Imaging the Earth using Green's theorem [1]

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1 Introduction

The Earth is a big place: its radius is about 6400 km. In comparison, the deepest boreholes drilled are about 10 km deep. We thus have little opportunity to take direct measurements or samples inside the Earth: it is mostly inaccessible. And even for the upper 10 km that we can sample, the cost of drilling deep boreholes is very high. This means that inferences about the Earth's interior are largely based on physical and chemical measurements taken at the Earth's surface, or even from space. Investigating the inside of the Earth thus resembles that classical black-box problem: determine the contents of a closed box when you can do anything except open the box.

If one knew physical fields, such as the gravitational field or the elastic wave field, inside the Earth, one could infer the local properties of the Earth by inserting the measured field into the equation that governs that field, and extract the physical parameters, such as the mass density, from the field equation. However, the fields are measured at the Earth's surface or sometimes even above that surface. One thus needs a recipe for propagating the measured field from its surface of observation into the Earth's interior. This is a problem where mathematics comes to the rescue in the form of Green's theorem. It relates measurements taken at a surface bounding a volume to the fields inside that volume. This principle is called *downward continuation*.

In the following we apply Green's theorem to a large class of physical systems and show that this theorem only relates measurements at a surface to measurements in the interior when the equations are the same regardless of whether one

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moves towards the future or towards the past. Such equations are said to be invariant for time reversal. We focus in particular on seismic imaging because this is the technique that provides the highest spatial resolution.

2 Green's theorem for general systems

Consider physical systems that satisfy the following partial differential equation for a field $u(\mathbf{r}, t)$ that is excited by sources $q(\mathbf{r}, t)$:

$$\sum_{n=0}^{N} a_n(\mathbf{r}) \frac{\partial^n u}{\partial t^n} = \nabla \cdot (B(\mathbf{r}) \nabla u(\mathbf{r}, t)) + q(\mathbf{r}, t). \quad (1)$$

This equation captures many specific equations. An example is the wave equation

$$\frac{1}{\kappa(\mathbf{r})} \frac{\partial^2 u}{\partial t^2} + \gamma(\mathbf{r}) \frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{1}{\rho(\mathbf{r})} \nabla u \right) + q(\mathbf{r}, t), \quad (2)$$

where κ is the bulk modulus, γ is a damping parameter and ρ is the density. Another example of equation (1) is the diffusion equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = \nabla \cdot (D(\mathbf{r}) \nabla u(\mathbf{r}, t)) + q(\mathbf{r}, t), \quad (3)$$

with $D(\mathbf{r})$ the diffusion constant. This equation is used to describe flow in porous media such as aquifers and hydrocarbon reservoirs. It also accounts for heat conduction and for diffusive spreading of pollutants. A variant of (3) is the Schrödinger equation that accounts for the dynamics of microscopic particles

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} - V(\mathbf{r}) \psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t). \quad (4)$$

Here $\hbar$ is Planck's constant divided by 2π, m is the mass of the particle and $V(\mathbf{r})$ the real potential in which the particle moves. In geophysics the gravitational potential plays an important role because it helps constrain the mass density inside the Earth. The gravitational field satisfies Poisson's equation

$$0 = \nabla^2 u(\mathbf{r}) - 4\pi G \rho(\mathbf{r}), \quad (5)$$

with G the gravitational constant. This equation does not depend on time.

Note that the applications (2)–(5) are special forms of the general equation (1). In these applications $B(\mathbf{r})$ is real, hence we use $B = B^*$ in the following, with the asterisk denoting complex conjugation. We also use the Fourier convention and notation $f(t) = \int f(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega$. With this, the general equation (1) reduces to

$$\sum_{n=0}^{N}(-\mathrm{i}\omega)^n a_n(\mathbf{r})u(\mathbf{r},\omega)$$
$$= \nabla \cdot (B(\mathbf{r})\nabla u(\mathbf{r},\omega)) + q(\mathbf{r},\omega). \qquad (6)$$

Each time derivative is replaced by a multiplication with $-\mathrm{i}\omega$. The treatment that follows is valid in the frequency domain. For brevity we omit the frequency dependence of variables.

