Extracting the Green's function of attenuating heterogeneous acoustic media from uncorrelated waves

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The Green's function of acoustic or elastic wave propagation can, for loss-less media, be retrieved by correlating the wave field that is excited by random sources and is recorded at two locations. Here the generalization of this idea to attenuating acoustic waves in an inhomogeneous medium is addressed, and it is shown that the Green's function can be retrieved from waves that are excited throughout the volume by spatially uncorrelated injection sources with a power spectrum that is proportional to the local dissipation rate. For a finite volume, one needs both volume sources and sources at the bounding surface for the extraction of the Green's functions. For the special case of a homogeneous attenuating medium defined over a finite volume, the phase and geometrical spreading of the Green's function is correctly retrieved when the volume sources are ignored, but the attenuation is not. © *2007 Acoustical Society of America.* [DOI: 10.1121/1.2713673]

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I. INTRODUCTION

The extraction of the Green's function by correlating waves excited by random sources that are recorded at two locations has recently received much attention. There are numerous derivations of this principle that are valid for closed systems[1] and for open systems (e.g., Refs. 2–4). Formulations of this principle are based either on random sources placed throughout a volume[1,5] or on sources that are located at a surface.[6–8] The extraction of the Green's function using random wave fields has been applied to ultrasound,[9–12] in seismic exploration,[13–15] in crustal seismology,[16–19] in ocean acoustics,[20–22] to buildings,[23,24] and in helioseismology.[25–27] The recent supplement of seismic interferometry[28] in Geophysics gives an overview of this field of research. Phrases that include passive imaging, correlation of ambient noise, extraction of the Green's function, and seismic interferometry have been proposed for this line of research. Recently the theory has been developed for the extraction of the Green's function for more general linear systems than acoustic or elastic waves.[29,30]

Many derivations of this principle are valid for systems that are invariant under time reversal. Several derivations invoke time-reversal invariance explicitly.[2,3,13] For acoustics waves in a flowing medium the time-reversal invariance is broken by the flow; this broken symmetry has been incorporated in the theory for the extraction of the Green's function.[31,32] Attenuation also breaks the invariance for time reversal. For homogeneous acoustic media[5,33] and for a homogeneous oceanic waveguide[21] attenuation has been incorporated into the theory for the extraction of the Green's function. Weaver and Lobkis[4] use complex frequency as a tool to force convergence on an integral over all sources.

Here I derive the principle of seismic interferometry for general attenuating, acoustic media, and extend earlier formulations for homogeneous media to include arbitrary heterogeneity in density, compressibility, and intrinsic attenuation. Section II introduces the basic equations and rederives a representation theorem of the correlation type for attenuating media. Section III shows that for an unbounded volume, or for a volume that is bounded by a surface where the pressure or normal component of the velocity vanishes, the Green's function can be extracted from waves excited by uncorrelated volume sources with a source strength that is proportional to the local dissipation rate. Section IV shows that for a bounded volume one needs, in general, both volume sources and surface sources in order to retrieve the correct Green's function. Section V illustrates the relative roles of the volume sources and surface sources by analyzing the special case of a homogeneous, attenuating medium, with a single reflector. In this special case, when volume sources are ignored, the phase and geometrical spreading of the Green's function are correctly reproduced by seismic interferometry, but the attenuation is not.

II. BASIC EQUATION FOR ACOUSTIC WAVES

Using the Fourier convention $f(t)=\int f(\omega)\exp(-i\omega t)d\omega$, the pressure p and particle velocity $\mathbf{v}$ for acoustic waves satisfy, in the frequency domain, the following coupled equations:

$$\nabla p - i\omega\rho\mathbf{v} = 0, \tag{1}$$

$$(\nabla\cdot\mathbf{v}) - i\omega\kappa p = q. \tag{2}$$

In these expressions ω is the angular frequency, ρ the mass density, and κ the compressibility. All expressions in this work are given in the frequency domain; for brevity this frequency-dependence is not denoted explicitly. It is assumed that only injection sources q are present. Body forces would render the right-hand side of expression (1) nonzero. For attenuating media, the compressibility κ is complex, this

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quantity can be decomposed in a real and imaginary part:

$$\kappa = \kappa_r(\mathbf{r},\omega) + i\kappa_i(\mathbf{r},\omega). \tag{3}$$

Because of the Kramers-Kronig relation (e.g., Refs. 34 and 35), the real and imaginary parts of the compressibility depend on frequency. In contrast to the treatment of de Hoop,[36] it is presumed that the mass density is real. In this general derivation the density and compressibility can be arbitrary functions of location and frequency.

