

General representation theorem for perturbed media and application to Green's function retrieval for scattering problems

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SUMMARY

Green's function reconstruction relies on representation theorems. For acoustic waves, it has been shown theoretically and observationally that a representation theorem of the correlation-type leads to the retrieval of the Green's function by cross-correlating fluctuations recorded at two locations and excited by uncorrelated sources. We extend the theory to any system that satisfies a linear partial differential equation and define an 'interferometric operation' that is more general than cross-correlation for the reconstruction. We analyse Green's function reconstruction for perturbed media and establish a representation theorem specifically for field perturbations. That representation is then applied to the general treatment of scattering problems, enabling interpretation of the contributions to Green's function reconstruction in terms of direct and scattered waves. Perhaps surprising, Green's functions that account for scattered waves cannot be reconstructed from scattered waves alone. For acoustic waves, retrieval of scattered waves also requires cross-correlating direct and scattered waves at receiver locations. The addition of cross-correlated scattered waves with themselves is necessary to cancel the spurious events that contaminate the retrieval of scattered waves from the cross-correlation of direct with scattered waves. We illustrate these concepts with numerical examples for the case of an open scattering medium. The same reasoning holds for the retrieval of any type of perturbations and can be applied to perturbation problems such as electromagnetic waves in conductive media and elastic waves in heterogeneous media.

Key words: Interferometry; Theoretical seismology; Wave scattering and diffraction.

1 INTRODUCTION

The extraction of Green's functions from wave field fluctuations has recently received considerable attention. The technique, known in much of the literature as interferometry, is described in tutorials (Curtis *et al.* 2006; Larose *et al.* 2006; Wapenaar *et al.* 2008) and has been applied to a large variety of fields including ultrasonics (Lobkis & Weaver 2001; Weaver & Lobkis 2001; Roux & Fink 2003; Malcolm *et al.* 2004), global (Campillo & Paul 2003; Sabra *et al.* 2005a; Shapiro *et al.* 2005; Ruigrok *et al.* 2008) and exploration (Bakulin & Calvert 2006; Miyazawa *et al.* 2008) seismology, helioseismology (Rickett & Claerbout 1999), medical imaging (Sabra *et al.* 2007), structural engineering (Snieder & Safak 2006; Thompson & Snieder 2006; Kohler *et al.* 2007) and ocean acoustics (Roux & Kuperman 2004; Sabra *et al.* 2005b). The theory relies on representation theorems (of either the convolution or correlation type) and allows for the retrieval of Green's functions for acoustic (Wapenaar & Fokkema 2006), elastic (Snieder 2002; Wapenaar *et al.* 2004; Van Manen *et al.* 2006) and electromagnetic (Wapenaar *et al.* 2006; Slob *et al.* 2007; Slob & Wapenaar 2009) waves. For acoustic media, the impulse response between two receivers is retrieved by cross-correlating and summing the signals

recorded by the two receivers for uncorrelated sources enclosing the studied system. This process, sometimes referred to as the virtual source method (Bakulin & Calvert 2006), is equivalent to having a source at one of the receiver locations. Further studies have extended the concept to a wide class of linear systems (Wapenaar & Fokkema 2004; Wapenaar *et al.* 2006; Snieder *et al.* 2007; Gouédard *et al.* 2008; Weaver 2008) and our work aims to accomplish the same objective.

We explore a general formulation of a representation theorem for any system that satisfies a linear partial differential equation (or, mathematically, for any field in the appropriate Sobolev space). In particular, this formulation involves no assumption of spatial reciprocity or time-reversal invariance. We introduce a bilinear interferometric operator as a means of reconstructing the Green's function. We study the influence of perturbations on the interferometric operator and thereby derive a general representation theorem for perturbed media. The perturbed field can be retrieved by using a process characterized by the interferometric operation, which is generally more complex than cross-correlation. For common systems, this interferometric operation can be simplified using the symmetry properties of differential operators. We apply the theory to scattering problems and illustrate the approach with an example involving

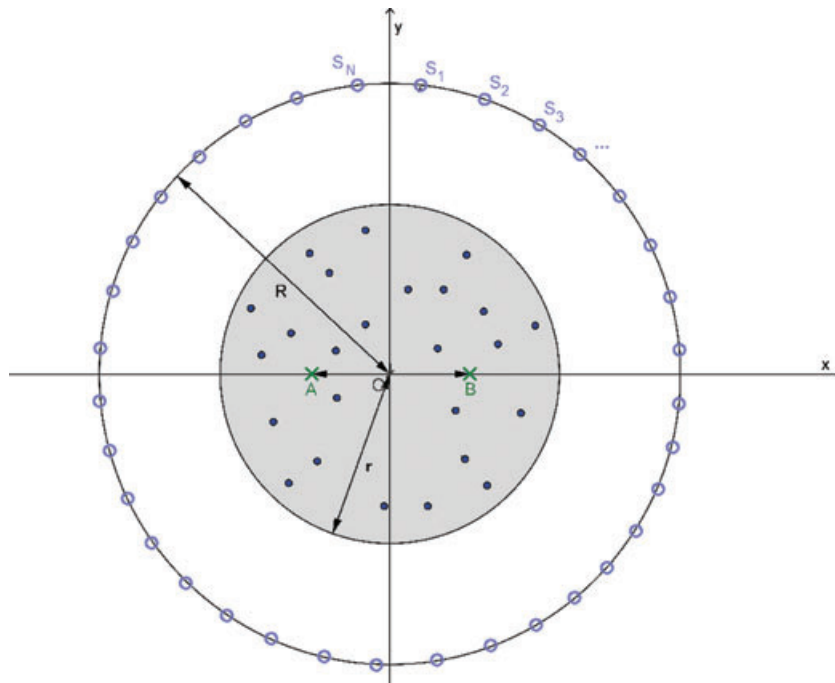


Figure 1. Two receivers, A and B , separated by a distance $d = 1.9$ km, are embedded in a 2-D acoustic scattering medium (unperturbed velocity $c_0 = 3.8$ km s $^{-1}$) characterized by n uniformly distributed isotropic point scatterers localized inside a circle of radius $r = 1.0$ km. A dense distribution of $N = 1000$ sources evenly spaced along a circle of radius $R = 4.0$ km surrounds the medium. The source is a band-limited signal with central frequency $\omega_0 = 100$ Hz and frequency range $\Delta\omega = 20$ Hz. For $n = 500$, the heterogeneous medium is considered strongly scattering. For $n = 10$, the scattering regime is weak.

scattered acoustic waves, obtaining a result that concurs with that published by Vasconcelos *et al.* (2009) on the representation theorem for scattering in acoustic media. In geophysics, applications of perturbation reconstruction exist in the areas of, for example, crustal seismology, seismic imaging, well monitoring and waveform inversion.

After exposing this general representation theorem for perturbed media, we give an innovative interpretation of Green's function reconstruction. To emphasize the connection between the general formulation and the particular case of scattering problems, we refer to field perturbation as scattered field and unperturbed field as direct field. Perturbation retrieval can be understood in terms of interferences between unperturbed fields and field perturbations. One might think that field perturbations can be reconstructed with contributions from just field perturbations alone. However, the retrieval of field perturbations requires the interferences with unperturbed fields. For acoustic media, this means that the scattering response between two receivers cannot be retrieved by cross-correlating only late coda waves. Here, the scattering response is defined as the superposition of the causal and acausal scattering Green's functions between the two points. In the numerical experiments (see Fig. 1), two receivers are embedded in a scattering medium and surrounded by sources that are activated separately and consequently, generate uncorrelated wavefields. The numerical scheme is based on computation of the analytical solution to the 2-D heterogeneous acoustic wave equation for a distribution of isotropic point scatterers (Groenenboom & Snieder 1995). In Fig. 2, we compare the actual scattering response for a source at the receiver location with the signal reconstructed by cross-correlating and summing the scattered waves recorded at the receiver positions. For a strongly scattering medium [average wavelength is of the same order as the scattering mean free path (Tourin

et al. 2000)], Fig. 2(a) shows that the reconstruction completely fails to retrieve the scattering response from cross-correlation of only the scattered waves recorded at the receiver locations. The reconstructed wave with only scattered waves is totally inaccurate: the early arrivals are non-physical because they do not respect causality, arriving before the minimum travelttime between the two receivers, while the late arrivals show no resemblance to the actual scattering response. Accurate retrieval of the scattered waves requires instead contributions from both direct and scattered waves, as shown in Fig. 2(b).

In this paper, we provide an interpretation of this result; one can find a similar approach by Halliday & Curtis (2009) and Snieder & Fleury (2010), the latter of which describes the case of multiple scattering by discrete scatterers. In Snieder & Fleury (2010), we identify different scattering paths, show their contributions to the retrieval of either physical or non-physical arrivals and analyse how cancellations occur to allow the scattering Green's function to emerge. Our interpretation, along with that given by Halliday & Curtis (2009), leads to the same important conclusion: the cross-correlation of purely scattered waves does not allow extraction of the correct scattered waves.

The paper is organized as follows. In Section 2, we describe the general systems under consideration and introduce the concept of perturbation. In Section 3, we define the interferometric operator and its relation to representation theorems, emphasizing the influence of perturbations on this operator. Section 4 presents the general representation theorems for perturbations that follow this approach. In Section 5, we apply this theory to interpret the reconstruction of Green's function perturbations; Section 6 offers discussions and conclusions.