Green's theorem follows by considering two different field states, $u_P(\mathbf{r})$ and $u_Q(\mathbf{r})$, that are excited by sources $q_P(\mathbf{r})$ and $q_Q(\mathbf{r})$, respectively. Take equation (1) for state P, multiply with u_Q^*, and integrate over volume. Next take the complex conjugate of (1) for state Q, multiply with u_P and integrate over volume. Subtracting these two volume integrals and applying Green's theorem to the terms containing $B(\mathbf{r})$ gives

$$\sum_{n=0}^{N}(-\mathrm{i}\omega)^n \int \mathcal{A}_n u_P u_Q^* \,\mathrm{d}V$$
$$= \oint B \left(\frac{\partial u_P}{\partial n}u_Q^* - u_P \frac{\partial u_Q^*}{\partial n} \right)\mathrm{d}S$$
$$+ \int \left(u_Q^* q_P - u_P q_Q^* \right)\mathrm{d}V, \qquad (7)$$

where $\partial/\partial n$ denotes the outward normal derivative to the surface S that bounds the volume over which we integrate and

$$\mathcal{A}_n(\mathbf{r}) = a_n(\mathbf{r}) - (-1)^n a_n^*(\mathbf{r}). \qquad (8)$$

In the following the Green's function $G(\mathbf{r},\mathbf{r}_0)$, defined as the solution of (6) to a point excitation, $q(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$, plays a key role. An important property of G is reciprocity:

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r}). \qquad (9)$$

Under suitable boundary conditions, this property is valid for all applications that follow.

Consider the case where u_P is source-free ($q_P = 0$) and u_Q is excited by a point source, $q_Q(\mathbf{r}) =$ $\delta(\mathbf{r} - \mathbf{r}_Q)$. Then $u_Q(\mathbf{r}) = G(\mathbf{r},\mathbf{r}_Q) = G(\mathbf{r}_Q,\mathbf{r})$. Using this in (7), denoting u_P by u, and replacing $\mathbf{r}_Q \to \mathbf{r}$ and $\mathbf{r} \to \mathbf{r}'$ gives

$$u(\mathbf{r}) = -\sum_{n=0}^{N}(-\mathrm{i}\omega)^n \int \mathcal{A}_n(\mathbf{r}')G^*(\mathbf{r},\mathbf{r}')u(\mathbf{r}')\,\mathrm{d}V'$$
$$+ \oint B(\mathbf{r}') \left(G^*(\mathbf{r},\mathbf{r}')\frac{\partial u}{\partial n'} - \frac{\partial G^*(\mathbf{r},\mathbf{r}')}{\partial n'}u \right)\mathrm{d}S'. \qquad (10)$$

3 Moving the field into the interior

Equation (10) is a powerful tool for propagating measurements taken at the boundary of a system into the interior of that system. This is of particular importance in Earth science. We first illustrate this principle for the acoustic wave equation (2), which is a prototype of the equations that govern seismic imaging. In the notation of (1) the wave equation (2) has $N = 2$, $a_2 = 1/\kappa$, $a_1 = \gamma$, $a_0 = 0$ and $B = 1/\rho$. The coefficients a_n enter (10) in a volume integral though the combination (8),

$$\mathcal{A}_n = \begin{cases} 2\mathrm{i}\,\mathrm{Im}\,(a_n) & \text{for } n \text{ even} \\ 2\,\mathrm{Re}\,(a_n) & \text{for } n \text{ odd}, \end{cases} \qquad (11)$$

where Re and Im denote the real and imaginary part, respectively. According to (11), a_2 does not contribute because κ is real. Consider first the case when there is no attenuation, so that $a_1 = \gamma = 0$ and (10) reduces to the *representation theorem*

$$u(\mathbf{r}) = \oint \frac{1}{\rho} \left(G^*(\mathbf{r},\mathbf{r}')\frac{\partial u}{\partial n'} - \frac{\partial G^*(\mathbf{r},\mathbf{r}')}{\partial n'}u \right)\mathrm{d}S'. \qquad (12)$$

This expression relates measurements at the surface in the integral on the right-hand side give the wave field in the interior on the left-hand side.

What happens if there is attenuation? In that case $a_1 = \gamma > 0$, and according to (10) and (11), equation (12) must be extended with a volume term $2\mathrm{i}\omega \int \gamma(\mathbf{r}')G^*(\mathbf{r},\mathbf{r}')u(\mathbf{r}')\,\mathrm{d}V$ on the right-hand side. This term contains the wave field in the interior that we seek to determine, so that this field does not follow from measurements at the surface only. In principle, attenuation makes seismic imaging impossible. In practice, attenuation in the Earth is weak so that the offending volume integral can be ignored.

Can diffuse fields be imaged? For the diffusion equation (3) the only nonzero terms in (1) are $a_1 = 1$ and $B = D$. Inserting these in (10) gives

$$u(\mathbf{r}) = 2\mathrm{i}\omega \int G^*(\mathbf{r}, \mathbf{r}')u(\mathbf{r}')\,\mathrm{d}V$$
$$+ \oint D(\mathbf{r}') \left(G^*(\mathbf{r}, \mathbf{r}')\frac{\partial u}{\partial n'} - \frac{\partial G^*(\mathbf{r}, \mathbf{r}')}{\partial n'}u \right)\mathrm{d}S'.$$

Just as for attenuating acoustic waves the right-hand side contains the unknown field in the interior. This means that measurements of diffusive fields taken at the surface cannot be used for imaging using Green's theorem.

