Geophys. J. Int. (1990) 101, 545–556

A perturbative analysis of non-linear inversion

Roel Snieder
Department of Theoretical Geophysics, University of Utrecht, PO Box 80.021, 3508 TA Utrecht, The Netherlands

Accepted 1989 November 30. Received 1989 November 20; in original form 1989 May 9

SUMMARY

In most geophysical forward problems, the data are related in a non-linear way to the model. Similarly, the inverse problem constitutes a non-linear mapping from the data to the model. In general, it is poorly understood which parts of the data contribute to the reconstruction of the model, and how the non-linearities are handled in the inversion. A perturbative treatment of non-line · inversion clarifies these issues. This perturbative treatment is first applied to · inverse scattering algorithm which has an exact solution, and is then generalized for a very wide class of non-linear inverse problems. It is shown that only the linear component in the data (the first Born approximation) contributes to the reconstruction of the model, and that the non-linear components in the data are being subtracted in the inversion. These arguments are generalized for a very wide class of non-linear inverse problems. The analysis can be used to derive non-linear inversion schemes. As an example an algorithm is derived which performs non-linear traveltime tomography without iteratively shooting or bending rays. The analysis has profound implications for the stability of non-linear inversions.

Key words: inverse scattering, non-linear inversion.

1 INTRODUCTION

In many problems of geophysics, quantum mechanics or non-destructive testing of materials it is impossible or undesirable to take direct samples of the object of interest. Instead, one can employ fields, such as elastic wavefields, magnetic fields, particle beams etc., to probe the unknown object. The mapping from the model to the measured fields is called the forward problem, while the mapping from the measured fields to the (unknown) model is known as the inverse problem. The forward and the inverse problem can either be linear or non-linear. Linear inverse problems are relatively well understood. In contrast to this, very little is known about the nature of non-linear inversion. The aim of this paper is to establish the effects of the non-linearities in non-linear inversion.

For the non-linear inverse problem in quantum mechanics, where one wants to reconstruct an unknown potential given a set of scattering data, exact inversion schemes (inverse scattering) have been formulated (e.g. Agranovich & Marchenko 1963; Newton 1981; Chadan & Sabatier 1989). These algorithms involve the solution of similar integral equations. It was poorly understood how these algorithms reconstruct the potential, and how the non-linearities are handled in the inversion. In Section 2, an inverse scattering method is used as a prototype of non-linear inverse problems. A perturbative treatment of the forward and inverse problem is the natural way to investigate the role of the linear and the non-linear aspects of both forward and the inverse problem. It is shown both analytically and numerically how the potential is reconstructed in an inverse scattering algorithm from the scattering data, and how the multiply scattered waves are handled in the inversion.

The perturbative analysis of inverse scattering methods can be generalized to any forward and inverse problem which have a regular perturbation expansion with non-vanishing first-order terms (Section 3). The analysis generalizes the ideas of Schmidt (1908), Moses (1956) and Prosser (1969), and covers the implications for non-linear inversion. This treatment leads to a set of recursive equations which can be used for the design of non-linear inversion methods. As an example, these equations are used in Section 4 to derive a non-linear tomographic inversion method which corrects for the second-order effects of ray bending without actually shooting or bending of rays.

It is known that some non-linear inverse wave propagation problems are very sensitive to errors, especially to timing errors (Koehler & Taner 1977; Koltracht & Lancaster 1988). In Section 5, the detrimental effects of errors on inverse scattering methods is illustrated with a numerical example. The results of Section 3 are used in

Section 5 to explain why exact non-linear inversion methods in general are very sensitive to errors in the data.

With realistic data sets, one can rarely apply exact inversion methods, because exact inversion methods in general are not applicable to noise contaminated data with a finite bandwidth, a limited number of sample points, etc. In practice, one can use methods of optimization to fit the data to synthetic data as a function of the model parameters. It is shown in Section 6 that the conclusions of this paper for exact inversion schemes can be extended to inversion methods where one fits the data to synthetics using methods of optimization.

2 A PERTURBATIVE TREATMENT OF INVERSE SCATTERING

As an example of the perturbative analysis of an exact non-linear inverse problem, inverse scattering as originally developed in quantum mechanics is discussed. Consider the wavefield $u(x, t)$ that satisfies the Plasma Wave Equation (PWE) (Balanis 1972);

$$u_{xx} - u_{tt} - \varepsilon m(x)u = 0. \tag{1}$$

The unknown potential (the model) is described by $\varepsilon m(x)$. The coefficient ε is used to facilitate a systematic perturbative treatment of the forward and the inverse problem. For simplicity it is assumed that

$$m(x) = 0, \qquad x < 0. \tag{2}$$

The model is illuminated by a delta pulse from the left

$$u(x, t) = \delta(t - x), \qquad (t < 0), \tag{3}$$

and the data $d(t)$ are the reflected waves recorded at $x = 0$:

$$d(t) = u(x = 0, t) - \delta(t - x). \tag{4}$$

The inverse problem consists of the reconstruction of the model $m(x)$, given the data $d(t)$. This inverse problem has an exact solution (Agranovich & Marchenko 1963; Burridge 1980), which can be found by solving the Marchenko equation

$$K(x, t) + d(x + t) + \int_{-t}^{x} K(x, \tau)d(\tau + t)\, d\tau = 0. \tag{5}$$

This integral equation needs to be solved for $K(x, t)$ for every x as a function of t. The data $d(t)$ act as the kernel for this integral equation. The model can then be found by differentiation:

$$\varepsilon m(x) = 2\frac{dK(x, x)}{dx}. \tag{6}$$

It is instructive to perform a perturbative treatment of the forward and inverse problem. For simplicity, the perturbative treatment is shown here only for the lowest orders; the generalization to higher orders is straightforward. Using the Lippman–Schwinger equation, a Born series for the data $d(t)$ can be derived:

$$d(t) = \varepsilon d_1(t) + \varepsilon^2 d_2(t) + \cdots. \tag{7}$$

In this expression, $d_1(t)$ is the first Born approximation which consists of the single reflected waves, $d_2(t)$ is the second Born approximation describing the double reflected waves, etc. It should be noted that $d(t)$ represents the recorded data, while the $d_i(t)$ denote the subsequent terms in the Born series as obtained from the theory. In an experiment one only measures the sum $d(t)$ of the single and multiple reflected waves. The terms $d_i(t)$ in the Born series can be expressed using the Green's function

$$G(x, t; x', t') = -\tfrac{1}{2}H(t - t' - |x - x'|), \tag{8}$$

$H(t)$ being the Heaviside function. This Green's function is the causal solution of

$$G_{xx}(x, t; x', t') - G_{tt}(x, t; x', t') = \delta(x - x')\,\delta(t - t'). \tag{9}$$

The first terms of the Born series are given by

$$d_1(t) = \int G(x = 0, t; x_1, t_1)m(x_1)\,\delta(t_1 - x_1)\, dx_1 dt_1, \tag{10a}$$

$$d_2(t) = \int G(x = 0, t; x_1, t_1)m(x_1)G(x_1, t_1; x_2, t_2) \times m(x_2)\,\delta(t_2 - x_2)\, dx_1 dt_1 dx_2 dt_2, \tag{10b}$$

$$d_3(t) = \int G(x = 0, t; x_1, t_1)m(x_1)G(x_1, t_1; x_2, t_2) \times m(x_2)G(x_2, t_2; x_3, t_3) \times m(x_3)\,\delta(t_3 - x_3) dx_1 dt_1 dx_2 dt_2 dx_3 dt_3. \tag{10c}$$