Following de Hoop[37] and Fokkema and van den Berg,[38] expressions (1) and (2) can be used to derive a representation theorem of the correlation type. The treatment given here generalizes earlier descriptions of the extraction of the Green's function[7,32] to include dissipation. Two wave states, labeled A and B, are considered that both satisfy expressions (1) and (2), and that are excited by forcing functions q_A and q_B, respectively. The subscripts A and B indicate the state for each quantity. A representation theorem of the correlation type is obtained by integrating the combination $(1)_A \cdot \mathbf{v}_B^* + (1)_B^* \cdot \mathbf{v}_A + (2)_A p_B^* + (2)_B^* p_A$ over volume, and applying Gauss' theorem. [The asterisk denotes complex conjugation, and $(1)_B^*$ stands, for example, for the complex conjugate of expression (1) for state B.] This gives

$$\oint (p_A \mathbf{v}_B^* + p_B^* \mathbf{v}_A) \cdot d\mathbf{S} = \int (q_B^* p_A + q_A p_B^*) dV - i\omega \times \int (\kappa^* - \kappa) p_A p_B^* dV, \tag{4}$$

where $\oint(\cdots)\cdot d\mathbf{S}$ denotes the surface integral over the surface that bounds the volume. Note that the last term is due to the attenuation; for loss-less media κ is real, and $\kappa^* - \kappa = 0$. The relative roles of the surface integral on the left-hand side and the volume integral in the last term play a crucial role in the following treatment. In the following the "surface" refers to the surface that bounds the volume. In the presence of cavities this surface may consist of disconnected pieces.

These representation theorems can be used to derive several properties of the Green's function $G(\mathbf{r},\mathbf{r}_0)$ that is the pressure response to an injection source $q(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_0)$. Setting

$$q_{A,B}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_{A,B}) \tag{5}$$

implies that the corresponding pressure states are given by

$$p_{A,B}(\mathbf{r}) = G(\mathbf{r},\mathbf{r}_{A,B}), \tag{6}$$

respectively.

Inserting the excitations (5) into expression (4), and using Eq. (1) to eliminate the velocity, one obtains

$$\begin{aligned} G^*(\mathbf{r}_A,\mathbf{r}_B) + G(\mathbf{r}_B,\mathbf{r}_A) &= 2\omega \int \kappa_i(\mathbf{r},\omega) G(\mathbf{r},\mathbf{r}_A) G^*(\mathbf{r},\mathbf{r}_B) dV \\ &+ \oint \frac{1}{i\omega\rho}(G^*(\mathbf{r},\mathbf{r}_B) \nabla G(\mathbf{r},\mathbf{r}_A) \\ &- G(\mathbf{r},\mathbf{r}_A) \nabla G^*(\mathbf{r},\mathbf{r}_B)) \cdot d\mathbf{S}. \end{aligned} \tag{7}$$

For brevity the frequency-dependence of G is suppressed. In the presence of intrinsic attenuation, reciprocity of acoustic waves still holds, hence

$$G(\mathbf{r}_A,\mathbf{r}_B) = G(\mathbf{r}_B,\mathbf{r}_A). \tag{8}$$

Expression (7) can therefore be written as

$$\begin{aligned} G^*(\mathbf{r}_B,\mathbf{r}_A) + G(\mathbf{r}_B,\mathbf{r}_A) &= 2\omega \int \kappa_i(\mathbf{r},\omega) G(\mathbf{r}_A,\mathbf{r}) G^*(\mathbf{r}_B,\mathbf{r}) dV \\ &+ \oint \frac{1}{i\omega\rho}(G^*(\mathbf{r}_B,\mathbf{r}) \nabla G(\mathbf{r}_A,\mathbf{r}) \\ &- G(\mathbf{r}_A,\mathbf{r}) \nabla G^*(\mathbf{r}_B,\mathbf{r})) \cdot d\mathbf{S}. \end{aligned} \tag{9}$$