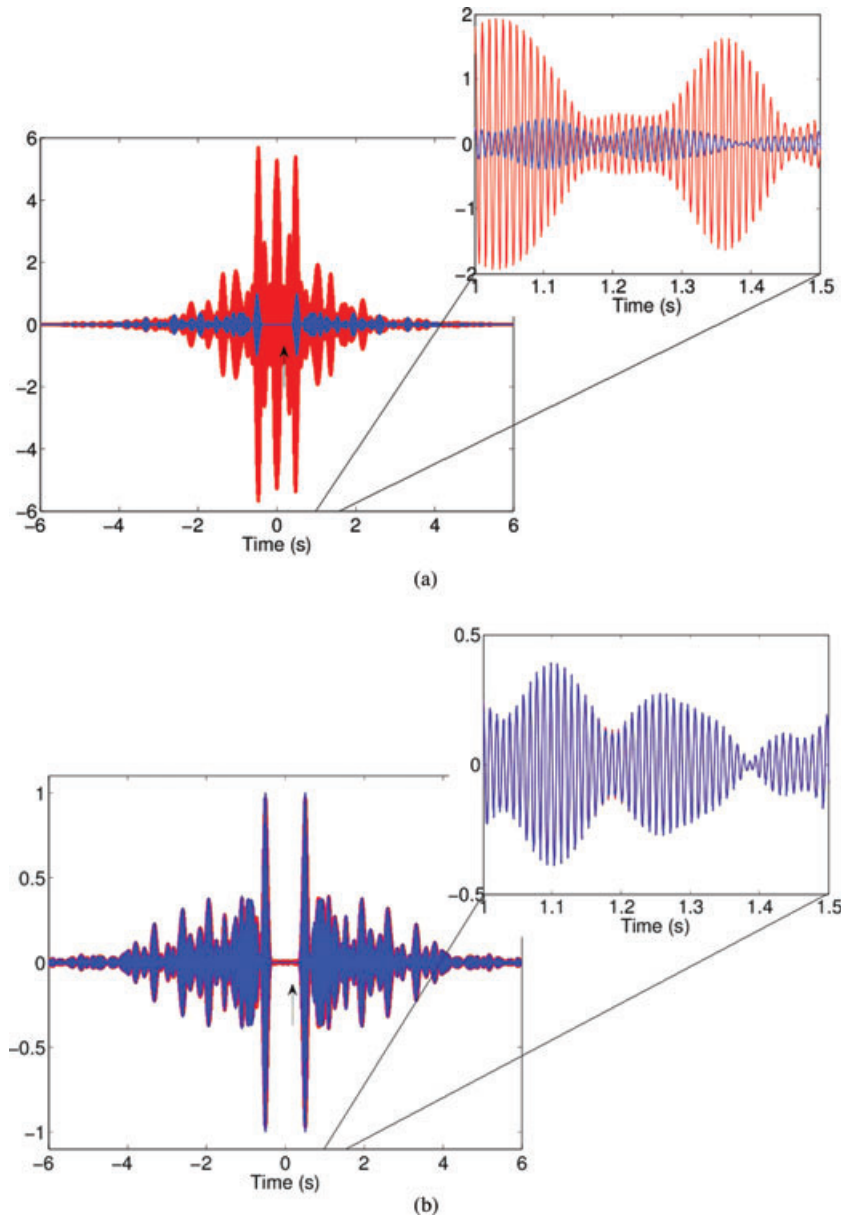


Figure 2. The blue curves show the actual scattering response (superposition of the causal and acausal scattering Green’s functions) between two points embedded in a strongly heterogeneous medium. The red curves represent the wave reconstructed by cross-correlating the waves recorded by two receivers at the same locations. Note the black arrow, which corresponds to the time of the first expected physical arrival. In panel (a), only scattered waves are cross-correlated. The reconstruction fails no matter how dense is the distribution of sources enclosing the medium. This failure of interferometry is not caused by restrictions of source distribution, aperture, or equipartitioning, but is a consequence of the missing contribution of recorded direct waves. In panel (b), both direct and scattered waves are cross-correlated. The latter result confirms that the scattering response can be retrieved by interferometry.

2 GREEN’S FUNCTION PERTURBATIONS FOR GENERAL SYSTEMS

Consider a general system governed by a linear partial differential equation in the frequency domain. To avoid the complexity of formalism that could obscure the main purpose of this paper, we leave the vector case for Appendix A. Let the complex scalar field $u_0(\mathbf{r}, \omega)$ be defined in a volume D_{tot} . One can adapt the result of this work to the time domain using the Fourier convention $u_0(\mathbf{r}, t) = \int u_0(\mathbf{r}, \omega) \exp(-j\omega t) d\omega$. Henceforth, we suppress the frequency dependence of variables and operators. The unperturbed

field $u_0(\mathbf{r})$ is a solution of the unperturbed equation

$$H_0(\mathbf{r}) \cdot u_0(\mathbf{r}) = s(\mathbf{r}), \tag{1}$$

where H_0 is the linear differential operator and s is the source term, associated with the unperturbed system. The dot denotes a tensor contraction when vectors and tensors are considered. For scalars, the dot reduces to multiplication of fields and action of operators on fields. For acoustic waves, one may define H_0 as the propagator for non-uniform density media: $H_0 = \nabla \cdot (\rho_0^{-1} \nabla) + \rho_0^{-1} \omega^2 / c_0^2$, where ρ and c denote density and velocity, respectively.

Assuming a perturbation of the system, confined to a subvolume D_V of the total domain D_{tot} , the perturbed field $u_1(\mathbf{r})$ follows from

$$H_1(\mathbf{r}) \cdot u_1(\mathbf{r}) = s(\mathbf{r}) \quad (2)$$

$$H_0(\mathbf{r}) \cdot u_1(\mathbf{r}) = V(\mathbf{r}) \cdot u_1(\mathbf{r}) + s(\mathbf{r}), \quad (3)$$

where V is the perturbation operator and $H_1 = H_0 - V$ is the linear differential operator associated with the perturbed system. For example, for acoustic waves, with a change in velocity for the medium, the perturbation operator is $V = \rho_0^{-1} \omega^2 / c_0^2 (1 - c_0^2 / c_1^2)$. Alternatively, a change in experimental conditions might imply a variation in density; then, a way to account for this perturbation is to consider $V = (\rho_0^{-1} - \rho_1^{-1}) \omega^2 / c_0^2 + \nabla \cdot ((\rho_0^{-1} - \rho_1^{-1}) \nabla)$ for homogeneous density changes [or follow Martin (2003) for inhomogeneous density changes]. One could also neglect attenuation in the medium in the first approximation and correct for it by introducing the perturbation $V = j\alpha$, where α is a real coefficient, function of the attenuation mechanisms. We are free to arbitrarily choose or even interchange the reference 0 and perturbed 1 states for any perturbation problem. Indeed, the perturbation need not necessarily introduce more complexity; its definition depends on the characteristics of the perturbation problem that one tries to solve.

For a problem to be well-defined, one needs to specify boundary conditions. Assume that the boundary conditions are unperturbed and consider a regular problem with homogeneous boundary conditions:

$$B(\mathbf{r}) \cdot u_{0,1}(\mathbf{r}) = 0, \quad \mathbf{r} \in \delta D_{tot}, \quad (4)$$

where B denotes the linear boundary condition operator that acts on the boundary δD_{tot} of total volume D_{tot} . One can, for example, apply the Sommerfeld radiation condition for acoustic waves. In general, however, the boundary conditions need not be limited to being homogeneous. In Appendix B, we extend our reasoning to any unperturbed boundary conditions.

The Green's functions $G_0(\mathbf{r}, \mathbf{r}_S)$ and $G_1(\mathbf{r}, \mathbf{r}_S)$ for both unperturbed and perturbed systems are defined as solutions for an impulsive source at location \mathbf{r}_S ,

$$s(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_S). \quad (5)$$

From the above equations, one obtains the familiar relation between unperturbed and perturbed Green's functions, known as the Lippmann–Schwinger equation (Rodberg & Thaler 1967):

$$G_1(\mathbf{r}, \mathbf{r}_S) = G_0(\mathbf{r}, \mathbf{r}_S) + \int_{D_{tot}} G_0(\mathbf{r}, \mathbf{r}_1) \cdot V(\mathbf{r}_1) \cdot G_1(\mathbf{r}_1, \mathbf{r}_S) d^3 r_1 \quad (6)$$

The perturbation operator V vanishes outside of D_V so that we can replace the volume of integration in eq. (6) by D_V , or by any subvolume D of total domain D_{tot} that contains D_V (see Fig. 3). Let's choose such a volume D and introduce the tensor notation,

$$(\mathbf{F} | \mathbf{O} | \mathbf{G})_D \equiv \int_D \mathbf{F}(\mathbf{r}) \cdot \mathbf{O}(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r}) d^3 r, \quad (7)$$

so that the Lippmann–Schwinger equation can be written as

$$G_1(\mathbf{r}, \mathbf{r}_S) = G_0(\mathbf{r}, \mathbf{r}_S) + (G_0(\mathbf{r}, \mathbf{r}_1) | V(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S))_D. \quad (8)$$

The variable of integration is always the space coordinate that is shared by all functions and operators present inside the round brackets. For brevity, we don't specify the domain of integration when equal to D and note $(\mathbf{F} | \mathbf{I} | \mathbf{G}) = (\mathbf{F} | \mathbf{G})$ for the case of the identity operator I . Finally, we define the Green's function perturbation or

scattering Green's function, that characterizes the field perturbation $u_S(\mathbf{r}) = u_1(\mathbf{r}) - u_0(\mathbf{r})$, as

$$G_S(\mathbf{r}, \mathbf{r}_S) = G_1(\mathbf{r}, \mathbf{r}_S) - G_0(\mathbf{r}, \mathbf{r}_S), \quad (9)$$

or

$$G_S(\mathbf{r}, \mathbf{r}_S) = (G_0(\mathbf{r}, \mathbf{r}_1) | V(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S)). \quad (10)$$

To clarify the terminology used throughout this paper, unperturbed field, perturbed field and field perturbation denote u_0 , u_1 and u_S , respectively.

3 DEFINITION OF THE INTERFEROMETRIC OPERATOR

To establish a representation theorem for perturbations, we first derive a general expression for Green's function retrieval by using a representation theorem of the correlation type (Wapenaar & Fokkema 2006). Given the volume of interest D , defined according to Fig. 3, consider two states of the field u , labelled A and B , governed by the partial differential equation $\mathcal{L}_{A,B}$,

$$\mathcal{L}_{A,B} : H(\mathbf{r}) \cdot u_{A,B}(\mathbf{r}) = s_{A,B}(\mathbf{r}), \quad (11)$$

where the subscript A,B refers to either state A or B . Following Fokkema & Van den Berg (1993) and Fokkema *et al.* (1996), we evaluate $(u_A | \overline{\mathcal{L}}_B) - (\overline{u}_B | \mathcal{L}_A)$, where \overline{f} denotes the complex conjugate of f (for an operator O , understand \overline{O} as changing in O every imaginary number j to $-j$); consequently,

$$(u_A | \overline{H} | \overline{u}_B) - (\overline{u}_B | H | u_A) = (u_A | \overline{s}_B) - (\overline{u}_B | s_A). \quad (12)$$

For impulsive sources, $s_{A,B}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_{A,B})$ and the fields $u_{A,B}(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_{A,B})$, the Green's functions in states A and B , so (12) becomes the general representation theorem of correlation-type for interferometry,

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = (G(\mathbf{r}, \mathbf{r}_A) | \overline{H}(\mathbf{r}) | \overline{G}(\mathbf{r}, \mathbf{r}_B)) - (\overline{G}(\mathbf{r}, \mathbf{r}_B) | H(\mathbf{r}) | G(\mathbf{r}, \mathbf{r}_A)). \quad (13)$$

This result is a general extension of the representation theorem in Snieder *et al.* (2007). To interpret and characterize the Green's function reconstruction more conveniently, we define the operator I_H ,

$$I_H \{f, g\} \equiv (f | \overline{H} | g) - (g | H | f), \quad (14)$$

so that the general representation theorem can be written as

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = I_H \{G(\mathbf{r}, \mathbf{r}_A), \overline{G}(\mathbf{r}, \mathbf{r}_B)\}. \quad (15)$$

