the group velocity in the direction of propagation are identical:

$$(\hat{\mathbf{n}}\cdot\mathbf{U}) = (\hat{\mathbf{n}}\cdot\mathbf{v}) = v. \tag{79}$$

This means that the group velocity differs from the phase velocity only because it also has a component along the wave fronts, as is illustrated in Fig. 6. In anisotropic media the energy flow is therefore not necessarily orthogonal to the wave fronts. It follows from the geometry of Fig. 6 and expression (77) that the angle between the group velocity and phase velocity is given by

$$\tan\psi = \frac{|\nabla_1 v|}{v}. \tag{80}$$

The nontrivial relationship between phase slowness and group velocity has interesting consequences. As an example, consider once again the group velocity at the points 1, 2, 3 and 4 in Fig. 7. The corresponding group velocities at these points are shown in the group velocity plane in Fig. 8. The group velocities indicated by the solid arrows have the same direction as the group velocity shown by the solid arrows in Fig. 7. When we move from point 1 through point 2 to point 3 in Fig. 7, the group velocity vector rotates in the *counterclockwise* direction. However, at point 3, the curve ω = constant has an inflection point, and the group velocity vector at point 4 is rotated *clockwise* compared to the group velocity vector at point 3. This means that the group velocity vector traces the folded curve that is shown in Fig. 8. At point 3 the group velocity is stationary for perturbations in the direction of propagation. Therefore point 3 corresponds to a caustic in the wave field. This example shows that caustics can form in a *homogeneous* elastic medium.

Such caustics can be observed experimentally. Fig. 9 shows snapshots of waves that have propagated through a rectangular block of silicon at three different times. (This figure was made available by *J.P. Wolfe* (1998) who also describes the experiment in detail.) The waves are excited at one side of the block. At $t = 1.76\ \mu s$ the waves have propagated to the opposing face where the wave field is measured, and at later times one sees the wave fronts spread over the face of the crystal. Note that the wave fronts have the same cusped structure as the group velocity shown in Fig. 8. This type of measurement provides information about the elasticity tensor of the crystal, which in turn gives information about the symmetry axes and elastic constants of the crystal.

It should be noted that in three dimensions the slowness and the group velocity as shown in Figs. 7 and 8 are described by surfaces that can have a very complex shape, which gives rise to caustics in the wave field with a complicated geometry (Helbig, 1994; Wolfe, 1998). The only ingredients used in this section were that the phase velocity is anisotropic and independent of frequency, and that the group velocity follows from expression (74). For this reason the theory of this section is not particular for elastic waves. In fact, the theory is equally applicable to other types of nondispersive anisotropic media. The analogue of the theory of this section applied to electromagnetic

Figure 8 The group velocity for the points 1, 2, 3 and 4 that are also shown in the previous figure. The solid arrows indicate the group velocity. Note the cusps in the group velocity surface, which correspond to caustics in the wave field.

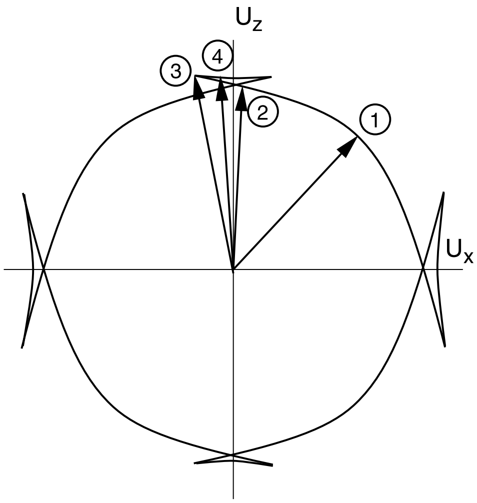
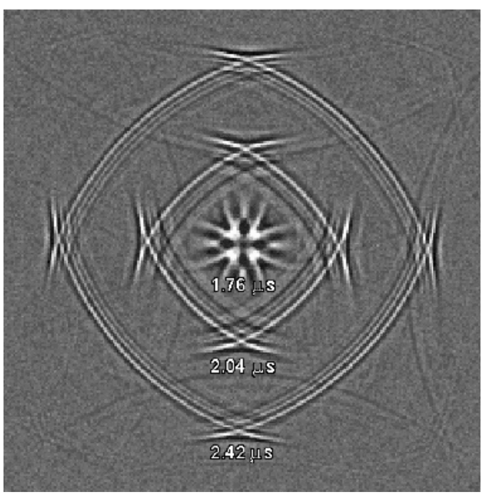


Figure 9 Snapshots of the wave field that has propagated through a homogeneous silicon crystal recorded at three times after excitation on the other side of the crystal. Details of the experiment are given by Wolfe (1998). (Courtesy of J.P. Wolfe.)



waves can for example be found in Chapter III and Section V.2 of *Kline and Kay* (1965).

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Note

1. When the strain energy is positive, the quantity $c_{iklj}v_i n_k v_l n_j$ is positive. It follows from this that Γ_{ij} is positive definitely so that the eigenvalues ρv^2 of $\mathbf{\Gamma}$ are positive.

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