The Schrödinger equation (4) is first order in time, and for this reason one might think that like for the diffusion equation one cannot infer the field values within a volume from measurements taken at the boundary. For this equation $N = 1$, $a_1 = \mathrm{i}\hbar$, $a_0 = -V$ and $B = -\hbar^2/(2m)$. According to (10) and (11), and assuming that the potential V is real, the volume integral depends on $\mathrm{Im}\,(a_0) = \mathrm{Im}\,(-V) = 0$ for $n = 0$ and on $\mathrm{Re}\,(a_1) = \mathrm{Re}\,(\mathrm{i}\hbar) = 0$ for $n = 1$. The volume integral thus vanishes and field values in the interior can be determined from field values measured at the boundary.

For the gravitational potential, field equation (5), all $a_n = 0$ and $B = 1$, so that in a source-free region the field satisfies

$$u(\mathbf{r}) = \oint \left(G(\mathbf{r}, \mathbf{r}')\frac{\partial u}{\partial n'} - \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'}u(\mathbf{r}') \right)\mathrm{d}S'. \tag{13}$$

The potential field does not depend on time, and as a result both the field u and the Green's function G are real functions, hence there are no complex conjugates in (13). This expression makes it possible to infer the gravitational field above the Earth when the field is known at the Earth's surface. Expression (13) can be used for *upward continuation*, where one infers the gravitational field at higher elevations from measurements taken at the Earth's surface. This can be used, for example, to compute the trajectories of satellites. Similarly, one can use this expression for *downward continuation* where one computes the gravitational field at lower elevations from measurements taken higher up. An application is to infer the gravitational field at the Earth's surface from measurements taken from satellites or aircraft. It

is, however, not possible to use (13) to compute the gravitational field inside the Earth. In the interior the mass density $\rho(\mathbf{r})$ is nonzero, and according to (5) the source $q(\mathbf{r})$ is nonzero. This violates the assumption $q_P = 0$ used in the derivation of equation (10). For this reason Green's theorem cannot be used to infer the mass density in the Earth from measurements taken at the surface.

In general, the property that the field in the interior follows from field measurements taken at the boundary is valid for systems that are *invariant for time reversal*. These are systems that obey equations that are invariant when time is reversed and t is replaced by $-t$. This is true for the wave equation in the absence of attenuation, but attenuation breaks the symmetry between past and future. The diffusion equation is not invariant under time reversal; heat diffuses away when moving to the future. Like the diffusion equation, the Schrödinger's equation is of first order in time, and one might think it is not invariant for time reversal. One can show, however, that when $\psi(\mathbf{r}, t)$ is a solution, then so is $\psi^*(\mathbf{r}, -t)$. According to the principles of quantum mechanics one cannot make a distinction between the wave function and its complex conjugate, hence the equation is effectively invariant for time reversal and, as we have seen, measurements at the surface suffice to determine the field in the interior.

4 Seismic imaging

In this section, we discuss the application of the representation theorem (12) to seismic imaging. A typical marine seismic experiment is shown in figure 1. A ship tows a streamer (dashed line). This is a long tube with hydrophones (pressure sensors) and/or geophones (motion sensors) that act as recording devices. An air gun (a device delivering an impulsive bubble of air) acts as a seismic source just behind the ship. The waves reflected by layers in the Earth are recorded by sensors in the streamer.

The water surface is a free surface, hence the pressure p vanishes there: $p(z = 0) = 0$. However, the particle motion does not vanish. According to Newton's law, the acceleration $\mathbf{a}$ is related to the pressure by $\rho\mathbf{a} = -\nabla p$. The vertical
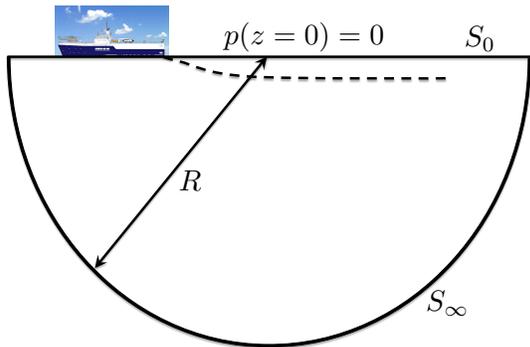
Figure 1: The geometry of a marine seismic survey.