The operation $I_H \{ \cdot, \cdot \}$ describes how Green's functions in a subvolume D 'interfere' to reconstruct the Green's function between the two points A and B . We consequently refer to I_H as the interferometric operator, associated with H , that acts on functions f and g and call the result of operation (14) an interference between f and g . The interferometric operation generalizes the concept of interferometry by cross-correlation for acoustic waves to a wider class of physical systems. For the specific case of acoustic waves in lossless media, the interferometric operation is the following volume integration:

$$I_H \{f, g\} = \int_D [f(\mathbf{r}) \nabla \cdot (\rho^{-1} \nabla g)(\mathbf{r}) - g(\mathbf{r}) \nabla \cdot (\rho^{-1} \nabla f)(\mathbf{r})] d^3 r. \quad (16)$$

Using Green's theorem, this volume integral becomes an integral over the bounding surface δD enclosing volume D :

$$I_H \{f, g\} = \oint_{\delta D} \rho^{-1}(\mathbf{r}) [f(\mathbf{r}) \nabla g(\mathbf{r}) - g(\mathbf{r}) \nabla f(\mathbf{r})] \cdot \hat{n} d^2 r, \quad (17)$$

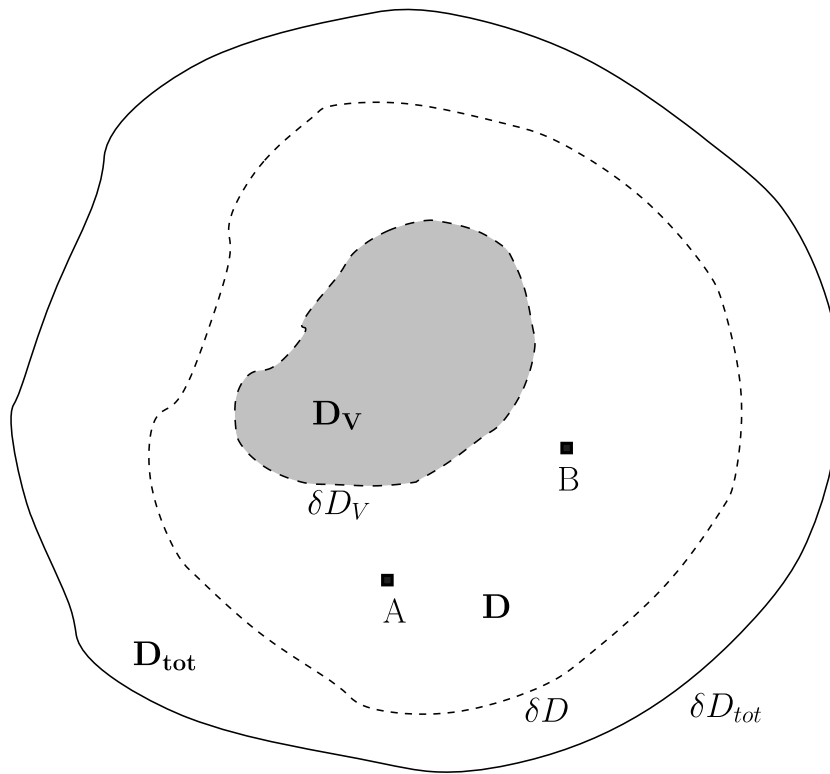


Figure 3. The studied physical system is defined over the total volume D_{tot} bounded by the surface δD_{tot} . Inside D_{tot} , the system is perturbed in the region D_V with boundary δD_V . The region of interest, for which we study the interferometric operator, is D . The volume D contains the domain of perturbation D_V and its boundary δD encloses two points A and B for which representation theorems (25) and (30) are defined.

where \hat{n} is the outward unit normal vector at \mathbf{r} . Then, eq. (15) retrieves the familiar representation theorem for acoustic waves (Wapenaar & Fokkema 2006):

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\delta D} \rho^{-1}(\mathbf{r}) [G(\mathbf{r}, \mathbf{r}_A) \nabla \overline{G}(\mathbf{r}, \mathbf{r}_B) - \overline{G}(\mathbf{r}, \mathbf{r}_B) \nabla G(\mathbf{r}, \mathbf{r}_A)] \cdot \hat{n} d^2r. \tag{18}$$

Returning to the general case, just as the unperturbed linear partial differential operator H_0 becomes $H_1 = H_0 - V$ after perturbing the system, the interferometric operators for unperturbed and perturbed systems I_0 and I_1 , relate in the following way:

$$I_0 = I_{H_0} \\ I_1 = I_0 - I_V. \tag{19}$$

Note that, in general, I_0 and I_1 differ; that is, the interferometric operator is perturbed for a perturbed system. The exception ($I_1 = I_0$) occurs when $I_V = 0$. Consider, for example, the acoustic case previously described. The unperturbed Green's function is retrieved using expression (18) and, for a perturbation in velocity only,

$$I_V\{f, g\} = \int_D \frac{\omega^2}{\rho_0(\mathbf{r})c_0(\mathbf{r})^2} \times \left[\left(1 - \left(\frac{c_0(\mathbf{r})}{c_1(\mathbf{r})} \right)^2 \right) - \left(1 - \left(\frac{c_0(\mathbf{r})}{c_1(\mathbf{r})} \right)^2 \right) \right] f(\mathbf{r})g(\mathbf{r})d^3r = 0, \tag{20}$$

so $I_1 = I_0$. If, instead, density rather than velocity is perturbed,

$$I_V\{f, g\} = \oint_{\delta D} (\rho_0^{-1}(\mathbf{r}) - \rho_1^{-1}(\mathbf{r})) \times [f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})] \cdot \hat{n} d^2r \neq 0. \tag{21}$$

Therefore, the interferometric operator changes ($I_1 \neq I_0$) with such a perturbation. Similarly, with a perturbation in attenuation,

$$I_V\{f, g\} = -2j \int_D \alpha(\mathbf{r})g(\mathbf{r})f(\mathbf{r})d^3r \neq 0. \tag{22}$$

These examples illustrate that, in general, the same interferometric operation cannot be used to reconstruct both perturbed and unperturbed Green's functions; we need to estimate the perturbation of the interferometric operator, I_V , to apply interferometry for perturbed media. As seen in eqs (21) and (22), the interferometric operator in general requires knowledge of medium properties for the perturbed system, a limiting factor because usually we know only the unperturbed medium properties. Eq. (20), however, is a specific example of an interferometric operator that remains unperturbed ($I_0 = I_1$) for nonzero perturbation. For benign cases such as this one, we need only know or estimate unperturbed medium properties and measure or extrapolate both perturbed and unperturbed fields, to reconstruct the Green's functions, which make these cases of major interest.

Before investigating such systems for which the interferometric operator is unperturbed ($I_V = 0$), let us discuss another characteristic of the interferometry operator. Starting by reformulating the general representation theorem for both perturbed and unperturbed media, we retrieve the Green's functions using

$$G_{0,1}(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_{0,1}(\mathbf{r}_A, \mathbf{r}_B) = I_{0,1}\{G_{0,1}(\mathbf{r}, \mathbf{r}_A), \overline{G}_{0,1}(\mathbf{r}, \mathbf{r}_B)\}. \tag{23}$$

This expression clearly depends on the properties of the interferometric operator and, according to definition (14), the reconstruction involves integration over the volume D . Because the integrand is a function of differential operators H_0 or H_1 and of the Green's functions between any point in D and points A or B , we need to know H_0 , V and the Green's functions for all points in the volume D to apply

the interferometric operator and retrieve the Green's functions between A and B . In particular, the estimation of the Green's functions for all points in D requires having receivers (or sources for spatially reciprocal systems) throughout the entire volume D . To apply interferometry in practice, this requirement for receivers (or sources) over the entire volume is yet more limiting than the need to estimate perturbations of the medium properties; it will severely restrict the possibility of retrieving even unperturbed Green's functions.

In practice, we are interested in systems for which we can reconstruct Green's functions with a limited number of sources and receivers. Just as for acoustic waves in eq. (18), we therefore aim for problems that enable us to transform the integration over volume D in expression (14) into integration over its boundary δD . This transformation allows significant reduction in the number of sources. In Appendix C, we show that this transformation can be done if and only if operators are self-adjoint. For non self-adjoint operators, an extension of representation theorem (23) may apply. We also demonstrate that the self-adjoint symmetry of the operators implies spatial reciprocity under specific boundary conditions. Spatially reciprocal systems are of major interest for interferometry applications because these systems allow for the permutation of the role of sources and receivers in the formulation of representation theorems (23) and are therefore favourable for Green's function retrieval. In addition, the transformation of volume integrals into surface integrals also constrains to just the surface δD the medium properties that must be known for the reconstruction. For perturbation problems that we are considering (see Fig. 3), we can always find a boundary of integration δD (for example, δD_{tot}) along which the system is unperturbed (there are no changes of the medium properties along δD). Then, under the assumption that H_0 and V are self-adjoint, the interferometric operation associated with this particular volume D can be reduced to an integration over δD and the interferometric operator is then unperturbed under the assumption that the properties of the medium are unchanged along this boundary. Consequently, we can reconstruct the perturbed Green's function independently of the perturbations in the rest of the volume. For example, for a perturbation of densities in an acoustic medium, expression (21) illustrates that the interferometric operator is unperturbed ($I_V = 0$) when the density is unchanged on the boundary δD . For an attenuative acoustic medium, however, expression (22) shows that the perturbation V breaks the self-adjoint symmetry of the operator H_0 . This prevents us to reduce the representation theorem to only a surface integral and links to further discussions on interferometry for dissipative media (Snieder 2007; Snieder *et al.* 2007).

To summarize, interferometry can be interpreted as the application of an interferometric operator. This technique is practical for systems characterized by self-adjoint operators and for perturbation problems where the interferometric operator is unperturbed.