Note that for loss-less media, because of the complex conjugates, the surface integral does not vanish when the system satisfies radiation boundary conditions at the surface. A similar relation has been derived for electromagnetic fields in conducting media.[39]

The left-hand side of expression (9) is the sum of the causal and acausal Green's functions. Wapenaar *et al.*[7] use this expression for loss-less media ($\kappa_i = 0$) to show that the sum of the causal and acausal Green's function can be obtained by cross-correlating the pressure fields that are due to uncorrelated random sources at the surface. The pressure field caused by these sources is transmitted to the points $\mathbf{r}_A$ and $\mathbf{r}_B$ in the interior by the Green's functions in the surface integral in Eq. (9). For attenuating media [$\kappa_i(\mathbf{r}) \neq 0$], this analysis is complicated by the presence of the volume integral in this expression.

III. INTERFEROMETRY WHEN THE SURFACE INTEGRAL VANISHES

This section analyzes the special case where the surface integral in expression (9) vanishes. This is the case when one of the following conditions is satisfied:

C1: The volume integration is over all space. For attenuating media the wave field vanishes exponentially at infinity, and the surface integral vanishes.
C2: The pressure vanishes at the surface ($G = 0$).
C3: The normal component of the velocity perpendicular to the surface vanishes at the surface. Because of expression (1) this implies that $\nabla G \cdot d\mathbf{S} = 0$.

When one of the conditions C1–C3 is satisfied, the pressure is related to the excitation by

$$p(\mathbf{r}_0) = \int G(\mathbf{r}_0,\mathbf{r}) q(\mathbf{r}) dV \tag{10}$$

and the representation theorem of the correlation type (9) reduces to

$$G^*(\mathbf{r}_B,\mathbf{r}_A)+G(\mathbf{r}_B,\mathbf{r}_A)=2\omega\int\kappa_i(\mathbf{r},\omega)G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dV. \tag{11}$$

Consider the situation where random pressure sources are present throughout the volume, and that these sources at different locations are uncorrelated:

$$q(\mathbf{r}_1,\omega)q^*(\mathbf{r}_2,\omega)=\kappa_i(\mathbf{r}_1,\omega)\delta(\mathbf{r}_1-\mathbf{r}_2)|S(\omega)|^2, \tag{12}$$

where $|S(\omega)|^2$ denotes the power spectrum of the random excitation. The excitation (12) is proportional to $\kappa_i(\mathbf{r},\omega)$, the imaginary part of the compressibility, which in turn is proportional to the local attenuation. This means that the excitation (12) supplies a random excitation of the pressure field that locally compensates for the attenuation. Multiplying expression (11) with $|S(\omega)|^2$ gives

$$\begin{aligned}(G^*(\mathbf{r}_B,\mathbf{r}_A)&+G(\mathbf{r}_B,\mathbf{r}_A))|S(\omega)|^2\\&=2\omega\int\kappa_i(\mathbf{r},\omega)|S(\omega)|^2G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dV\\&=2\omega\int\int\kappa_i(\mathbf{r}_1,\omega)\delta(\mathbf{r}_1-\mathbf{r}_2)\\&\quad\times|S(\omega)|^2G(\mathbf{r}_A,\mathbf{r}_1)G^*(\mathbf{r}_B,\mathbf{r}_2)dV_1dV_2\\&=2\omega\int G(\mathbf{r}_A,\mathbf{r}_1)q(\mathbf{r}_1)dV_1\left(\int G(\mathbf{r}_B,\mathbf{r}_2)q(\mathbf{r}_2)dV_2\right)^*\\&=2\omega p(\mathbf{r}_A)p^*(\mathbf{r}_B).\end{aligned} \tag{13}$$

Expression (12) is used for the third identity, above, and expression (10) for the last one. The sum of the causal and acausal Green's function thus follows from correlating the pressure fields caused by the random volume sources:

$$G^*(\mathbf{r}_B,\mathbf{r}_A)+G(\mathbf{r}_B,\mathbf{r}_A)=\frac{2\omega}{|S(\omega)|^2}p(\mathbf{r}_A)p^*(\mathbf{r}_B). \tag{14}$$

As in seismic interferometry for loss-less media,[1,4,7] one needs to divide by the power spectrum of the excitation to remove the imprint of this excitation on the recorded pressure $p(\mathbf{r}_A)$ and $p(\mathbf{r}_B)$.