component of this expression is given by

$$\rho a_z = -\frac{\partial p}{\partial z}. \qquad (14)$$

We use this relation in the representation theorem (12) for the pressure p. For the boundary we take the combination of the sea surface S_0 and a hemisphere S_∞ with radius R (figure 1). In the presence of a tiny amount of attenuation the pressure p and Green's function G decay as $\exp(-\alpha R)$, with α an attenuation coefficient, and the contribution of S_∞ vanishes as $R \to \infty$. The closed surface integral thus reduces to the contribution of the free surface S_0. Since that surface is horizontal, the normal derivative is just the derivative in the $-z$-direction. (Choosing a coordinate system with positive z pointing down.) As the pressure vanishes at the free surface, expression (12) reduces to $p(\mathbf{r}) = -\int_{S_0} \rho^{-1}(\mathbf{r}')G^*(\mathbf{r},\mathbf{r}')(\partial p(\mathbf{r}')/\partial z')\,dS'$. Eliminating the vertical derivative of the pressure with expression (14) gives

$$p(\mathbf{r},\omega) = \int_{S_0} G^*(\mathbf{r},\mathbf{r}',\omega)a_z(\mathbf{r}',\omega)\,dS', \qquad (15)$$

having restored the frequency dependence. This formula relates the pressure in the subsurface to the motion recorded at the sea surface.

The reader may have wondered why the complex conjugate of G was used, since most of the expressions hold also when the complex conjugation is not applied. The time-domain Green's function is related to the frequency domain Green's function by $G(\mathbf{r},\mathbf{r}',t) = \int G(\mathbf{r},\mathbf{r}',\omega)e^{-i\omega t}\,d\omega$, hence the time-reversed Green's function satisfies. $G(\mathbf{r},\mathbf{r}',-t) = \int G^*(\mathbf{r},\mathbf{r}',\omega)e^{-i\omega t}\,d\omega$. This means that $G^*(\mathbf{r},\mathbf{r}',\omega)$ corresponds in the time domain to the time-reversed Green's function $G(\mathbf{r},\mathbf{r}',-t)$. As a consequence, equation (15) corresponds in the time domain to

$$p(\mathbf{r},t) = \int_{S_0} G(\mathbf{r},\mathbf{r}',-t) \star a_z(\mathbf{r}',t)\,dS', \qquad (16)$$

where the star $(\star)$ denotes convolution. The Green's function $G(\mathbf{r},\mathbf{r}',t)$ is causal, meaning that it is only nonzero after the point source acts at $t = 0$. It then moves the waves forward in time away from the point of excitation. Consequently, the time-reversed Green's function $G(\mathbf{r},\mathbf{r}',-t)$ is nonzero only for $t < 0$, and it propagates the wave backward in time. In (16) the time reversed Green's function $G(\mathbf{r},\mathbf{r}',-t)$ is convolved with the recorded acceleration. This means that it takes the waves that are recorded at the streamer and propagates them backward in time. This is a desirable property; in order to find the reflectors in the Earth one needs to know the wave field at the moment when it was reflected off the reflectors. The recorded waves thus need to be propagated back in time to know them at earlier times as they were reflecting inside the Earth. This is the reason why the time-reversed Green's function is used, and ultimately this is the reason why the theory presented here used the complex conjugate $G^*(\mathbf{r},\mathbf{r}',\omega)$ instead of $G(\mathbf{r},\mathbf{r}',\omega)$. If we had used $G(\mathbf{r},\mathbf{r}',t)$ instead of $G(\mathbf{r},\mathbf{r}',-t)$, equation (16) would give the pressure field inside the Earth *after* it has been recorded at the receivers. This field does not give information about the interaction of waves with reflectors *before* the waves propagated to the surface where they are recorded. For this reason the theory in section 3 is based on G^* rather than G.

5 A chicken and egg problem

As shown here, Green's theorem makes it possible to infer the value of a physical field in the interior of the Earth from measurements taken at the Earth's surface, or above that surface. There is, however, a catch. In order to downward continue

fields measured at the Earth's surface, one must know the Green's function, see for example (15). For the wave equation (2) the space and time derivative fields are multiplied by the mass density and bulk modulus of the Earth, respectively. The Green's function needed for downward continuation of seismic waves thus depends on the properties of the Earth, but it is these properties that one seeks to determine. We therefore need the properties of the Earth to determine the properties of the Earth!

Fortunately, there is a way out off this conundrum. It turns out that for seismic imaging it suffices to have an estimate of the Green's function that positions the wavefronts at more or less the correct location. Such an estimated Green's function is computed from a smooth velocity model. The velocity used is obtained from a procedure called *velocity estimation* where one determines a smooth velocity model from measured arrival times from reflected seismic waves. The success of the seismic method in the hydrocarbon industry shows that this procedure works in practice.

Further Reading

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