4 REPRESENTATION FOR GREEN'S FUNCTION PERTURBATIONS

In the previous section, we established a general representation theorem for perturbed systems. Here, we derive a representation for field perturbations. This general representation differs from the traditional representation theorem for the special case of scattered acoustic waves (Vasconcelos *et al.* 2009) because, in general, we must take into account the perturbation of the interferometric operator. The perturbation of Green's function, defined in Section 2, can be retrieved by interferometry by taking the difference of the two

eqs (23) for the perturbed and unperturbed states to give

$$G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) = I_1\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} - I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \quad (24)$$

Using relation (19) between unperturbed and perturbed interferometric operators, we have

$$G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) = I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} - I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} - I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}. \quad (25)$$

Eq. (25) is a general representation theorem for field perturbations. In addition, the interferometric operator is bilinear, that is, $I_H\{\alpha f, g\} = \alpha I_H\{f, g\}$, $I_H\{f, g+h\} = I_H\{f, g\} + I_H\{f, h\}$ and $I_H\{f+g, h\} = I_H\{f, h\} + I_H\{g, h\}$. We exploit the bilinearity of I_0 and expand $I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ in terms of unperturbed fields and field perturbations:

$$I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} = I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \quad (26)$$

This decomposition allows for the identification of different types of interference between unperturbed Green's functions and Green's function perturbations. Then, inserting eq. (26) into representation theorem (25), gives

$$G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} - I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}. \quad (27)$$

Representation theorem (27) illustrates that the retrieval of Green's function perturbations requires a combination of not just interferences between Green's function perturbations, but interferences between both unperturbed Green's functions and Green's function perturbations. In Section 5, we analyse the individual contributions of the different terms on the right-hand side of eq. (27) to the reconstruction. Note in particular the term $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ that represents the interference between perturbed Green's functions associated with the operator V and accounts for the perturbation of the interferometric operator. We prefer to consider situations for which $I_V = 0$ because in such cases,

$$G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}. \quad (28)$$

Representation theorem (28) is a function of only the unperturbed interferometric operator I_0 and, consequently, depends only on the properties of the unperturbed medium. For these special cases, such as lossless acoustic media with velocity perturbation, the perturbation retrieval does not require an estimation of the perturbation V .

Now, let us return to the general case where I_V can be nonzero and establish another form of representation theorem for perturbations, one that characterizes only the causal Green's function perturbation,

$G_S(\mathbf{r}_B, \mathbf{r}_A)$, rather than the superposition of the causal and acausal functions, $G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B)$. This representation will help us analysing the individual contribution of the interference between direct and scattered fields to the partial retrieval of the scattered field $G_S(\mathbf{r}_B, \mathbf{r}_A)$. Rearranging relation (23) for unperturbed systems and inserting it into eq. (10) yields

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= (I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \\ &\quad \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + \overline{G}_0(\mathbf{r}_1, \mathbf{r}_B))|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A) \\ &= I_0\{(G_0(\mathbf{r}, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &\quad + (\overline{G}_0(\mathbf{r}_1, \mathbf{r}_B)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)). \end{aligned} \quad (29)$$

Using once again expression (10), which defines the Green's function perturbation, we identify the first term on the right-hand side of (29) with $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$ to obtain

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &\quad + (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)). \end{aligned} \quad (30)$$

This representation theorem for perturbations generalizes to any physical system the representation theorem for the special case of acoustic waves (Vasconcelos *et al.* 2009),

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= \oint_{\partial D} \rho_0^{-1}(\mathbf{r})[G_S(\mathbf{r}, \mathbf{r}_A)\nabla\overline{G}_0(\mathbf{r}, \mathbf{r}_B) \\ &\quad - \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\nabla G_S(\mathbf{r}, \mathbf{r}_A)] \cdot \hat{\mathbf{n}}d^2r \\ &\quad + \int_D \overline{G}_0(\mathbf{r}, \mathbf{r}_B)V(\mathbf{r})G_1(\mathbf{r}, \mathbf{r}_A)d^3r. \end{aligned} \quad (31)$$

Our derivation of eq. (30) assumes that $D_V \subset D$. However, eq. (30) can be extended to any perturbation domain $D_V \subset D_{\text{tot}}$. Rewrite eq. (10) by specifying the domain of integration,

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= (G_0(\mathbf{r}_B, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \cap D} \\ &\quad + (G_0(\mathbf{r}_B, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \setminus D}, \end{aligned} \quad (32)$$

where $D_V \cap D$ and $D_V \setminus D$ denote the intersection and complement of D in D_V , respectively. The derivation of eq. (29) can be applied to the first term of the right-hand side of eq. (32) and

$$\begin{aligned} &(G_0(\mathbf{r}_B, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \cap D} \\ &= I_0\{(G_0(\mathbf{r}, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \cap D}, \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &\quad + (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A))_{D_V \cap D}. \end{aligned} \quad (33)$$

For the second term, because it can be shown that $\mathbf{r}_1 \notin D$ implies a modification of relation (23) such that $G_0(\mathbf{r}_B, \mathbf{r}_1) = I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$,

$$\begin{aligned} &(G_0(\mathbf{r}_B, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \setminus D} \\ &= I_0\{(G_0(\mathbf{r}, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))_{D_V \setminus D}, \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \end{aligned} \quad (34)$$

The sum of eqs (33) and (34) reduces to the original representation theorem (30). Note that for a domain of perturbation D_V outside of D , the representation theorem reduces to

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \quad (35)$$

Representation theorems (25) and (30) offer the possibility of extracting field perturbations (e.g. scattered waves) between points A and B , as if one of these points acts as a source. They allow calculation of perturbation propagation between these two points without the need for a physical source at either of the two locations. These representation theorems have potential for estimating

not only perturbations in fields but perturbations in medium properties by treating expression (30) as an integral equation for the perturbation V given the field perturbation G_S . They can therefore be used for detecting, locating, monitoring and modelling medium perturbations. In geoscience, this theory has application to a diversity of techniques including passive imaging using seismic noise, seismic migration, modelling for inversion of electromagnetic data and remote monitoring of hydrocarbon reservoirs, aquifers and CO₂ injection for carbon sequestration.

5 ANALYSIS OF THE DIFFERENT CONTRIBUTIONS TO THE RETRIEVAL OF PERTURBATIONS

Here, we analyse the different terms that contribute to representation theorem (27) for perturbations. In particular, we interpret the contribution of the interference between field perturbations, corresponding to the term $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$ and explain why perturbations cannot be reconstructed by using solely the interference between perturbations; that is, the reconstruction of perturbations requires knowledge of the unperturbed fields for the system. We show that the contribution of the interference between unperturbed fields and field perturbations, corresponding to the terms $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$ and $I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$, is essential for the retrieval process. These contributions are responsible for retrieving the field perturbations plus extra volume terms. For some cases, these extra volume terms are purely spurious and contaminate the retrieval process. The interference between just the field perturbations is necessary to cancel these extra volumes terms. To a certain extent, the cancellation mechanism involved in the reconstruction process can be connected to the general optical theorem as discussed below.

5.1 Partial retrieval of field perturbations

First, consider the contributions of the interferences between unperturbed fields and field perturbations. Rearranging the terms in representation theorem (30), we have the two following expressions, eq. (37) being the negative conjugate of eq. (36):

$$\begin{aligned} &I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &= G_S(\mathbf{r}_B, \mathbf{r}_A) - (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)), \end{aligned} \quad (36)$$

$$\begin{aligned} &I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} \\ &= -\overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) + (G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B)). \end{aligned} \quad (37)$$

Eqs (36) and (37) show that the terms $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$ and $I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$ contribute to the causal and acausal Green's function perturbation between A and B , respectively. Note, however, the two additional volume integrals that depend on the perturbation operator:

$(\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A))$ and $(G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B))$. Their presence thus leads to a partial retrieval of field perturbations. The retrieval of field perturbations is incomplete because the two volume integrals $(\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A))$ and $(G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B))$ can both reconstruct missing contributions of the estimate of the Green's function perturbation and contaminate this estimate with spurious contributions (called spurious arrivals by Snieder *et al.* 2008). In general, we cannot neglect the contributions of these extra volume terms because

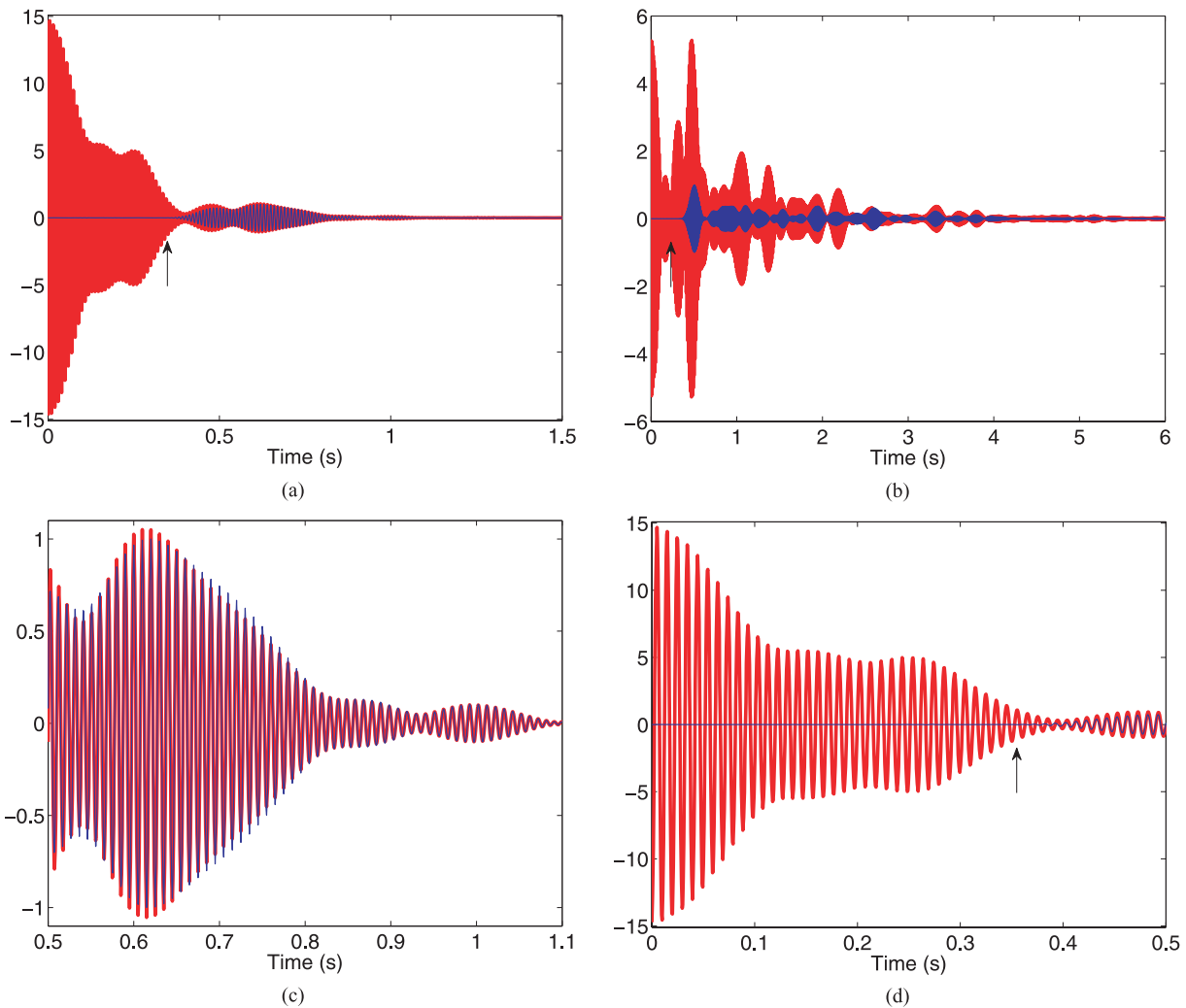


Figure 4. The causal part of the actual scattering response (blue curves) between two points embedded in heterogeneous media is compared to the reconstructed wave (red curves) obtained by cross-correlating direct and scattered waves recorded by two receivers at the same locations. Panels (a) and (b) show the signals for a weakly and strongly scattering medium, respectively. Panel (c) and (d) provide zooms on the late and early parts of experiment in weakly scattering regime, respectively. In both scattering regimes, the reconstruction is inaccurate. The weakly scattering case, however, suggests a partial retrieval of the scattering response: the reconstructed and reference signals are similar in their late parts (Panel c) while the early part of the reconstructed signal (i.e., the portion before the time of the direct arrival, denoted by the arrow) is purely erroneous (Panel d) and contains only the spurious arrivals.

they do not vanish regardless of the subspace D under consideration. Depending on the perturbation V , however, their contribution can be relatively small. For cases such as the acoustic problems formulated by Snieder *et al.* (2008), Snieder & Fleury (2010) and Vasconcelos *et al.* (2009) in some of their examples, the contributions of these volume integrals only contribute to spurious arrivals while all of the physical arrivals arise from $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$ and $I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$. The summation of eqs (36) and (37) thus gives a retrieval of the field perturbation, $G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B)$, contaminated with extra volume terms.