The integral kernel $K(x, t)$ is implicitly also a function of the model $\varepsilon m(x)$, and can also be expanded in a Taylor series in the parameter ε:

$$K(x, t) = \varepsilon K_1(x, t) + \varepsilon^2 K_2(x, t) + \varepsilon^3 K_3(x, t) + \cdots. \tag{11}$$

Inserting (11) and the Born series (7) in the Marchenko equation (5), and equating the coefficients of equal powers of ε gives the following hierarchy of equations

$$K_1(x, t) = -d_1(x + t), \tag{12a}$$

$$K_2(x, t) = -d_2(x + t) - \int_{-t}^{x} K_1(x, \tau)d_1(\tau + t)\, d\tau, \tag{12b}$$

$$K_3(x, t) = -d_3(x + t) - \int_{t}^{x} K_2(x, \tau)d_1(\tau + t)\, d\tau - \int_{-t}^{x} K_1(x, \tau)d_2(\tau + t)\, d\tau, \tag{12c}$$

$$K_n(x, t) = -d_n(x + t) - \sum_{j=1}^{n-1} \int_{-t}^{x} K_j(x, \tau)d_{n-j}(\tau + t)\, d\tau. \tag{12d}$$

These equations are also denoted by the following abbreviated notation: $K_1 = -d_1$, $K_2 = -d_2 - K_1 d_1$, $K_3 = -d_3 - K_2 d_1 - K_1 d_2$, etc. The power series (11) can also be inserted in expression (6) for the reconstructed model. Equating the coefficients of equal power of ε gives

$$2\frac{dK_1(x, x)}{dx} = m(x), \tag{13}$$

$$2\frac{dK_n(x, x)}{dx} = 0, \qquad n \geq 2. \tag{14}$$

According to (13), the model $m(x)$ is completely determined by the first-order contribution K_1. The higher order terms K_n $(n \geq 2)$ do not contribute to the reconstruction of the model; see (14). According to (12a), the first-order term K_1 depends only on the first Born approximation d_1 of the data. This means that the reconstructed model is completely determined by the first Born approximation, and that the non-linear components in the data (d_n, with $n \geq 2$) do not contribute to the reconstruction of the model.

So what happens to the non-linear components d_n $(n \geq 2)$ in the data? Repeated recursion of the equations (12) gives

$$K_2 = -d_2 + d_1 d_1, \tag{15a}$$

$$K_3 = -d_3 + d_2 d_1 + d_1 d_2 - d_1 d_1 d_1, \tag{15b}$$

$$K_n = -d_n + \sum_{j=2}^{n} (-1)^j \sum_{i_1 + \cdots + i_j = n} d_{i_1} \cdots d_{i_j}. \tag{15c}$$

In each of these equations, the higher order Born approximation $-d_n$ is being cancelled by repeated iterations of lower order Born terms. For example, in (15a) the double scattered waves d_2 are being cancelled by the iteration $d_1 d_1$ of the single scattered waves. The same happens for the higher order terms. This means that in the reconstruction of the model using the Marchenko equation, only the first Born approximation contributes to the reconstruction of the model, and that the non-linear components in the data are being cancelled by repeated iteration of lower order Born terms.

A numerical example illustrates these principles. The potential is chosen to be non-zero in two thin regions as shown in Fig. 1. This potential basically consists of two regions where the potential is non-zero. Waves can bounce back and forth between the two sides of the potential. The first, second, and third Born approximations from (10) are shown in Fig. 2, together with their sum. In the third Born approximation (the dotted line), the wave that has bounced back and forth once between the sides of the potential can clearly be seen around $t = 300$.

The first-order contribution to $K_1(x, x)$ computed with (10a) and (12a) is shown in Fig. 3. [$K(x, x)$ is shown here rather than the reconstructed model, because the relevant effects are easier to see before the differentiation (6).] This function is after the differentiation (13) indistinguishable

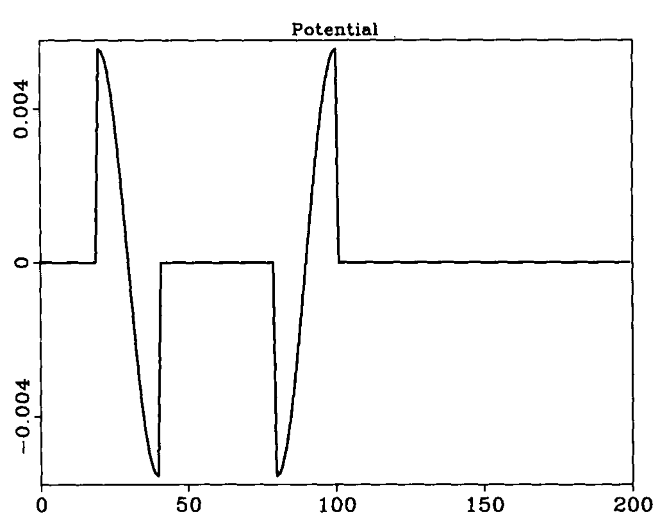

Figure 1. Potential used in the numerical example.

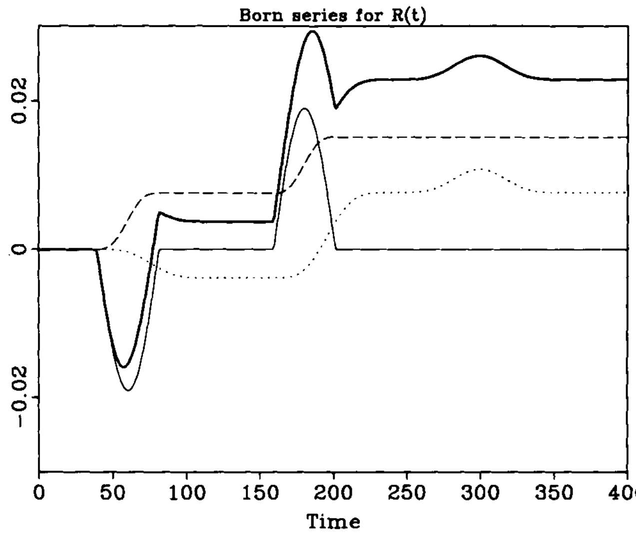

Figure 2. The first (thin solid line); second (dashed line) and third (dotted line) Born approximations and their sum (thick solid line).

from the true potential in Fig. 1. This confirms that only the linear component d_1 contributes to the reconstruction of the potential.

The second-order contributions to $K_2(x, x)$ computed with (10a, b) and (12b) are shown in Fig. 4. The second-order Born approximation $-d_2$ is being cancelled by the iterated term $-K_1 d_1$; the sum of these terms is shown by the thick solid line. This confirms that the second-order Born term d_2 is being subtracted in the inversion by the iterated first Born approximation, and that there is no net contribution of second-order terms to the reconstruction of the model.