IV. WHEN THE SURFACE INTEGRAL IS NONZERO

In practical applications, none of the conditions C1–C3 might be satisfied. This is, in fact, the case in formulations of seismic interferometry where the Green's function is extracted by correlating pressure fields that are excited by uncorrelated sources at the surface that bounds the volume (e.g., Ref. 7). This section investigates the relative roles of the surface and volume integrals in expression (9). For simplicity, I use, following Wapenaar *et al.*,[7] that the surface is far from the region of interest and that $\nabla G(\mathbf{r},\mathbf{r}_0)\cdot d\mathbf{S}=ikG(\mathbf{r},\mathbf{r}_0)dS=(i\omega/c)G(\mathbf{r},\mathbf{r}_0)dS$. Inserting this in Eq. (9) gives

$$G^*(\mathbf{r}_B,\mathbf{r}_A)+G(\mathbf{r}_B,\mathbf{r}_A)=I_V(\mathbf{r}_B,\mathbf{r}_A)+I_S(\mathbf{r}_B,\mathbf{r}_A), \tag{15}$$

where the volume integral $I_V(\mathbf{r}_B,\mathbf{r}_A)$ is given by

$$I_V(\mathbf{r}_B,\mathbf{r}_A)=2\omega\int\kappa_i(\mathbf{r},\omega)G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dV, \tag{16}$$

and the surface integral $I_S(\mathbf{r}_B,\mathbf{r}_A)$ by

$$I_S(\mathbf{r}_B,\mathbf{r}_A)=2\oint\frac{1}{\rho c}G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dS. \tag{17}$$

In many applications, the attenuation is weak $(\kappa_i\ll\kappa_r)$, and one might think that the volume integral is small compared to the surface integral. This, however, is not the case. Because of the attenuation, the surface integral decreases exponentially with increasing surface area, and goes to zero while the volume integral is finite. According to expression (15) the sum of the volume integral and the surface integral is independent of the size of the volume. This implies that the volume integral and the surface integral are, in general, both needed for the extraction of the Green's function. The stationary phase analysis of Sec. V shows that, for the special case of a homogeneous medium, this is indeed the case.

V. STATIONARY PHASE ANALYSIS OF THE SURFACE INTEGRAL AND VOLUME INTEGRAL

To better understand the relative roles of the volume and surface integrals of expressions (16) and (17), the special case of a homogeneous, attenuating medium is analyzed in this section, and the volume and surface integrals are solved in the stationary phase approximation. For a homogeneous medium, Eqs. (1) and (2) can be combined to give

$$\nabla^2p+\omega^2\kappa\rho p=i\omega\rho q. \tag{18}$$

The wave number is therefore given by

$$k=\omega\sqrt{\kappa\rho}. \tag{19}$$

Weak attenuation is considered; in this case the wave number is to first order in κ_i/κ_r given by

$$k=\omega\sqrt{\kappa_r\rho}\left(1+\frac{i\kappa_i}{2\kappa_r}\right). \tag{20}$$

The phase velocity thus is given by

$$c=\frac{\omega}{k_r}=1/\sqrt{\kappa_r\rho}, \tag{21}$$

and the imaginary component of the wave number by

$$k_i=\omega\kappa_i\rho c/2. \tag{22}$$

The Green's function solution for expression (18) is equal to

$$G(R)=-i\omega\rho\exp(-k_iR)\frac{e^{ikR}}{4\pi R}. \tag{23}$$

The geometry for the stationary phase analysis is shown in Fig. 1. A coordinate system is used whose origin is at the midpoint of the receiver positions $\mathbf{r}_A$ and $\mathbf{r}_B$ and whose z axis points along the receiver line. The distance between these points is denoted by R; hence $\mathbf{r}_A=(0,0,-R/2)$, and $\mathbf{r}_B=(0,0,R/2)$. A volume that is bounded by a surface at distance L from the origin is considered. The stationary phase analysis follows the treatment of Ref. 5. The stationary phase point of the integrals in expressions (16) and (17) is located