To get insight into the physical meaning of this partial reconstruction, let us particularize the general description of eqs (36) and (37) to the case of acoustic waves in which direct waves interfere with scattered waves. Fig. 4 illustrates the reconstruction obtained by cross-correlating just direct and scattered waves for both weakly and strongly scattering media (Figs 4a and b, respectively). Interestingly, for a weakly scattering medium [average wavelength is

much smaller than the scattering mean free path], Fig. 4(c) shows a reconstructed signal that fully retrieves the late portion of the scattering response. The early part of the signal, however, contains strong nonphysical arrivals, prior to the true first arrival (arrow), as seen in Fig. 4(d). These observations suggest that while the signal reconstructed by cross-correlating direct and scattered waves does contain the scattering response, it is contaminated by spurious arrivals. Fig. 4(a) shows that, for a strongly scattering medium, the reconstructed signal is contaminated so severely that no similarities can be found between the reconstructed and reference signals; the contribution of the spurious arrivals dominates the reconstruction. In summary, because the physical nature of the spurious arrival is the same for both weakly and strongly scattering media, cross-correlating direct and scattered waves retrieves the scattered waves but generates unexpected arrivals that can be more intense than the useful signal. These spurious arrivals, corresponding to the extra volume terms, must cancel in order for the retrieval of scattered waves to be completed.

5.2 Cancellation of the extra volume terms

The interference between direct and scattered waves, that is, the first two terms in (27), partially retrieves the scattered waves. We are interested in studying the mechanism for cancelling the extra volume terms described in the previous subsection. According to representation theorem (27), completion of the reconstruction requires the additional contributions from the interferences $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$ and $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$. In the introduction, we showed numerically that the interference between scattered waves alone does not correctly retrieve scattered waves. Taken individually, the interference between unperturbed fields and field perturbations, $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$ and $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}$, the interference between just the field perturbations $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$, or the interference $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ do not reconstruct field perturbations. The summation of all their contributions, however, is expected to accurately retrieve the perturbations and, consequently, cancel the extra volume terms.

We develop the following relation for the interference between field perturbations by rewriting $I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}$:

$$\begin{aligned}
 I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &= I_0\{(G_0(\mathbf{r}, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)), \\
 &\quad (\overline{G}_0(\mathbf{r}, \mathbf{r}_2)|\overline{V}(\mathbf{r}_2)|\overline{G}_1(\mathbf{r}_2, \mathbf{r}_B))\} = (I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \\
 &\quad \overline{G}_0(\mathbf{r}, \mathbf{r}_2)\}|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)|\overline{V}(\mathbf{r}_2)|\overline{G}_1(\mathbf{r}_2, \mathbf{r}_B)\} \\
 &= (([G_0(\mathbf{r}_2, \mathbf{r}_1) - \overline{G}_0(\mathbf{r}_1, \mathbf{r}_2)]|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)|\overline{V}(\mathbf{r}_2)| \\
 &\quad \times \overline{G}_1(\mathbf{r}_2, \mathbf{r}_B)) \\
 &= ((G_0(\mathbf{r}_2, \mathbf{r}_1)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)|\overline{V}(\mathbf{r}_2)|\overline{G}_1(\mathbf{r}_2, \mathbf{r}_B)) \\
 &\quad - ((\overline{G}_0(\mathbf{r}_1, \mathbf{r}_2)|\overline{V}(\mathbf{r}_2)|\overline{G}_1(\mathbf{r}_2, \mathbf{r}_B)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)),
 \end{aligned} \tag{38}$$

where we used expression (10) for field perturbations in the first identity, the bilinearity of I_0 in the second identity and representation theorem (23) in the third identity; so that finally,

$$\begin{aligned}
 I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &= (G_S(\mathbf{r}_1, \mathbf{r}_A)|\overline{V}(\mathbf{r}_1)|\overline{G}_1(\mathbf{r}_1, \mathbf{r}_B)) \\
 &\quad - (\overline{G}_S(\mathbf{r}_1, \mathbf{r}_B)|V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)).
 \end{aligned} \tag{39}$$

We next show that the interaction between Green's function perturbations indirectly retrieves the Green's function perturbation by contributing to the cancellation of the extra volume terms. We identify the right-hand side of eq. (39) as the complement of the contributions $-(\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A))$ and $(G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B))$ in eqs (36) and (37); that is, the summation of these integrals retrieves the term $-I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$. For cases where $I_V = 0$, the interaction between perturbations entirely cancels the extra volume terms,

$$\begin{aligned}
 I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &+ (G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B)) \\
 &\quad - (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)) = 0
 \end{aligned} \tag{40}$$

and the reconstruction is then completed by summing the contributions from eqs (36), (37) and (39) [the sum reduces to representation

theorem (28)]. For the general case ($I_V \neq 0$),

$$\begin{aligned}
 I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &+ (G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B)) \\
 &\quad - (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)) \\
 &= -I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}
 \end{aligned} \tag{41}$$

and the summation of eqs (36), (37) and (39) gives

$$\begin{aligned}
 (36) + (37) + (39) &= G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) \\
 &\quad + I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}.
 \end{aligned} \tag{42}$$

The retrieval is incomplete and does not produce the Green's function perturbation because of the term $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ that still contaminates the right-hand side of eq. (42). Accurate reconstruction requires an additional estimate of this interaction between perturbed fields associated to V .

In any case, one of the direct consequence for scattering problems is that we cannot reconstruct the scattering Green's function by merely using the contribution of scattered waves alone. This explains the failure of interferometry based solely on the interference of scattered waves, as shown in Fig. 2. The interference between Green's function perturbations nevertheless plays a fundamental role in the retrieval of the perturbation because they are needed to cancel the extra volume terms. Our numerical experiments illustrate this observation for scattered acoustic waves (Fig. 5). For both weakly and strongly scattering media, combining the contributions of both interference between direct and scattered waves and interference between just scattered waves cancels the spurious arrivals and reconstructs the superposition of the causal and acausal scattering Green's functions (see Figs 5e and f). Note, additionally, that in order for this experiment to be successful, the distribution of sources must be sufficiently dense on a close surface surrounding the receivers (see numerical set-up description in Fig. 1). Considerations of narrow aperture and limited number of sources are independent problems that limit the accuracy of reconstructions (Fan & Snieder 2009; Snieder 2004).

5.3 Connection with the general optical theorem

Above, we emphasize the process that leads to the reconstruction of perturbations. Interestingly, for problems with unperturbed interferometric operators, the interference between field perturbations alone contributes entirely to the cancellation of the extra volume terms that arise from the interferences between unperturbed fields and field perturbations in the reconstruction process, and rewriting eq. (40) gives

$$\begin{aligned}
 (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)) &- (G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B)) \\
 &= I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}.
 \end{aligned} \tag{43}$$

Eq. (43) with $\mathbf{r}_A = \mathbf{r}_B$ directly connects to the work of Carney *et al.* (2004) on the optical theorem that gives a similar relationship between scattering amplitude and extinguished power for scattering of scalar waves in an arbitrary background. Carney *et al.* (2004) show how their derivation provides insight into the interference mechanisms that ensure energy conservation in scattering. For $\mathbf{r}_A = \mathbf{r}_B$, eq. (43) suggests a relation between cancellation of the extra volume terms and conservation of scattering energy. In a sense, we can also interpret this mechanism as an extension of the general optical theorem, as has been suggested for acoustic waves (Snieder *et al.* 2008, 2009). The general optical theorem (Marston 2001; Schiff 1968) concerns the scattering amplitude $f_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$ of scattered waves with wave number k and unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$