The third-order contributions to $K_3(x, x)$ computed with (10a, b, c) and (12c) are shown in Fig. 5. The cubic term $-d_3$ (the dash–dotted line) is being cancelled by iterated terms $-K_2 d_1$ and $-K_1 d_2$ of lower order Born approximations, so that the third-order Born approximation d_3 does

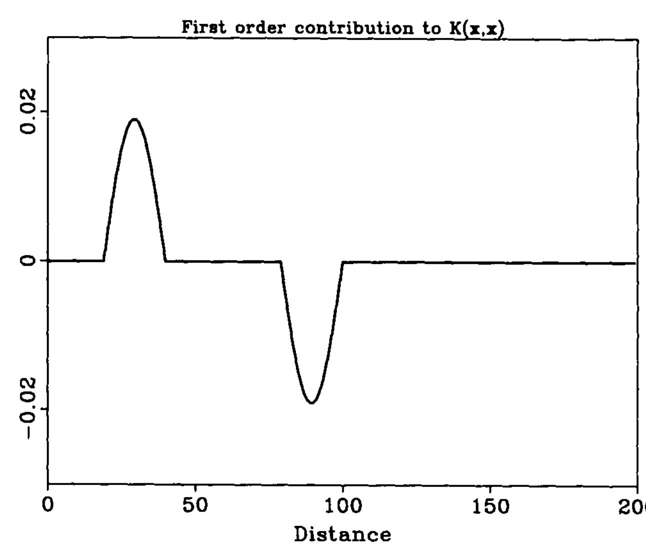

Figure 3. The first-order contribution $K_1(x, x)$ to the reconstruction of the potential. The derivative $2(dK_1(x, x)/dx)$ is indistinguishable from the true potential in Fig. 1.

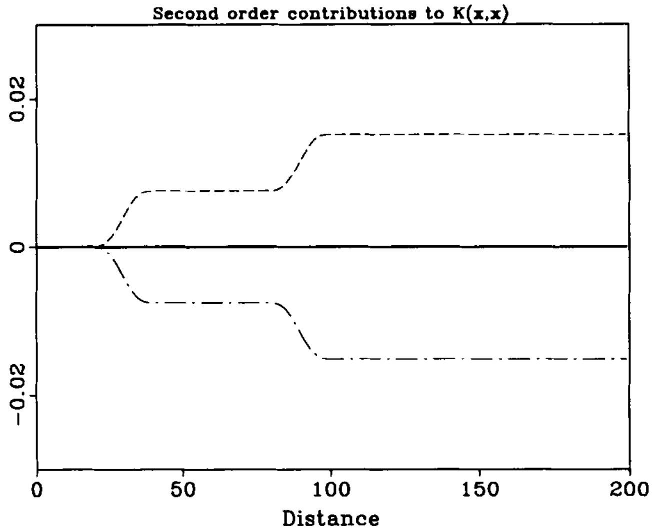

Figure 4. The second-order contributions $-R_2$ (dash–dotted line) and $-K_1R_1$ (dashed line) to the reconstruction of the potential, with their sum $K_2(x, x)$ (thick solid line).

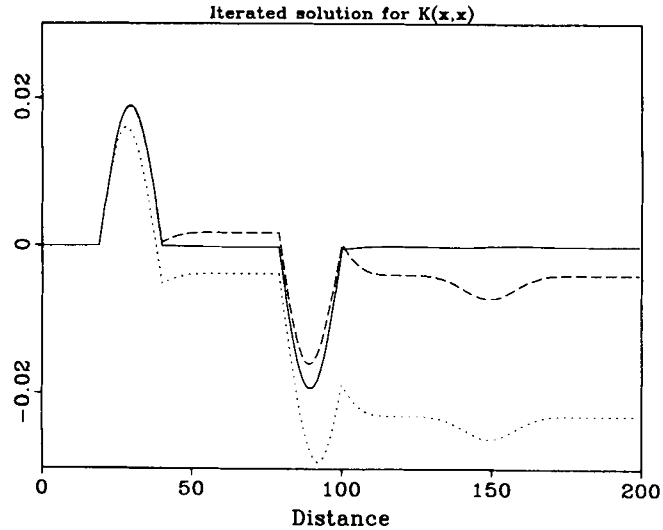

Figure 6. Iterative solution of the Marchenko equation after one (dotted line), two (dashed line), and three iterations (solid line). Only terms up to third order have been taken into account.

not contribute to the reconstruction of the potential. The sum of these terms (the thick solid line) is equal to zero. The slight deviation from zero around $t = 100$ is caused by numerical errors. This is due to the fact that the subtraction of the non-linear components from the data is a numerically unstable process.

As shown by Ge (1987), the Marchenko equation can efficiently be solved by iteration. In the iterative method of Ge (1987) the starting value is

$$\tilde{K}^{(1)}(x, t) = -d(x + t), \tag{16a}$$

and subsequent iterations are given by

$$\tilde{K}^{(n+1)}(x, t) = -d(x + t) - \int_{-t}^{x} \tilde{K}^{(n)}(x, \tau) d(\tau + t)\, d\tau, \quad n \geq 1. \tag{16b}$$

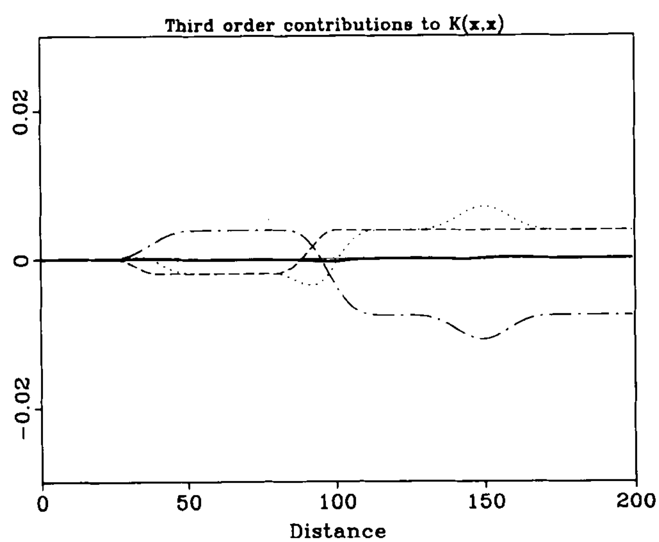

Figure 5. The third-order contributions $-R_3$ (dash–dotted line), $-K_1R_2$ (dashed line) and $-K_2R_1$ (dotted line) to the reconstruction of the potential, with their sum $K_3(x, x)$ (thick solid line).

The solution of the Marchenko equation after the first three iterations is shown in Fig. 6. In this computation only terms up to third order are taken into account. The first iteration (the dotted line) amounts to a Born inversion of the data. The non-linear components in the data are after one iteration incorrectly mapped to artifacts in the data, such as for example the bump around $x = 150$. After several iterations these artifacts are removed from the reconstructed model. The result of this iterative process (the solid line) is after the differentiation (6) indistinguishable from the true potential. In this example, three iterations were sufficient, because only terms up to third order were taken into account. It is shown in Snieder (1990) that after N iterations all scattering effects up to Nth order are correctly handled.