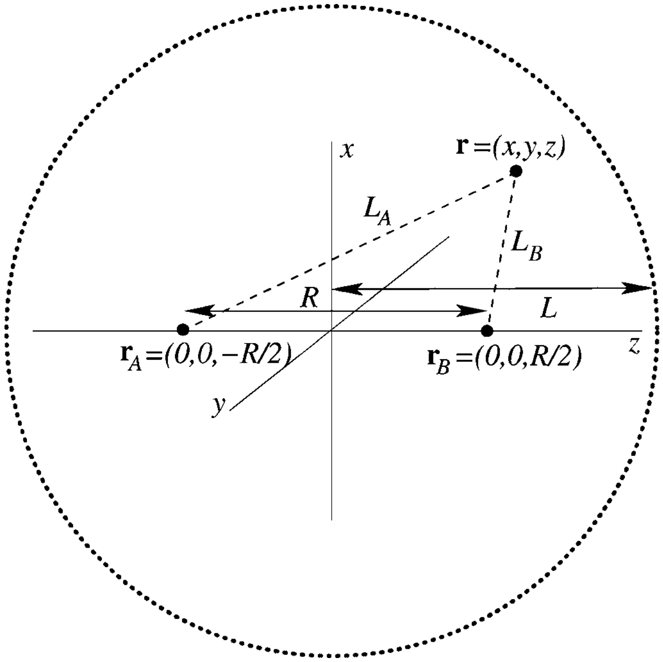

FIG. 1. Definition of geometric variables for the stationary phase evaluation of integrals $I_V(\mathbf{r}_B,\mathbf{r}_A)$ and $I_S(\mathbf{r}_B,\mathbf{r}_A)$. The volume is bounded by a sphere with radius L, as denoted by the dotted line.

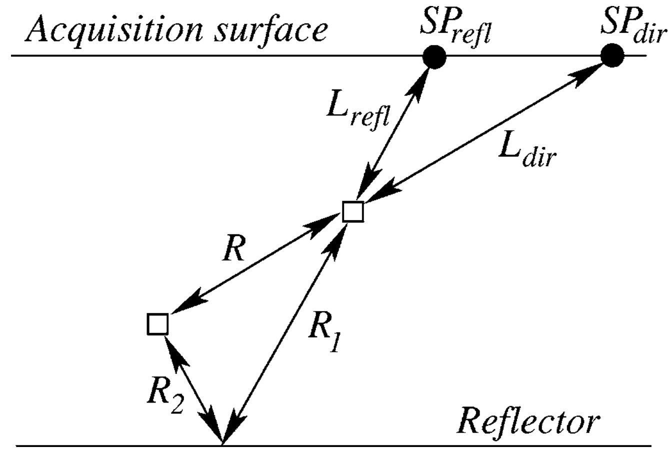

FIG. 2. Two receivers (open squares) that are located between the acquisition surface and a reflector, and the stationary phase points SP_{dir} and SP_{refl} of the direct and reflected waves, respectively.

on the z axis ($x=y=0$). Following Ref. 5, the points to the right of $\mathbf{r}_B$ (for which $z>R/2$) give the causal Green's function $G(\mathbf{r}_B,\mathbf{r}_A)$, while the points to the left of $\mathbf{r}_A$ (for which $z<-R/2$) give the acausal Green's function $G^*(\mathbf{r}_B,\mathbf{r}_A)$. In the following only the contribution of integration points for which $z>R/2$ is treated; this gives only the causal Green's function. Because of this limitation, the corresponding surface and volume integrals are denoted with the superscript (+).