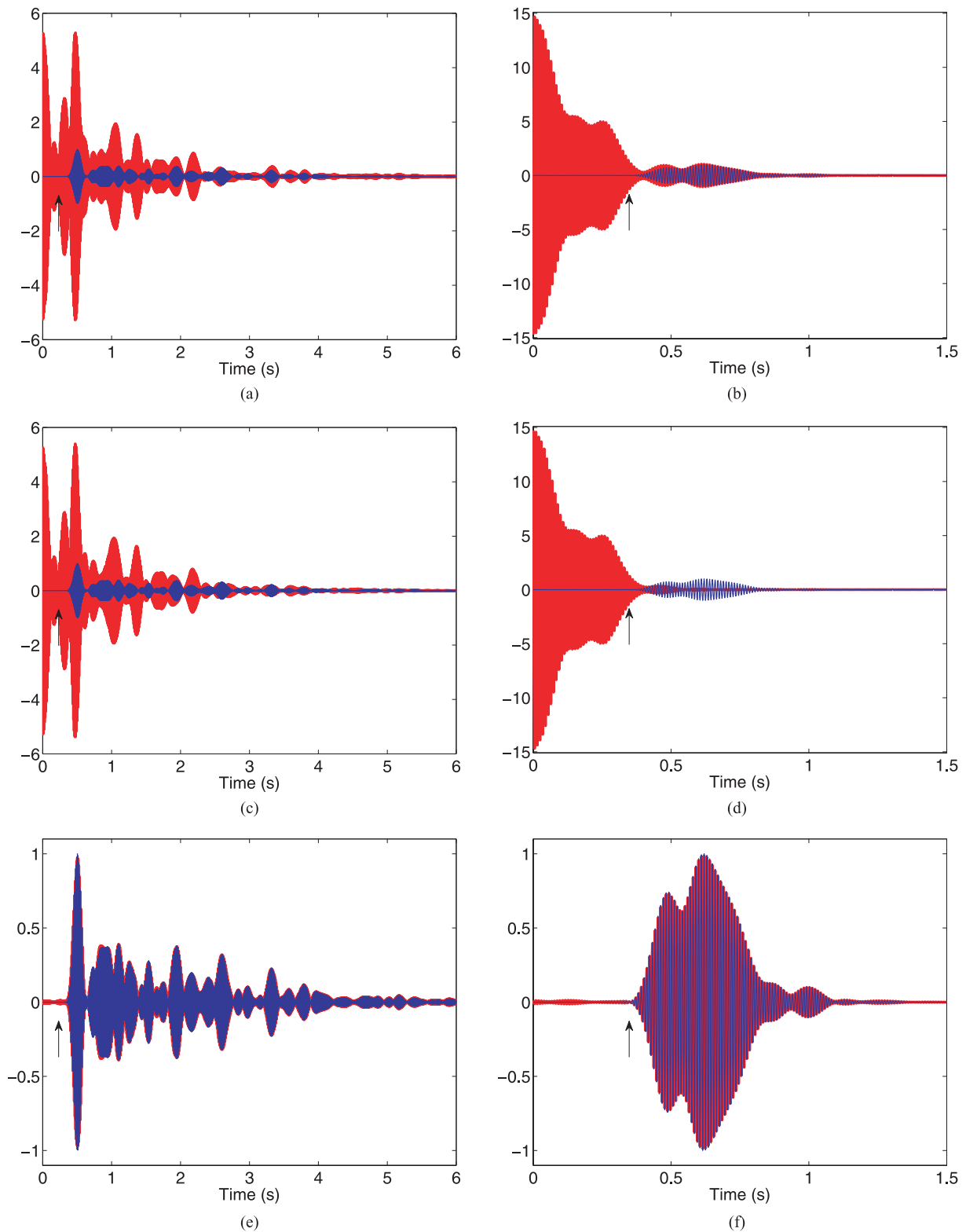


Figure 5. The blue curves show the causal part of the scattering response between two points embedded in heterogeneous acoustic media. The red curves correspond to the reconstructed signals for the different individual contributions discussed in Section 5. For strongly scattering media (left column), the summation of the reconstructed signal by cross-correlating direct and scattered waves (a) with that obtained by cross-correlating scattered waves (c) leads to the retrieval of the scattering response and cancellation of the spurious arrivals (e). Similarly, (b), (d) and (f) show success of the reconstruction for weakly scattering media (right column).

representing the directions of the outgoing and incoming waves, respectively. With a far-field approximation in expression (17), the interferometric operator for the constant density wave equation ($\rho_0 = 1$) becomes

$$I_0\{f, g\} = -2jk \oint_{\delta D} f(\mathbf{r})g(\mathbf{r})d^2r \quad (44)$$

for a homogenous medium as the unperturbed state ($G_0(\mathbf{r}, \mathbf{r}_S) = -\frac{e^{jk\|\mathbf{r}-\mathbf{r}_S\|}}{4\pi\|\mathbf{r}-\mathbf{r}_S\|}$). With the medium perturbed by a unique scattering object positioned at \mathbf{r}_x , the scattering Green's function is in the far field given by

$$G_S(\mathbf{r}, \mathbf{r}_S) = -4\pi G_0(\mathbf{r}, \mathbf{r}_x)f_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_S)G_0(\mathbf{r}_x, \mathbf{r}_S). \quad (45)$$

If A and B are far from the scatterer and δD is a sphere centred at \mathbf{r}_x with radius R , the interference between scattered Green's functions is

$$\begin{aligned} I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &= -2jk \oint_{\delta D} G_0(\mathbf{r}_x, \mathbf{r}_A)\overline{G}_0(\mathbf{r}_x, \mathbf{r}_B) \\ &\quad \times f_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B)(4\pi)^2 G_0 \\ &\quad \times (\mathbf{r}, \mathbf{r}_x)\overline{G}_0(\mathbf{r}, \mathbf{r}_x)d^2r. \end{aligned} \quad (46)$$

The integration over the sphere δD is related to an integration over solid angle by $d^2r = R^2 d\hat{n}$ and $(4\pi)^2 G_0(\mathbf{r}, \mathbf{r}_x)\overline{G}_0(\mathbf{r}, \mathbf{r}_x) = R^{-2}$ so that

$$\begin{aligned} I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} &= -2jk G_0(\mathbf{r}_x, \mathbf{r}_A)\overline{G}_0(\mathbf{r}_x, \mathbf{r}_B) \\ &\quad \oint f_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B)d\hat{n}. \end{aligned} \quad (47)$$

In the far-field approximation for the scattering Green's function, one can modify previously established equations by using where necessary expression (45) instead of (10) for field perturbation. Consequently, the extra volume terms introduced in eqs (36) and (37) are

$$\begin{aligned} (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|V(\mathbf{r})|G_1(\mathbf{r}, \mathbf{r}_A)) \\ = -4\pi \overline{G}_0(\mathbf{r}_x, \mathbf{r}_B)f_k(\hat{\mathbf{n}}_B, \hat{\mathbf{n}}_A)G_0(\mathbf{r}_x, \mathbf{r}_A), \end{aligned} \quad (48)$$

$$\begin{aligned} (G_0(\mathbf{r}, \mathbf{r}_A)|\overline{V}(\mathbf{r})|\overline{G}_1(\mathbf{r}, \mathbf{r}_B)) \\ = -4\pi G_0(\mathbf{r}_x, \mathbf{r}_A)\overline{f}_k(\hat{\mathbf{n}}_A, \hat{\mathbf{n}}_B)\overline{G}_0(\mathbf{r}_x, \mathbf{r}_B) \end{aligned} \quad (49)$$

and we thus retrieve the general optical theorem from eq. (43):

$$f_k(\hat{\mathbf{n}}_B, \hat{\mathbf{n}}_A) - \overline{f}_k(\hat{\mathbf{n}}_A, \hat{\mathbf{n}}_B) = \frac{2jk}{4\pi} \oint f_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B)d\hat{n}. \quad (50)$$

This interpretation of the cancellations, however, is limited to problems with unperturbed interferometric operators. For general systems, the extra volume terms do not cancel by summing the interferences associated with the unperturbed operator H_0 . Unless the interferometric operator is unperturbed ($I_V = 0$), the interference associated with V on the right-hand side of eq. (42) still contaminates the perturbations we desire to reconstruct by adding the contributions from eqs (36), (37) and (39). In general, we have to evaluate the contribution of $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ to cancel the extra volume terms and reconstruct the exact field perturbations. Thus as stated in Section 3, because the perturbation operator is usually unknown, interferometry appears practical for perturbation problems only with an interferometric operator that is unperturbed.

In summary, we have shown that the scattering response cannot be retrieved by cross-correlating scattered waves alone. To reconstruct scattered waves, we need to consider the contribution from cross-correlation of direct and scattered waves. The key to the ability to

cancel the extra volume terms and succeed in the reconstruction for any kind of perturbation problem is that we consider systems for which the interferometric operator is unperturbed, $I_V = 0$.

6 DISCUSSION AND CONCLUSION

We have derived a representation theorem for general systems and in particular for perturbed media. This makes it possible to retrieve Green's functions and their perturbations for a large variety of linear differential systems that include acoustic, elastic and electromagnetic waves (we show the extension to vector fields in Appendix A). We investigate the reconstruction of Green's functions, applying an interferometric operator to unperturbed fields and field perturbations. This mathematical description of interferometry simplifies the analysis of the reconstruction of perturbations: we interpret this process as summing contributions from different types of interference between perturbations and unperturbed Green's functions. In geophysics, this description can be applied to a series of problems. For example, one can extend conventional interferometry techniques for seismic waves to some possible applications in imaging and inverse problems: the representation theorem can be related to sensitivity kernels used in waveform inversion (Tarantola & Tarantola 1987), in imaging (Colton & Kress 1998), or in tomography (Woodward 1992); the theorem also allows the establishment of formal connections with seismic migration (Clearbout 1985) and with inverse scattering methods (Beylkin 1985; Borcea *et al.* 2002).

Our study of the retrieval of perturbations differs from previous work because we show explicitly that not only fields are perturbed but the operator itself changes when the medium is perturbed. For most general systems, we would need to modify the interferometric process used for the reconstruction after the application of a perturbation. We obtain this fundamental result after deriving the perturbation of the interferometric operator. Our analysis emphasizes the importance of systems for which the interferometric operator is unperturbed because these systems appear to offer the prospect for practical application of interferometry. In these cases, reconstruction of the Green's function perturbations does not require knowledge or estimation of the perturbations of the medium properties. We also demonstrate that perturbations cannot be retrieved by measuring only field perturbations; knowledge of the unperturbed state of the studied system is essential as well. Perturbations are reconstructed by combining interferences between field perturbations and unperturbed fields. The contribution from interference of field perturbations alone complete the reconstruction from interference of unperturbed fields with field perturbations.

Simulations for scattering acoustic media clearly show the importance of direct arrivals in the extraction of scattering responses and verify the failure to reconstruct scattering Green's function by cross-correlating just scattered waves. These results are intriguing and should be carefully considered when designing applications because our result appears to be counterintuitive with respect to many results obtained in seismology. Campillo & Paul (2003), for example, have shown that cross-correlation of just late coda in earthquake data, allows for retrieval of direct surface waves. Also, Stehly *et al.* (2008) have used the coda of the cross-correlation of seismic noise for improving the reconstruction of direct Green's functions. Direct waves are thus efficiently reconstructed by windowing, cross-correlating and averaging just late coda waves that mainly contain scattered waves. So, can a similar method be used for the extraction of the scattering component of Green's function? Our study of representation theorems for scattered waves suggests that the extension of the correlation technique to the reconstruction of scattered

waves is not necessarily straightforward because of the the role of direct waves in the retrieval process. In geoscience, Campillo & Paul (2003), Halliday & Curtis (2008), Roux *et al.* (2005) and Shapiro *et al.* (2005) have shown that direct surface waves are beautifully extracted by interferometry; but examples of reconstruction of scattered surface waves are still lacking. There are clear challenges in the retrieval of scattered waves but, again, the general formulation of the representation theorem for perturbed media states that we can in principle retrieve any and all perturbations for a given system.