Note that the first iteration approximates the potential quite well for small x values ($x \approx 25$), but for larger values of x more iterations are needed for the removal of the non-linear effects. The reason for this is that the multiply scattered waves need time to bounce back and forth, so that the non-linearities are more pronounced in the later part of the signal. The later part of the signal is mapped in the first iteration to larger x values; this means that in the iterative process the model is more strongly updated for large x values than for small x values.

The fact that only the first Born approximation d_1 contributes to the reconstruction of the potential does not imply that it suffices to perform a Born inversion of the data. The reason for this is that one does not measure the first Born approximation, but that one measures the sum of both the linear and the non-linear components in the data. A Born inversion of the raw data maps the non-linearities into artifacts in the reconstructed model. This can be seen in Fig. 6. The first iteration (the dotted line) amounts to a Born inversion of the data. The multiply reflected waves lead to artifacts in the model after a Born inversion, see for example the bump around $x = 150$. In subsequent iterations, these artifacts are subtracted from the reconstructed model; see the solid line in Fig. 6. It is thus crucial that the non-linear components are subtracted from the data; for the example in this section this is achieved in an implicit fashion by solving the Marchenko equation (5).

3 A GENERAL PERTURBATIVE TREATMENT OF NON-LINEAR INVERSION

The arguments presented in the previous section for the inverse problem for the Plasma Wave Equation using the Marchenko algorithm can be generalized for a very wide class of non-linear inverse problems. Suppose we have data $d(\varepsilon)$, and a model εm. The parameter ε facilitates a systematic perturbative treatment. The data $d(\varepsilon)$ are in general a non-linear function of the model εm, this is the forward problem F which constitutes a non-linear mapping from the model εm to the data $d(\varepsilon)$:

$$d(\varepsilon)=F(\varepsilon m)=\varepsilon F_1 m+\varepsilon^2 F_2 m^2+\varepsilon^3 F_3 m^3+\cdots. \qquad (17)$$

Just as in Section 2, the forward problem F has been expanded in a Taylor series in ε. The term $\varepsilon F_1 m$ is called the first Born approximation, the term $\varepsilon^2 F_2 m^2$ the second Born approximation, etc. The inverse problem I consists of a mapping from the data $d(\varepsilon)$ to the model εm. This mapping is also non-linear and has a perturbation expansion similar to (17):

$$\varepsilon m=I[d(\varepsilon)]=I_1 d+{}^2I_2 d^2+{}^3I_3 d^3+\cdots, \qquad (18)$$

the operators I_n indicating operators which act on the data n times. Note that it is tacitly assumed that the forward and the inverse problem have regular perturbation expansions.

The forward and inverse mappings F and I are in general operators, which may include differentiations, integrations or other operators. In the example of the previous section, the operator F_2 is according to (10b) given by

$$\begin{aligned}d_2(t)&\equiv(F_2 m^2)(t)\\&=\int G(x=0,t;x_1,t_1)m(x_1)\delta(t_1-x_1)G(x_1,t_1;x_2,t_2)\\&\quad\times m(x_2)\,\delta(t_2-x_2)dx_1dt_1dx_2dt_2.\end{aligned} \qquad (19)$$

As can be seen in (19), terms like m^2 simply indicate that the operator acts on the model twice, it does not necessarily mean the square is taken. The same applies to the operators in the inverse mapping (18). For the inverse mapping of the previous section, the terms I_n follow from the Marchenko equation (5) and the differentiation (6). The Marchenko equation can be iterated by writing the Marchenko equation (5) in the form $K=-d-Kd$ and by inserting this expression for K repeatedly in the right-hand side. This gives explicitly

$$\begin{aligned}K(x,t)&=-d(x+t)+\int_{-t}^{x}d(x+\tau)d(\tau+t)\,d\tau\\&\quad+\sum_{j=2}^{\infty}(-1)^j\int_{-t}^{x}d\tau_1\int_{-\tau_1}^{x}d\tau_2\cdots\int_{-\tau_{j-1}}^{x}d\tau_j\\&\quad\times d(x+\tau_1)d(\tau_1+\tau_2)\cdots d(\tau_j+t).\end{aligned} \qquad (20)$$

The first term depends linearly on the data and therefore leads to the operator I_1, the second term depends quadratically on the data and determines I_2, etc. The potential follows by applying the differentiation (6) to $K(x,t)$. This gives for the first- and second-order inverse operators:

$$(I_1 d)(x)=-2\frac{d}{dx}d(2x), \qquad (21)$$

$$(I_2 d^2)(x)=2\frac{d}{dx}\int_{-x}^{x}d^2(x+\tau)\,d\tau=4d^2(2x), \qquad (22)$$

where it is used in the last line that $d(0)=0$. It can thus be seen that for a specific problem, the operators F_n and I_n can be given a definite meaning.

The effect of the non-linearities in the inversion can be seen by inserting (17) in (18). Equating the coefficients of equal powers of ε gives the following equations:

$$m=I_1F_1m, \qquad (23a)$$

$$0=(I_1F_2+I_2F_1F_1)m^2, \qquad (23b)$$

$$0=(I_1F_3+I_2F_1F_2+I_2F_2F_1+I_3F_1F_1F_1)m^3, \qquad (23c)$$

$$0=\left[\sum_{j=1}^{n-1}I_j\sum_{i_1+\cdots+i_j=n}F_{i_1},F_{i_2}\cdots F_{i_j}+I_nF_1^n\right]m^n. \qquad (23d)$$

The above equations can be interpreted in the same way as in the preceding section. The left-hand side of these equations constitutes according to (18) the contribution of the terms of different order in the inversion. Only the left-hand side of (23a) is non-zero. This equation states that a Born inversion I_1 applied to the first Born approximation $F_1 m$ leads to the reconstruction of the model. The fact that the left-hand side of the other equations is zero implies that the non-linear components in the data such as $F_2 m^2$ do not give a contribution to the reconstruction of the model. Just as in Section 2, this happens because in the inversion the non-linear components are subtracted from the data. For example, in equation (23b) the first term indicates a Born inversion (I_1) of the component in the data which has a quadratic dependence on the model ($F_2 m^2$). If the non-linear inversion scheme is to reproduce the model m exactly, then this quadratic term is to be cancelled in the inversion by the application of the operator $I_2F_1F_1$. This means that just as with the Marchenko equation, only the first Born approximation contributes to the reconstruction of the model, and that the non-linearities are subtacted from the data. The same argument applies to the higher order equations like (23cd).

The equations (23) can be used to design non-linear inversion algorithms. In these equations, the forward operators F_i are supposed to be known. These equations should hold for every model m. From (23a) and the knowledge of F_1, one can usually deduce the inverse Born operator I_1 such that

$$I_1=F_1^{-1}. \qquad (24)$$

Given I_1, one can deduce I_2 from (23b) since this equation should hold for every model m. Given I_1 and I_2, one can determine I_3 from (23c). This procedure can in principle be continued to every desired order. More generally, it follows from (23d) and (24) that

$$I_n=-\sum_{j=1}^{n-1}I_j\sum_{i_1+\cdots+i_j=n}F_{i_1}F_{i_2}\cdots F_{i_j}I_1^n, \qquad (25)$$

so that I_n can be determined if $F_1 \cdots F_n$ and $I_1 \cdots I_{n-1}$ are known. For example, it is possible to derive (21) and (22) from the general expressions (23) using the operators F_1 and F_2 defined by $d_1 = F_1 m$ (10a) and $d_2 = F_2 m^2$ (10b). In this way it is possible to derive the lowest order inversion operators (21) and (22) for the Plasma Wave Equation without actually using the Marchenko equation or some of the other integral equations of exact inverse scattering.