Both the surface and volume integrals contain a double integration over the transverse x and y coordinates. As shown in the Appendix, the stationary phase approximation of the surface and volume integrals gives

$$I_V^{(+)}(\mathbf{r}_B,\mathbf{r}_A)=-i\omega\rho[\exp(-k_iR)-\exp(-2k_iL)]\frac{e^{ikR}}{4\pi R}, \quad (24)$$

and

$$I_S^{(+)}(\mathbf{r}_B,\mathbf{r}_A)=-i\omega\rho\exp(-2k_iL)\frac{e^{ikR}}{4\pi R}. \quad (25)$$

The sum of the surface and volume integrals indeed gives the causal Green's function:

$$I_S^{(+)}(\mathbf{r}_B,\mathbf{r}_A)+I_V^{(+)}(\mathbf{r}_B,\mathbf{r}_A)=-i\omega\rho\exp(-k_iR)\frac{e^{ikR}}{4\pi R} = G(\mathbf{r}_B,\mathbf{r}_A). \quad (26)$$

Expressions (24) and (25) show that neither the volume integral nor the surface integral gives the Green's function, but that the sum does. Equation (16) suggests that for weak attenuation the volume integral can be ignored, because this integral is proportional to $\kappa_i \ll \kappa_r$. Expressions (24) and (25) show, however, for the special case of a homogeneous medium that as long as $k_iL=O(1)$, the volume integral and the surface integral in general have comparable strength. (As shown in expression (A5), the volume integral $I_V^{(+)}$ is proportional to κ_i/k_i, which according to expression (22) has a finite value as $\kappa_i\to 0$.) The relative contribution of the surface integral and the volume integral is weighted by $\exp(-2k_iL)$. As the volume occupies all space ($L\to\infty$), the surface integral vanishes ($I_S^{(+)}\to 0$) and the volume integral is given by $I_V^{(+)}(\mathbf{r}_B,\mathbf{r}_A)=-i\omega\rho\exp(-k_iR)e^{ikR}/4\pi R=G(\mathbf{r}_B,\mathbf{r}_A)$. This is the special case treated in Sec. III because in this limit the surface integral vanishes because of the large distance L traversed by the attenuating waves that are correlated.

Equation (26) shows that in the frequency domain the Green's function can be retrieved from the cross correlation of waves excited by a combination of volume sources and surface sources. A similar result was obtained in the frequency domain in expression (10) of Ref. 5 where an infinite volume is needed. Expression (26) of Ref. 33 gives a time-domain formulation of the retrieval of the Green's function. In the latter studies sources in a homogeneous attenuating medium were integrated over an infinite volume. Because of the infinite integration region, the surface integral (25) did not contribute in those studies. The relative role of the surface integral and the volume integral is important because in some applications sources are present only on a finite surface (e.g., Ref. 14).

In this example, only the direct wave arrives, and ignoring the volume integral leads to an overall amplitude error. Next, the example of interferometry for both the direct wave and a reflected wave is considered. Sources are placed on the acquisition surface shown in Fig. 2. Both the direct wave and a reflected wave propagate to receivers indicated with open squares. The points SP_{dir} and SP_{refl} shown in Fig. 2 indicate the stationary phase source locations for the direct and reflected waves, respectively.[8]

The direct wave contains contributions $\exp(-k_iL_{dir})$ $\times\exp(-k_i(L_{dir}+R))$ from the attenuation at the stationary points. Following the stationary phase analysis of Ref. 8, and taking the attenuation terms into account gives a contribution of the surface integral to the direct wave that is given by

$$u_{\mathrm{dir}} \propto e^{-2k_i L_{\mathrm{dir}}} e^{-k_i R} \frac{e^{ikR}}{R}. \tag{27}$$

The reflected waves have a contribution at the stationary phase point from the attenuation $\exp(-k_i L_{\mathrm{refl}})\exp(-k_i(L_{\mathrm{refl}}+R_1+R_2))$, the reflected wave obtained from the surface integral satisfies in the stationary phase approximation[8]

$$u_{\mathrm{refl}} \propto e^{-2k_i L_{\mathrm{refl}}} r e^{-k_i(R_1+R_2)} \frac{e^{ik(R_1+R_2)}}{R_1+R_2}, \tag{28}$$

where r is the reflection coefficient of the interface. Both the direct and reflected waves thus obtained have the correct phase and geometrical spreading, but both contain an amplitude term $[\exp(-2k_i L_{\mathrm{dir}})$ and $\exp(-2k_i L_{\mathrm{refl}})$, respectively$]$ that is due to neglecting the volume integrals. Since these amplitude terms are different for the direct wave and the reflected waves, neglecting the contribution of the volume integrals disrupts the relative amplitude of the different arrivals.