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APPENDIX A: EXTENSION TO VECTOR SPACES AND $n \times n$ DIFFERENTIAL OPERATORS

Here, we extend our reasoning to vector fields by using the tensor notation previously introduced. Consider the unperturbed vector field $\mathbf{u}_0(\mathbf{r})$, defined in the vector space of dimension n , which is a solution of equation

$$\mathbf{H}_0(\mathbf{r}) \cdot \mathbf{u}_0(\mathbf{r}) = \mathbf{s}(\mathbf{r}), \quad (\text{A1})$$

where \mathbf{H}_0 and \mathbf{s} are the $n \times n$ linear differential operator and the $n \times 1$ source vector, respectively. For elastic waves, the operator is

$$\mathbf{H}_0 = \rho\omega^2\boldsymbol{\delta} + \nabla \cdot \mathbf{c} \cdot \nabla, \quad (\text{A2})$$

where \mathbf{c} is the elasticity tensor and $\boldsymbol{\delta}$ the Kronecker tensor; \mathbf{u}_0 is the displacement vector and \mathbf{s} is the body force per unit of volume. For electromagnetic waves in isotropic media (ϵ , permittivity; σ , conductivity; μ , permeability), the operator is

$$\mathbf{H}_0 = \begin{bmatrix} (j\omega\epsilon - \sigma)\boldsymbol{\delta} & \nabla \times \\ \nabla \times & -j\omega\mu\boldsymbol{\delta} \end{bmatrix} \quad \text{with}$$

$$\mathbf{u}_0 = \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{h}_0 \end{bmatrix}; \quad \mathbf{s} = \begin{bmatrix} \mathbf{j}^e \\ \mathbf{j}^m \end{bmatrix}, \quad (\text{A3})$$

where \mathbf{e} and \mathbf{h} denote the electric and magnetic fields; \mathbf{j}^e and \mathbf{j}^m are the electric and magnetic current densities. We give two examples of systems for which our reasoning applies; further cases of study can be found in Wapenaar *et al.* (2006). The perturbed field $\mathbf{u}_1(\mathbf{r})$ satisfies

$$\mathbf{H}_0(\mathbf{r}) \cdot \mathbf{u}_1(\mathbf{r}) = \mathbf{V}(\mathbf{r}) \cdot \mathbf{u}_1(\mathbf{r}) + \mathbf{s}(\mathbf{r}), \quad (\text{A4})$$

where \mathbf{V} is the perturbation operator. Elastic waves can be perturbed in the presence of viscosity ($\boldsymbol{\eta}$ tensor), in which case we write \mathbf{V} as

$$\mathbf{V} = -j\omega\nabla \cdot \boldsymbol{\eta} \cdot \nabla. \quad (\text{A5})$$

A change in medium properties ($\delta\epsilon$, $\delta\sigma$, $\delta\mu$) influences electromagnetic waves by a perturbation

$$\mathbf{V} = \begin{bmatrix} (\delta\sigma - j\omega\delta\epsilon)\boldsymbol{\delta} & 0 \\ 0 & j\omega\delta\mu\boldsymbol{\delta} \end{bmatrix}. \quad (\text{A6})$$

Assume a regular problem with unperturbed homogeneous boundary conditions. We relate the Green's tensors $\mathbf{G}_1(\mathbf{r}, \mathbf{r}_S)$ and $\mathbf{G}_0(\mathbf{r}, \mathbf{r}_S)$ by using the Lippmann–Schwinger equation:

$$\mathbf{G}_1(\mathbf{r}, \mathbf{r}_S) = \mathbf{G}_0(\mathbf{r}, \mathbf{r}_S) + (\mathbf{G}_0(\mathbf{r}, \mathbf{r}_1) | \mathbf{V}(\mathbf{r}_1) | \mathbf{G}_1(\mathbf{r}_1, \mathbf{r}_S)); \quad (\text{A7})$$

let the perturbation of the Green's tensor $\mathbf{G}_S(\mathbf{r}, \mathbf{r}_S)$ be given by $\mathbf{G}_S(\mathbf{r}, \mathbf{r}_S) = \mathbf{G}_1(\mathbf{r}, \mathbf{r}_S) - \mathbf{G}_0(\mathbf{r}, \mathbf{r}_S)$.

The new bilinear interferometric operator I_H now acts on matrices,

$$I_H\{\mathbf{F}, \mathbf{G}\} = (\mathbf{F}^T | \overline{\mathbf{H}} | \mathbf{G}) - (\mathbf{G}^T | \mathbf{H} | \mathbf{F}), \quad (\text{A8})$$

where \mathbf{F}^T denotes the transpose of the matrix \mathbf{F} . We introduce the unperturbed and perturbed interferometric operators as

$$\begin{aligned} I_0\{\mathbf{F}, \mathbf{G}\} &= I_{H_0}\{\mathbf{F}, \mathbf{G}\} \\ I_1\{\mathbf{F}, \mathbf{G}\} &= I_{H_0}\{\mathbf{F}, \mathbf{G}\} - I_V\{\mathbf{F}, \mathbf{G}\}. \end{aligned} \quad (\text{A9})$$

Consequently, the general representation theorem for vector systems becomes

$$\mathbf{G}_{0,1}(\mathbf{r}_B, \mathbf{r}_A) - \mathbf{G}_{0,1}^H(\mathbf{r}_A, \mathbf{r}_B) = I_{0,1}\{\mathbf{G}_{0,1}(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_{0,1}(\mathbf{r}, \mathbf{r}_B)\}, \quad (\text{A10})$$

where $\mathbf{G}_{0,1}^H$ denotes the hermitian conjugate of $\mathbf{G}_{0,1}$. For elastic waves,

$$I_0\{\mathbf{F}, \mathbf{G}\} = \oint_{\delta D} (\mathbf{F}^T(\mathbf{r}) \cdot \mathbf{c}(\mathbf{r}) \cdot \nabla \cdot \mathbf{G}(\mathbf{r}) - \mathbf{G}^T(\mathbf{r}) \cdot \mathbf{c}(\mathbf{r}) \cdot \nabla \cdot \mathbf{F}(\mathbf{r})) \cdot \hat{\mathbf{n}} d^2r \quad (\text{A11})$$

and

$$\begin{aligned} I_V\{\mathbf{F}, \mathbf{G}\} \\ = j\omega \int_D (\mathbf{F}^T(\mathbf{r}) \cdot \nabla \cdot \boldsymbol{\eta}(\mathbf{r}) \cdot \nabla \cdot \mathbf{G}(\mathbf{r}) + \mathbf{G}^T(\mathbf{r}) \cdot \nabla \cdot \boldsymbol{\eta}(\mathbf{r}) \cdot \nabla \cdot \mathbf{F}(\mathbf{r})) d^3r. \end{aligned} \quad (\text{A12})$$

For electromagnetic waves,

$$\begin{aligned} I_0\{\mathbf{F}, \mathbf{G}\} &= \oint_{\delta D} \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & \times \\ \times & 0 \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} d^2r \\ &+ 2 \int_D \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} -j\omega\epsilon\boldsymbol{\delta} & 0 \\ 0 & j\omega\mu\boldsymbol{\delta} \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) d^3r \end{aligned} \quad (\text{A13})$$

and

$$I_V\{\mathbf{F}, \mathbf{G}\} = 2 \int_D \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} j\omega\delta\epsilon\boldsymbol{\delta} & 0 \\ 0 & -j\omega\delta\mu\boldsymbol{\delta} \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) d^3r. \quad (\text{A14})$$

The corresponding representation theorem for electromagnetic waves is not necessarily the most suitable for applying interferometry because of the volume integrations in (A13) and (A14) that depend on the electromagnetic parameters ϵ , μ , $\delta\epsilon$, $\delta\mu$ in the entire volume D . In Appendix C, we show how to use a transformation \mathbf{K} to obtain an extended representation theorem (C8). For the general perturbation problem,

$$\begin{aligned} \mathbf{K}^H \cdot \mathbf{G}_{0,1}(\mathbf{r}_B, \mathbf{r}_A) - \mathbf{G}_{0,1}^H(\mathbf{r}_A, \mathbf{r}_B) \cdot \mathbf{K} \\ = I'_{0,1}\{\mathbf{G}_{0,1}(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_{0,1}(\mathbf{r}, \mathbf{r}_B)\}, \end{aligned} \quad (\text{A15})$$

where $I'_0 = I_{KH_0}$ and $I'_1 = I'_0 + I_{KV}$. For electromagnetic waves,

$$\mathbf{K} = \begin{bmatrix} -j\mathbf{I} & 0 \\ 0 & j\mathbf{I} \end{bmatrix} \quad (\text{A16})$$

gives

$$\mathbf{K} \cdot \mathbf{H}_0 = \begin{bmatrix} (\omega\epsilon + j\sigma)\delta & -j\nabla \times \\ j\nabla \times & \omega\mu\delta \end{bmatrix} \quad (\text{A17})$$

and

$$\mathbf{K} \cdot \mathbf{V} = \begin{bmatrix} -(j\delta\sigma + \omega\delta\epsilon)\delta & 0 \\ 0 & -\omega\delta\mu\delta \end{bmatrix}. \quad (\text{A18})$$

The associated interferometric operators are

$$I'_0\{\mathbf{F}, \mathbf{G}\} = j \oint_{\delta D} \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & \times \\ \times & 0 \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} d^2r \\ + 2j \int_D \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} -\sigma\delta & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) d^3r \quad (\text{A19})$$

and

$$I_{K \cdot V}\{\mathbf{F}, \mathbf{G}\} = 2j \int_D \mathbf{G}^T(\mathbf{r}) \cdot \begin{bmatrix} \delta\sigma\delta & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathbf{F}(\mathbf{r}) d^3r. \quad (\text{A20})$$

The volume integration in the corresponding representation theorem for electromagnetic waves only depends on the conductivity σ and its perturbation $\delta\sigma$.