Note that it is tacitly assumed in the derivation that the first-order term F_1 is non-zero. This is not always the case. In the theory of the magnetic jerk of Backus (1983), the forward problem is of the form

$$d(\varepsilon) = \varepsilon^2 \iint K(r_1, r_2) m(r_1) m(r_2) dr_1 dr_2. \tag{26}$$

For this problem the first-order operator F_1 is zero and can therefore not be inverted. For such a problem an inversion in the form of the power series (18) does not exist. To see this, consider the scalar relation $d(\varepsilon) = \varepsilon^2 m^2$ with the exact inverse $\varepsilon m = \sqrt{d(\varepsilon)}$, the inverse can clearly not be written as a power series of the form (18). For these kind of problems the theory is not applicable.

Although the equations (24) and (25) can in principle be used to find the inverse operators I_i, this procedure becomes increasingly complicated for the higher order terms. In practice, the design of non-linear inversion schemes will be most useful for problems which are only weakly non-linear, so that a few terms of the series $I(d) = I_1 d + I_2 d^2 + \cdots$ suffice for the accurate reconstruction of the model m. It can be seen from (24) and (25) that in order to derive the inverse operators $I_1, \ldots, I_n$, only the operators $F_1, \ldots, F_n$ are needed; the higher order operators of the forward problem need not be known. An example of the design of correction for non-linear effects in a weakly non-linear inverse problem is presented in the next section.

4 CORRECTIONS FOR RAY BENDING IN TOMOGRAPHY

Traveltime tomography is a linear problem if the rays are assumed to be fixed. However, the rays depend on the (unknown) velocity structure, so that ray bending effects make the problem non-linear. As the simplest prototype of this problem, consider a body with unknown velocity, which is being probed by rays in all directions ϕ, and with all ray offsets p; see Fig. 7. The rays travel between sources and receivers on different sides of the inhomogeneity and the employed sources and receivers are separated by a length L.

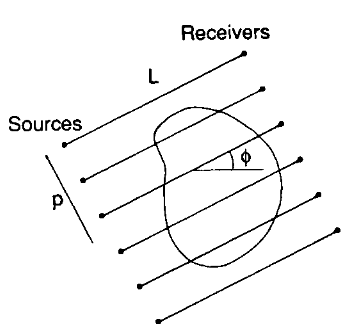

Figure 7. Geometry of the tomographic experiment.

The traveltime $T(p, \phi)$ for ray offset p and ray direction ϕ is given by

$$T(p, \phi) = \int_{\Gamma(p,\phi)} \frac{1}{c[\mathbf{r}(s)]} ds, \tag{27}$$

where the integral is over the true ray $\Gamma(p, \phi)$ connecting a source and receiver with offset p and azimuth ϕ. The position of the ray $\mathbf{r}(s)$ is a function of the length s along the ray. Now assume that the velocity $c(\mathbf{r})$ can be decomposed in a constant reference velocity c_0, and a small slowness perturbation $\varepsilon m(\mathbf{r})$:

$$\frac{1}{c(\mathbf{r})} = \frac{1}{c_0} + \varepsilon m(\mathbf{r}). \tag{28}$$

The slowness perturbation εm is the model to be determined.

In order to design an algorithm which corrects for ray bending, it is necessary to determine the lowest order terms of the forward problem, i.e., to prescribe the operators F_1 and F_2. The position $\mathbf{r}(s)$ of a ray depends on the velocity model. Therefore, this position is a function of the slowness perturbation εm:

$$\mathbf{r}(s) = \mathbf{r}_0(s) + \varepsilon \mathbf{r}_1(s) + \varepsilon^2 \mathbf{r}_2(s) + \cdots. \tag{29}$$

In this expression $\mathbf{r}_0(s)$ describes the straight reference ray through the velocity model c_0. Inserting (29) in (28), and Taylor expanding $m(\mathbf{r})$ gives

$$\frac{1}{c(\mathbf{r})} = \frac{1}{c_0} + \varepsilon m[\mathbf{r}_0(s)] + \varepsilon^2 \mathbf{r}_1(s) \cdot \nabla m[\mathbf{r}_0(s)] + \cdots. \tag{30}$$

If the ray is deflected by the velocity variation, one needs only to account for the displacement of the ray perpendicular to the reference ray in the unperturbed medium, hence it is assumed that

$$\mathbf{r}_i \perp \mathbf{r}_0, \qquad i \geq 1. \tag{31}$$

The total length of the ray is increased by the ray bending; the line increment ds along the true ray is related to the line increment ds_0 along the straight reference ray through the following relation:

$$\begin{aligned}(ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= d\mathbf{r}_0(s_0) \cdot d\mathbf{r}_0(s_0) + \varepsilon d\mathbf{r}_0(s_0) \cdot d\mathbf{r}_1(s_0) \\ &\quad + \varepsilon^2 [d\mathbf{r}_1(s_0) \cdot d\mathbf{r}_1(s_0) + 2 d\mathbf{r}_0(s_0) \cdot d\mathbf{r}_2(s_0)] + \cdots.\end{aligned} \tag{32}$$

Using (31) this leads to

$$ds = \left(1 + \frac{1}{2} \varepsilon^2 \frac{d\mathbf{r}_1}{ds_0} \cdot \frac{d\mathbf{r}_1}{ds_0}\right) ds_0 + \cdots. \tag{33}$$

With (30) and (33) one finds that the traveltime $T(p, \phi)$ is to order ε^2 given by

$$\begin{aligned}T(p, \phi) &= \int_{\Gamma_0(p,\phi)} \frac{ds}{c_0} + \varepsilon \int_{\Gamma_0(p,\phi)} m[\mathbf{r}_0(s)] \\ &\quad + \varepsilon^2 \int_{\Gamma_0(p,\phi)} \left\{ \mathbf{r}_1(s) \cdot \nabla m[\mathbf{r}_0(s)]\, ds + \frac{1}{2c_0} \frac{d\mathbf{r}_1}{ds} \cdot \frac{d\mathbf{r}_1}{ds} \right\} ds + \cdots.\end{aligned} \tag{34}$$

The velocity perturbation and the ray length are now related through (30) and (31) to the position of the unperturbed ray, therefore the integrals in this expression and the following expressions are over the straight reference rays $\Gamma_0(p, \phi)$. The data $d(p, \phi, \varepsilon)$ are the delay times due to the slowness perturbation:

$$d(p, \phi, \varepsilon) = \varepsilon \int_{\Gamma_0(p,\phi)} m[\mathbf{r}_0(s)]\, ds + \varepsilon^2 \int_{\Gamma_0(p,\phi)} \left\{ \mathbf{r}_1(s) \cdot \nabla m[\mathbf{r}_0(s)] + \frac{1}{2c_0} \frac{d\mathbf{r}_1}{ds} \cdot \frac{d\mathbf{r}_1}{ds} \right\} ds + \cdots. \quad (35)$$