It is interesting to compare this result with expressions (1) and (18) of Sabra *et al.*,[21] who show that for a homogeneous attenuating oceanic wave guide with source placed at a surface of constant depth that the phase of the different arrivals is correctly produced by the cross correlation, but the amplitude is not in the presence of attenuation. Expressions (27) and (28) presented here describe what happens when the sources are placed on a surface only. It is the absence of volume sources in Ref. 21 that leads to an incorrect estimate attenuation in the Green's function estimated from cross correlation.

VI. DISCUSSION

The derivation in this work shows that the Green's function of attenuating acoustic waves in a heterogeneous medium can be extracted by cross-correlating measurements of the pressure that is excited by random sources. As shown in Secs. III and IV, the Green's function can, however, be computed from the cross correlation when the random pressure field is excited by sources that are distributed throughout the volume, and that have a source strength that is proportional to the local dissipation rate (which is proportional to κ_i). Volume sources are also required for the extraction of the Green's function of the diffusion equation,[40] which is another example of a system that is not invariant for time-reversal.

The physical reason that the excitation must be proportional to the local dissipation rate is that the extraction of the Green's function is based on the equilibration of energy. This condition is necessary for the fluctuation-dissipation theorem, which relates the response of a dissipative system (the Green's function) to the fluctuations of that system around the equilibrium state.[41,42] Acoustic, dissipative, waves can be in equilibrium only when the excitation of the waves matches the local dissipation rate. If this were not the case, there would be a net energy flow, and the system would not be in equilibrium. The equilibrium of energy,[43] also referred to as equipartitioning, has been shown to be essential for the accurate reconstruction of the Green's function (e.g., Refs. 11, 30, and 44).

When none of the conditions C1–C3 of Sec. III is satisfied, the sum of the causal and acausal Green's function is, according to expression (15) given by the sum of the volume integral and a surface integral. The physical reason is that in equilibrium, the sources at the surface must be supplemented with sources within the volume that are proportional to the local dissipation rate if the system is to be in equilibrium. In some applications this condition can be realized. For example, Weaver and Lobkis[9] extract the Green's function from the wave field that is excited by thermal fluctuations throughout the volume of their sample. The need to have sources throughout the volume in addition to sources at the surface is impractical in applications where one seeks to extract the Green's function for two points in the interior by placing sources at the bounding surface only (e.g., Refs. 13 and 14).

Roux *et al.*[33] show that for a homogeneous infinite acoustic medium one needs to correct for a factor ω^{-1}. They note that this term is due to their assumed attenuation mechanism $[\mathrm{Im}(c)=\text{constant}]$. In the formulation of this work, such a correction term is hidden in condition (12) which states that the power of the sources is proportional to the local attenuation rate. In this work, $\kappa_i(\mathbf{r},\omega)$ can be an arbitrary function of position and frequency, but, as long as condition (12) is satisfied the Green's function can be extracted by cross correlation. In practical applications the source spectrum may not satisfy this condition. In that case there is no energy balance, and the Green's function is not correctly retrieved. This may be an important limitation in practical applications.

In practical situations, attenuation is present, and the contribution of the volume integral is often ignored, yet seismic interferometry seems to be able to retrieve the Green's function well (e.g., Refs. 13 and 14). For the special case of a homogeneous medium, the contribution of the surface integral to the Green's function is given by expression (25). This contribution has the correct phase and geometrical spreading $[\exp(ik_r R)/R]$, but incorrect attenuation $[\exp(-2k_i L)$ instead of $\exp(-k_i R)]$. This suggests that when seismic interferometry for attenuating systems is used by summing over sources at the surface only, the correct phase and geometrical spreading are recovered, but that the attenuation is not. According to expressions (15)–(17) one needs for a general inhomogeneous attenating medium both volume sources and surface sources for the extraction of the Green's function. It is known that multiple scattering by a boundary[45] or by internal scatterers[46] can compensate for a deficit of sources needed for focusing by time-reversal. This raises the unsolved question to what extent multiple scattering can compensate for the lack of volume sources.