Following the same reasoning as for scalar fields, the two representation theorems for field perturbations are

$$\mathbf{G}_S(\mathbf{r}_B, \mathbf{r}_A) - \mathbf{G}_S^H(\mathbf{r}_A, \mathbf{r}_B) = I_1\{\mathbf{G}_1(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}_B)\} \\ - I_0\{\mathbf{G}_0(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \quad (\text{A21})$$

and

$$\mathbf{G}_S(\mathbf{r}_B, \mathbf{r}_A) = I_0\{\mathbf{G}_S(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \\ + (\mathbf{G}_0^H(\mathbf{r}, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}) | \mathbf{G}_1(\mathbf{r}, \mathbf{r}_A)). \quad (\text{A22})$$

This leads to the same analysis of contributions to the Green's function reconstruction as in Section 5 by applying the following decomposition:

$$I_0\{\mathbf{G}_S(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} = \mathbf{G}_S(\mathbf{r}_B, \mathbf{r}_A) \\ - (\mathbf{G}_0^H(\mathbf{r}, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}) | \mathbf{G}_1(\mathbf{r}, \mathbf{r}_A)) \quad (\text{A23})$$

$$I_0\{\mathbf{G}_0(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_S(\mathbf{r}, \mathbf{r}_B)\} = -\mathbf{G}_S^H(\mathbf{r}_A, \mathbf{r}_B) \\ + (\mathbf{G}_0^T(\mathbf{r}, \mathbf{r}_A) | \overline{\mathbf{V}}(\mathbf{r}) | \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}_B)) \quad (\text{A24})$$

$$I_0\{\mathbf{G}_S(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_S(\mathbf{r}, \mathbf{r}_B)\} = (\mathbf{G}_S^T(\mathbf{r}_1, \mathbf{r}_A) | \overline{\mathbf{V}}(\mathbf{r}_1) | \overline{\mathbf{G}}_1(\mathbf{r}_1, \mathbf{r}_B)) \\ - (\mathbf{G}_S^H(\mathbf{r}_1, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}_1) | \mathbf{G}_1(\mathbf{r}_1, \mathbf{r}_A)). \quad (\text{A25})$$

APPENDIX B: TREATMENT OF GENERAL UNPERTURBED BOUNDARY CONDITIONS

Here, we generalize the results of this paper to any unperturbed boundary conditions. For boundary conditions that remain un-

changed after perturbing the system, both perturbed and unperturbed fields fulfill equation

$$B(\mathbf{r}) \cdot u_{0,1}(\mathbf{r}) = f(\mathbf{r}), \quad \mathbf{r} \in \delta D_{tot} \quad (\text{B1})$$

where B denotes the linear boundary condition operator that acts on the boundary δD_{tot} of total volume D_{tot} . In particular, the unperturbed and perturbed Green's functions, $G_0(\mathbf{r}, \mathbf{r}_S)$ and $G_1(\mathbf{r}, \mathbf{r}_S)$, between points r and r_S each satisfy eq. (B1). To account for this, the relation (8) between unperturbed and perturbed Green's functions is modified as follows:

$$G_1(\mathbf{r}, \mathbf{r}_S) = G_0(\mathbf{r}, \mathbf{r}_S) + (G_0(\mathbf{r}, \mathbf{r}_1) | \mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S)) \\ - \mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S)), \quad (\text{B2})$$

where \mathcal{G} is a solution of the homogeneous unperturbed system with boundary conditions (B1):

$$H_0(\mathbf{r}) \cdot \mathcal{G}(\mathbf{r}) = 0. \quad (\text{B3})$$

One can verify that this new formulation satisfies boundary conditions (B1) by applying operator B to eq. (B2). The perturbation of Green's function $G_S(\mathbf{r}, \mathbf{r}_S)$ satisfies a different expression:

$$G_S(\mathbf{r}, \mathbf{r}_S) = (G_0(\mathbf{r}, \mathbf{r}_1) | \mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S)) \\ - \mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_S)). \quad (\text{B4})$$

The main results of this paper, however, remain unchanged. We introduce the interferometric operator and derive the same general representation theorem (23) as for homogeneous boundaries. Additional derivations are needed in order to demonstrate expression (30). Consider eq. (B4) for $G_S(\mathbf{r}_A, \mathbf{r}_B)$ and insert the general representation theorem for unperturbed media $G_0(\mathbf{r}_B, \mathbf{r}_1) = I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} + \overline{\mathbf{G}}_0(\mathbf{r}_1, \mathbf{r}_B)$ to obtain

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = (I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} | \mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)) \\ + (\overline{\mathbf{G}}_0(\mathbf{r}_1, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)) \\ - \mathcal{G}(\mathbf{r}_B) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)) \\ = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \\ + (\overline{\mathbf{G}}_0(\mathbf{r}_1, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)) \\ + I_0\{\mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \\ - \mathcal{G}(\mathbf{r}_B) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)). \quad (\text{B5})$$

Additionally,

$$I_0\{\mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \\ = (\mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)) | \underbrace{\overline{\mathbf{H}}_0(\mathbf{r}) | \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)}_{\delta(\mathbf{r}-\mathbf{r}_B)}) \\ - (\overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B) | \underbrace{H_0(\mathbf{r}) | \mathcal{G}(\mathbf{r}) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A))}_{=0}) \\ = \mathcal{G}(\mathbf{r}_B) \cdot (\mathbf{V}(\mathbf{r}_1) | G_1(\mathbf{r}_1, \mathbf{r}_A)). \quad (\text{B6})$$

Summing these two equations yields

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B)\} \\ + (\overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_B) | \mathbf{V}(\mathbf{r}) | G_1(\mathbf{r}, \mathbf{r}_A)). \quad (\text{B7})$$

Eq. (B7) is identical to representation theorem (30), which holds for unperturbed homogeneous boundary conditions. By analogy, one can show that all the results presented in Section 5 holds for any type of unperturbed boundary conditions.

APPENDIX C: PROPERTIES OF SELF-ADJOINT DIFFERENTIAL OPERATOR: VOLUME/SURFACE INTEGRALS AND SPATIAL RECIPROCITY

For multi-dimensional space, the interferometric operator is defined as

$$I_H\{F, G\} = (F^T | \overline{H} | G) - (G^T | H | F) \\ = \int_D (F^T \cdot \overline{H} \cdot G - G^T \cdot H \cdot F) dV \quad (C1)$$

and the general representation theorem is

$$G(r_B, r_A) - G^H(r_A, r_B) = I_H\{G(r, r_A), \overline{G}(r, r_B)\}. \quad (C2)$$

In Section 3, we explain why for practical applications, it is useful to convert volume into surface integral to reduce the integration over the sub-volume D to its bounding surface δD . This reasoning extends to vector fields. In this appendix, we show how this relates to the concept of self-adjoint operator. We introduce what is sometimes referred to as extended Green's identity in the literature (Lanczos 1996) and define the adjoint \tilde{H} of a linear differential operator H : the adjoint is the unique operator such that for any pair of vectors (f, g) , an operator, P_H exists and

$$\int_D (g^T \cdot H \cdot f - f^T \cdot \tilde{H} \cdot g) dV \\ = - \oint_{\delta D} P_H(f, g) \cdot \hat{n} dS = \text{boundary term.} \quad (C3)$$

A differential operator is self-adjoint if $H = \tilde{H}$. For self-adjoint operators, eq. (C1) can be written using the extended Green's identity and consequently,

$$I_H\{F, G\} = \oint_{\delta D} P_H(F, G) \cdot \hat{n} dS, \quad (C4)$$

so the general representation theorem becomes

$$G(r_B, r_A) - G^H(r_A, r_B) = \oint_{\delta D} P_H(G(r, r_A), \overline{G}(r, r_B)) \cdot \hat{n} dS. \quad (C5)$$

For self-adjoint operators, in order to effectively extract the Green's function between two points A and B , we need to know the operator P_H , which depends on the properties of the system, and the Green's functions on an enclosing surface δD . For more general systems ($H \neq \tilde{H}$), relation (C5) is no longer valid, but we can always decompose the interferometric operator into surface and volume integrals and express the representation theorem as

$$G(r_B, r_A) - G^H(r_A, r_B) = \oint_{\delta D} P_H(G(r, r_A), \overline{G}(r, r_B)) \cdot \hat{n} dS \\ + \int_D G^T(r, r_A) \cdot (\overline{H} - \tilde{H}) \cdot \overline{G}(r, r_B) dV. \quad (C6)$$

We can also possibly find a spacially independent transform K of the operator H so that both the physics of the system is conserved and $K \cdot H$ is self-adjoint. This leads to a modified representation theorem that is even more general and allows us to apply practically interferometry in many cases. We discuss an example in Appendix A. Consider the linear systems

$$K \cdot H(r) \cdot G(r, r_{A,B}) = K \cdot I \delta \cdot (r - r_{A,B}) \quad (C7)$$

and apply a reciprocity relation of the correlation-type to obtain

$$K^H \cdot G(r_B, r_A) - G^H(r_A, r_B) \cdot K = I_{K \cdot H}\{G(r, r_A), \overline{G}(r, r_B)\}. \quad (C8)$$

Eq. (C8) is a general representation theorem that allows extensions of representation theorem (C2) (corresponding to $K = I$). The matrix K being chosen so that $K \cdot H$ is self-adjoint, the representation theorem reduces to a formulation with only surface integrals,

$$K^H \cdot G(r_B, r_A) - G^H(r_A, r_B) \cdot K \\ = \oint_{\delta D} P_{K \cdot H}(G(r, r_A), \overline{G}(r, r_B)) \cdot \hat{n} dS. \quad (C9)$$

Note that the results of this paper do not require space- and time-reciprocity. This means that the order of spatial coordinates matters in the relations we establish. To facilitate the use and interpretation of representation theorems in practice, we desire systems that are spatially reciprocal, as holds for particular boundary conditions and symmetry of linear differential operators. For example, consider a representation theorem of the convolution type. By analogy with the representation theorem (C2) of the correlation type, we get

$$G(r_B, r_A) - G^T(r_A, r_B) \\ = \int_D (G^T(r, r_A) \cdot H(r) \cdot G(r, r_B) \\ - G^T(r, r_B) \cdot H(r) \cdot G(r, r_A)) dV. \quad (C10)$$

For operators such that $\overline{H} = \tilde{H}$, which include self-adjoint real operators, for example, the wave operator, use of Green's identity (C3) with $D = D_{tot}$ yields

$$G(r_B, r_A) - G^T(r_A, r_B) = \oint_{\delta D_{tot}} P_H(G(r, r_A), G(r, r_B)) \cdot \hat{n} dS. \quad (C11)$$

Depending on boundary conditions, all of the components of the tensor on the right-hand side of eq. (C11) vanish and consequently, we obtain $G(r_B, r_A) = G^T(r_A, r_B)$, that is, spatial reciprocity. Typically, the components of $\oint_{\delta D_{tot}} P_H(G(r, r_A), G(r, r_B)) \cdot \hat{n} dS$ will go to zero if some of the components of the Green's tensors $G(r, r_{A,B})$ or their derivatives vanish at δD_{tot} . For acoustic waves in lossless media, the Sommerfield radiation and free surface conditions lead to spatial reciprocity. Acoustic systems with free boundaries, however, are of limited interest because we cannot practically apply interferometry. Indeed, in this case, eq. (C5) shows that $G(r_B, r_A) - \overline{G}(r_A, r_B) = 0$ for the pressure field; that is, for self-adjoint operators, free boundaries always lead to reconstruction of a null pressure field.

In summary, for systems with both appropriate boundary conditions and symmetric linear differential operator, representation theorem C5 is valid and spatial reciprocity applies. In this case, the general representation theorem is

$$G(r_B, r_A) - \overline{G}(r_B, r_A) = \oint_{\delta D} P_H(G^T(r_A, r_S), G^H(r_B, r_S)) \cdot \hat{n} dS. \quad (C12)$$

Given eq. (C12), we retrieve the Green's function between points A and B by applying the interferometric operator to the Green's functions recorded at A and B for sources S on the boundary δD . This is today's commonly used setup for interferometry.