This expression does not yet give the explicit non-linear relation between the data $d(p, \phi, \varepsilon)$ and the model $\varepsilon m(\mathbf{r})$, because the ray perturbation $\varepsilon \mathbf{r}_1(s)$ in (35) depends on the slowness perturbation $\varepsilon m(\mathbf{r})$. To make this relation explicit, one needs to use the equation of kinematic ray-tracing:

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{r}}{ds}\right) = \nabla \frac{1}{c}. \quad (36)$$

Inserting (28) for the slowness, and (29) for the ray position, one finds for the term proportional to ε:

$$\frac{dm}{ds}\frac{d\mathbf{r}_0}{ds} + m\frac{d^2\mathbf{r}_0}{ds^2} + \frac{1}{c_0}\frac{d^2\mathbf{r}_1}{ds^2} = \nabla m. \quad (37)$$

Using the fact that the unperturbed rays are straight ($d^2\mathbf{r}_0/ds^2 = 0$), and decomposing the gradient of the slowness in components parallel and perpendicular to the reference ray

$$\nabla m = \nabla_\perp m + \frac{dm}{ds}\frac{d\mathbf{r}_0}{ds}, \quad (38)$$

one finds that

$$\frac{d^2\mathbf{r}_1}{ds^2} = c_0 \nabla_\perp m. \quad (39)$$

The beginning and end of each ray is fixed, so that $\mathbf{r}_1(0) = \mathbf{r}_1(L) = 0$. Using these boundary conditions, the solution of (39) is given by

$$\mathbf{r}_1(s) = \int_0^L K(s, s')\nabla_\perp m(s')\, ds', \quad (40)$$

where it is understood that the integral is taken over the straight reference ray. The integral kernel $K(s, s')$ is given by

$$K(s, s') = -\left(1 - \frac{s}{L}\right)s'c_0, \qquad 0 < s' < s,$$
$$K(s, s') = -s\left(1 - \frac{s'}{L}\right)c_0, \qquad s < s' < L. \quad (41)$$

Inserting (40) in (35), the relation between the data $d(\varepsilon)$ and the model εm is up to order ε^2.

$$d(p, \phi, \varepsilon) = \varepsilon \int_{\Gamma_0(p,\phi)} m(s)ds + \varepsilon^2 \left\{ \int_{\Gamma_0(p,\phi)} ds \int_{\Gamma_0(p,\phi)} ds' \times K(s,s')\nabla_\perp m[\mathbf{r}_0(s)] \cdot \nabla_\perp m[\mathbf{r}_0(s')] + \frac{1}{2c_0} \int_{\Gamma_0(p,\phi)} ds \int_{\Gamma_0(p,\phi)} ds' \int_{\Gamma_0(p,\phi)} ds'' \times \frac{dK(s, s')}{ds}\frac{dK(s, s'')}{ds} \nabla_\perp m[\mathbf{r}_0(s')] \cdot \nabla_\perp m[\mathbf{r}_0(s'')] \right\}. \quad (42)$$

This relation can be simplified by using

$$\frac{1}{2c_0}\int_0^L ds \frac{dK(s, s')}{ds}\frac{dK(s, s'')}{ds} = -\tfrac{1}{2}K(s', s''), \quad (43)$$

which follows by explicit integration of (41). This implies that the increase of the traveltime due to the increase of the ray length by the bending cancels half the decrease of the traveltime due to the fact that the ray travels preferentially through regions of high velocity. Using (43) one finds that

$$d(p, \phi, \varepsilon) = \varepsilon \int_{\Gamma_0(p,\phi)} m(s)\, ds + \tfrac{1}{2}\varepsilon^2 \int_{\Gamma_0(p,\phi)} ds \times \int_{\Gamma_0(p,\phi)} ds' K(s, s')\nabla_\perp m[\mathbf{r}_0(s)] \cdot \nabla_\perp m[\mathbf{r}_0(s')]. \quad (44)$$

The first-order operator F_1 for the forward problem is therefore

$$(F_1 m)(p, \phi) = \int_{\Gamma_0(p,\phi)} m[\mathbf{r}_0(s)]\, ds. \quad (45)$$

This is just the Radon transform (Chapman 1987). According to (23a), the inverse operator I_1 is the inverse of F_1, so that I_1 is the inverse Radon transform. The explicit form of this operator is shown by Chapman (1987) and the numerical implementation of this operator can be found in Chapman & Cary (1986).

The second-order forward operator F_2 is given by

$$(F_2 m^2)(p, \phi) = \frac{1}{2}\int_{\Gamma_0(p,\phi)} ds \int_{\Gamma_0(p,\phi)} ds' K(s, s') \times \nabla_\perp m[\mathbf{r}_0(s)] \cdot \nabla_\perp m[\mathbf{r}_0(s')]. \quad (46)$$

Using this expression and (45), one can determine the second-order inverse operator I_2. According to (25) I_2 is given by

$$I_2 = -I_1 F_2 I_1 I_1. \quad (47)$$

It is convenient to define a model $\tilde{m}$ which is the inverse Radon transform of the data:

$$\tilde{m} = I_1 d. \quad (48)$$

Using this in (47), the application of I_2 to the data can be written as

$$I_2 d^2 = -I_1 F_2 \tilde{m}^2. \quad (49)$$

This equation can be simplified by introducing the traveltime perturbation $\delta T^{(2)}$ due to ray bending in the model $\tilde{m}$:

$$\delta T^{(2)}(p, \phi) \equiv F_2 \tilde{m}^2 = \frac{1}{2}\int_{\Gamma_0(p,\phi)} ds \int_{\Gamma_0(p,\phi)} ds' K(s, s') \times \nabla_\perp \tilde{m}[\mathbf{r}_0(s)] \cdot \nabla_\perp \tilde{m}[\mathbf{r}_0(s')]. \quad (50)$$

From (49) and (50) one finds the second-order correction to the model by applying an inverse Radon transform I_1 to

$\delta T^{(2)}$:

$$I_2 d^2 = -I_1 \delta T^{(2)}. \tag{51}$$

Note that the second-order correction $-I_1\delta T^{(2)}$ to the model $\tilde{m}$ is subtracted rather than added because the effect of ray bending is removed from the data.

The second-order algorithm for non-linear tomography thus is as follows.

(i) Apply an inverse Radon transform to the data; this gives a slowness model $\tilde{m}$.
(ii) Compute the traveltime perturbation $\delta T^{(2)}(p, \phi)$ for this slowness model using (50).
(iii) Apply an inverse Radon transform to $\delta T^{(2)}$ (51); according to (18) this slowness correction should be subtracted from $\tilde{m}$.

In this way it is possible to correct for the quadratic effects due to ray bending without iteratively shooting new rays.

5 THE EFFECT OF ERRORS

The theory of Section 3 is not only useful for designing non-linear inversion methods, it also has implications for the effects of errors on non-linear inversion. This is illustrated with the inversion of scattering data for the Plasma Wave Equation using the Marchenko equation. As an example of the effect of errors, the reflection data for the potential of Fig. 1 are subjected to the time-stretch shown in Fig. 8. The time error thus introduced is not very large in an absolute sense, but is larger than the duration of the reflected waves. This time error has an effect on the data similar to static time delays in reflection seismics due to uncertainties in the structure of the weathering layer.