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APPENDIX: STATIONARY PHASE ANALYSIS OF THE INTEGRATION OVER THE TRANSVERSE COORDINATES

The integrals $I_V(\mathbf{r}_B,\mathbf{r}_A)$ and $I_S(\mathbf{r}_B,\mathbf{r}_A)$ of expressions (16) and (17) contain, in the geometry of Fig. 1, an integration over the x and y coordinates. This Appendix considers the contribution from integration points $z>R/2$. These points lead to the causal Green's function. The contribution from integration points $z<-R/2$ leads to the acausal Green's function, which can be obtained by complex conjugation of the results derived here.

For the Green's function of the homogeneous medium of expression (23),

$$G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r}) = \left(\frac{\rho\omega}{4\pi}\right)^2 e^{-k_i(L_A+L_B)}\frac{e^{ik_r(L_A+L_B)}}{L_A L_B}, \tag{A1}$$

where $L_{A,B}=|\mathbf{r}-\mathbf{r}_{A,B}|$, as shown in Fig. 1. The phase term $\exp(ik_r(L_A+L_B))$ of expression (A1) is oscillatory as a function of the transverse coordinates x and y. The phase is stationary along the z axis ($x=y=0$). For fixed z, near the stationary phase point, the lengths L_A and L_B are, to second order in x and y, given by

$$L_A = \sqrt{x^2+y^2+(z+R/2)^2} \approx (z+R/2) + \frac{1}{2}\frac{x^2+y^2}{(z+R/2)}, \tag{A2}$$

and

$$L_B = \sqrt{x^2+y^2+(z-R/2)^2} \approx (z-R/2) + \frac{1}{2}\frac{x^2+y^2}{(z-R/2)}. \tag{A3}$$

These expressions are valid for integration points $z>R/2$. In the stationary phase approximation[47] of the integration of expression (A1) over x and y, these approximations for L_A and L_B are used in the phase term $\exp(ik_r(L_A+L_B))$. In the stationary phase approximation, the attenuation and geometrical spreading terms $\exp(-k_i(L_A+L_B))/L_A L_B$ are evaluated at the stationary phase point $x=y=0$, where $L_{A,B}=z\pm R/2$. The integral of expression (A1) over the transverse coordinates is, in the stationary phase approximation, given by[47]

$$\begin{aligned}\int\int G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dxdy &= \left(\frac{\rho\omega}{4\pi}\right)\frac{e^{-2k_i z}}{z^2-R^2/4}e^{ik_rR}\int\int \exp\left(-\frac{ik_r}{2}\left(\frac{R}{z^2-R^2/4}\right)\right.\\ &\quad\left.\times(x^2+y^2)\right)dxdy\\ &= \left(\frac{\rho\omega}{4\pi}\right)\frac{e^{-2k_i z}}{z^2-R^2/4}e^{ik_rR}\left(e^{-i\pi/4}\sqrt{\frac{2\pi(z^2-R^2/4)}{k_rR}}\right)^2\\ &= -\frac{i\rho^2\omega c}{8\pi}e^{-2k_i z}\frac{e^{ik_rR}}{R}.\end{aligned} \tag{A4}$$

Inserting this result in Eq. (17), and setting $z=L$, gives expression (25) for the contribution of the surface $z=L$ to the surface integral $I_S(\mathbf{r}_B,\mathbf{r}_A)$. In order to obtain the contribution of the region $R/2<z<L$ to the volume integral of expression (16), one needs to integrate Eq. (A4) over z:

$$\begin{aligned}I_V^{(+)}(\mathbf{r}_B,\mathbf{r}_A) &= 2\omega\kappa_i\frac{-i\rho^2\omega c}{8\pi}\frac{e^{ik_rR}}{R}\int_{R/2}^{L}e^{-2k_i z}dz\\ &= -\frac{i\rho^2\omega^2 c\kappa_i}{2k_i}\frac{e^{ik_rR}}{R}(e^{-k_iR}-e^{-2k_iL}).\end{aligned} \tag{A5}$$

Using Eq. (22) to eliminate κ_i/k_i gives expression (24).

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