The first, second and third time-stretched Born approximations are shown in Fig. 9. This figure is a time-stretched version of Fig. 2. The first-order contribution to the reconstruction of the potential is shown in Fig. 10. Differentiation of this function according to (6) leads to a stretched version of the true potential.

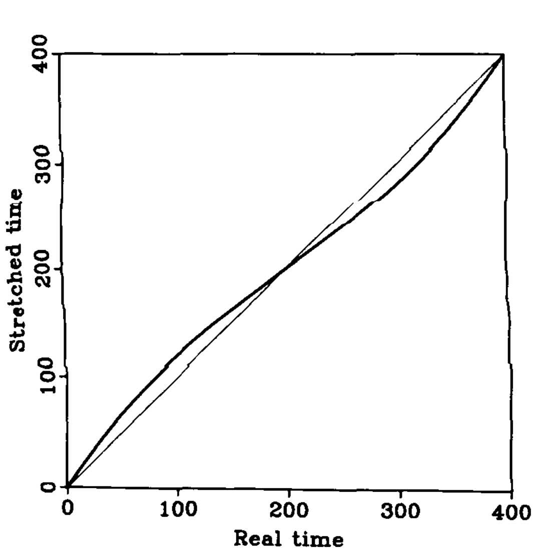

Figure 8. Time stretch used for the example for the effect of timing errors.

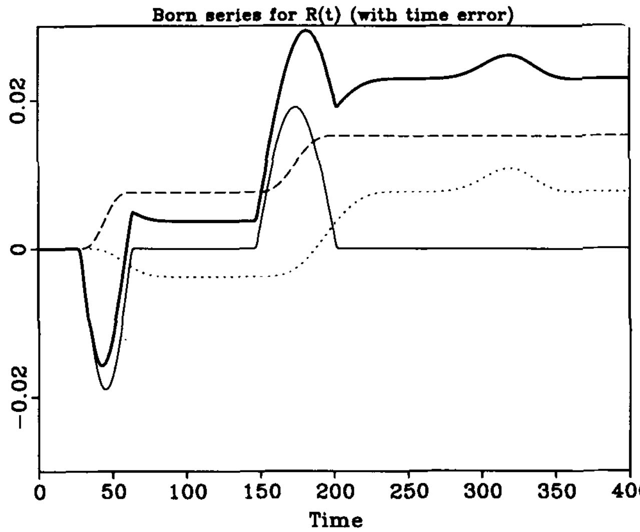

Figure 9. The first (thin solid line), second (dashed line) and third (dotted line) Born approximations and their sum (thick solid line) after the timing error of Fig. 8.

The second-order and third-order contributions to $K(x, x)$ using the time-stretched data are shown in the Figs 11 and 12. These figures should be compared with the Figs 4 and 5. Note that the second-order contributions $-d_2$ and $-K_1 d_1$ in Fig. 11 do not cancel each other, so that there is an erroneous net second-order contribution to the reconstruction of the potential. This effect is even more pronounced for the third-order effects shown in Fig. 12. The terms $-d_3$, $-K_1 d_2$ and $-K_2 d_1$ do not cancel each other. Note that the wave that bounced back and forth once between the sides of the potential, (the bump around $x = 150$) is not subtracted from the data in the inversion. In fact, the sum of all third-order contributions (which should be zero) is of the same order of magnitude as the third-order Born term R_3 itself. This means that if one neglects in this example the erroneous time-stretch, one may just as well refrain from

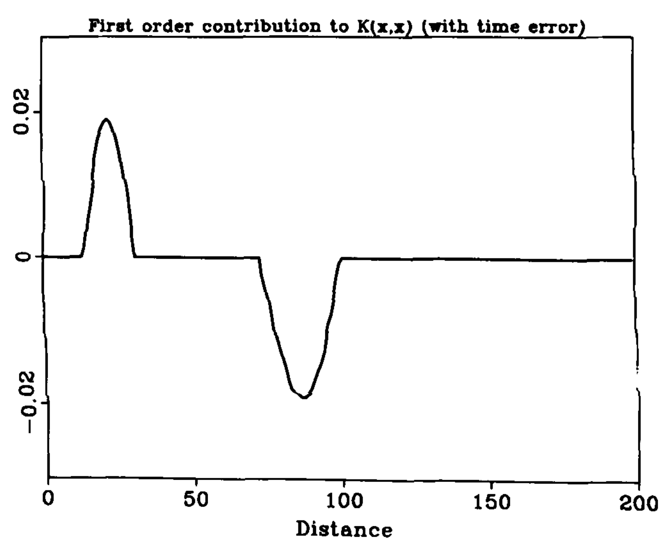

Figure 10. The first-order contribution $K_1(x, x)$ to the reconstruction of the potential for the reflection data with a timing error. The derivative $2(dK_1(x, x)/dx)$ yields a stretched version of the true potential in Fig. 1.

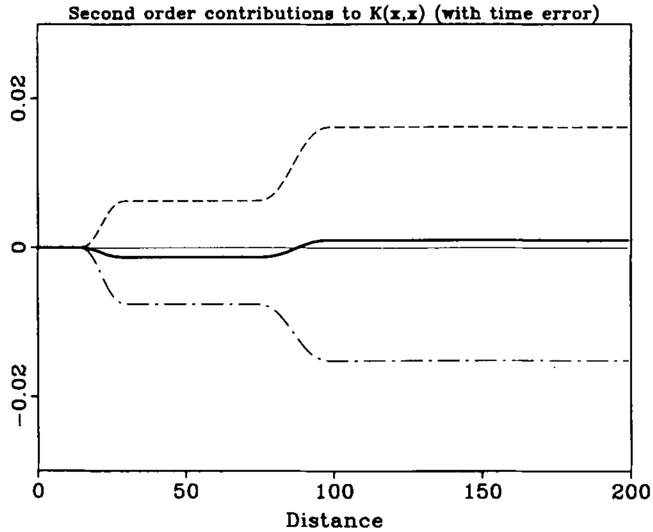

Figure 11. The second-order contributions $-R_2$ (dash–dotted line) and $-K_1R_1$ (dashed line) to the reconstruction of the potential for the reflection data with a timing error, with their sum $K_2(x, x)$ (thick solid line).

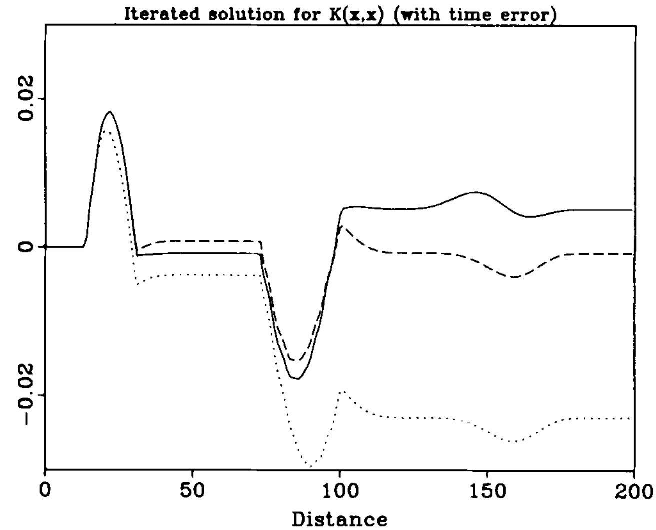

Figure 13. Iterative solution of the Marchenko equation after one (dotted line), two (dashed line), and three iterations (solid line) for the reflection data with a timing error. Only terms up to third order have been taken into account.

performing a non-linear inversion. The iterated solution of the Marchenko equation using the time-stretched data is shown in Fig. 13; this figure should be compared with Fig. 6. Note that the iterated solution contains artifacts due to the incorrectly handled non-linearities.

A similar inversion was performed with data which had erroneously been multiplied with a time dependent amplitude factor $\exp-(t-200)/400$. This factor crudely models the effect of anelastic attenuation. The third-order contributions for this example are shown in Fig. 14. Just as in Fig. 12 the non-linearities are not subtracted correctly from the data, and the net third-order effect after inversion is of the same order as the non-linearity itself.

There is a simple reason why errors in the data have such a detrimental effect on non-linear inversion in general. As shown in Section 3, the non-linear components in the data are in non-linear inversion subtracted from the data (either implicitly or explicitly). This subtraction of the non-linearities in the data is a highly unstable process. In general for non-linear data, the linear component in the data determines the model. Since the model determines the non-linear components in the data, this means that intricate internal relations exist between the linear component and the different non-linear components in the data. In general, it is extremely difficult to make these relations explicit, but these relations do exist for every non-linear problem. For example, in invariant imbedding techniques the complete non-linear response is retrieved by a recursive application of the Born approximation (Tromp & Snieder 1989). Errors in the data have the effect of destroying the internal relations

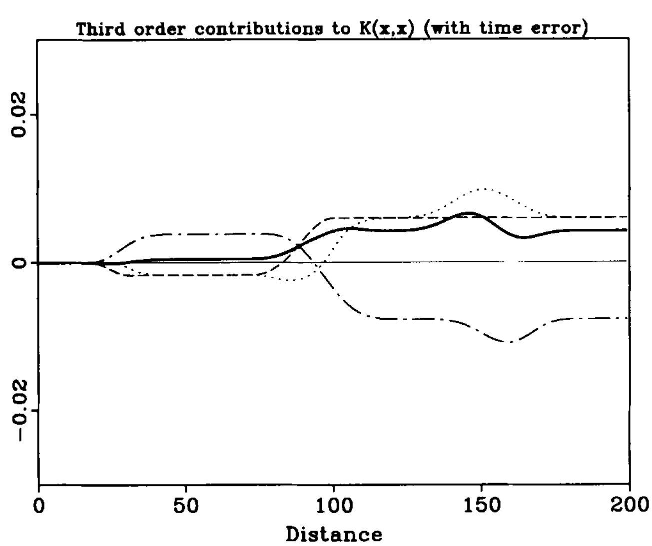

Figure 12. The third-order contributions $-R_3$ (dash–dotted line), $-K_1R_2$ (dashed line) and $-K_2R_1$ (dotted line) to the reconstruction of the potential for the reflection data with a timing error, with their sum $K_3(x, x)$ (thick solid line).

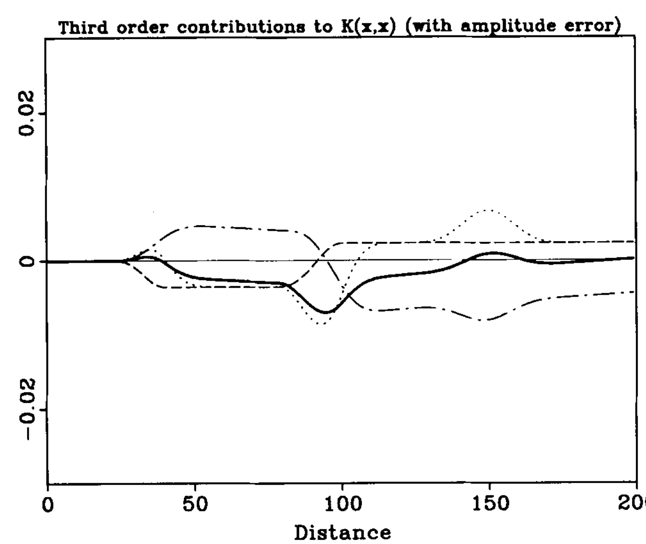

Figure 14. The third-order contributions $-R_3$ (dash–dotted line), $-K_1R_2$ (dashed line) and $-K_2R_1$ (dotted line) to the reconstruction of the potential for the reflection data with an amplitude error, with their sum $K_3(x, x)$ (thick solid line).

between the linear component in the data and the different non-linear components. This is the reason why for example in Fig. 11 the different contributions do not cancel each other, because after the time-stretch the arrival times of the multiply reflected waves cannot be inferred from the single reflected waves. Mathematically stated, the iterated first-order term $d_1 d_1$ does not cancel the quadratic term d_2 any more after application of the time-stretch of Fig. 8.

One may define 'errors' as parts of the data that are not contained in the employed theory. For a seismic experiment one may think for example of timing errors due to velocity variations in the weathering layer which are not included in the employed velocity model. The obvious solution to handle errors is to incorporate the physical effect which causes the error in some way in the theory. This does not necessarily have to be done by describing the true physical process which causes the error, an accurate parametrization may be sufficient. For example, if one ignores attenuation of seismic waves, the amplitude decay will lead to erroneous amplitudes. This can in principle be remedied by incorporating attenuation in the inversion. Ignoring relatively modest errors in the data may lead to severe artifacts in the models obtained by non-linear inversion.

In exploration seismics and global seismology there are several sources of error which may hamper non-linear inversion when they are not taken into account. Important sources of error include static time shifts, amplitude errors for either the source or the receivers, and most importantly, errors in the employed velocity model.

6 NON-EXACT INVERSION METHODS

In the preceding sections it was assumed that in the inversion the model can be exactly reconstructed. For example, the relation (18) implicitly assumes that an exact and unambiguous operator exists which maps the data into the true model. For many inverse problem such an operator is not known. Furthermore, realistic data sets frequently do not contain all the required information for the complete reconstruction of the model. For example, bandlimited seismic reflection data carry only information on the spatial wavelengths in the impedance which match the wavelengths of the employed waves, and on the trend in the velocity (Jannane *et al.* 1989). As an additional complication, realistic data are contaminated with noise.

A relation like (18) can therefore not be used for realistic geophysical data. Instead of the exact inverse relation (18) one normally uses some estimator E of the inverse mapping:

$$\tilde{m}(\varepsilon) = E[d(\varepsilon)]. \tag{52}$$

The operator E describes how the data $d(\varepsilon)$ are mapped into the estimated model $\tilde{m}(\varepsilon)$. This operator can take many different forms, and is often not known in an explicit form. For example, the operator E may describe a regularized least-squares fit of the model synthetics to the data. In this case the operator E is not prescribed explicitly, but implicitly the rules for performing the least-squares inversion constitute the mapping E from the data $d(\varepsilon)$ to the estimated model $\tilde{m}(\varepsilon)$. Note that this mapping does not necessarily lead to the true model; this is the reason that $\tilde{m}(\varepsilon)$ is used in the left-hand side of (52) rather than εm. The estimated model may depend non-linearly on the true model, hence the notation $\tilde{m}(\varepsilon)$.

The estimator E is in general a non-linear operator acting on the data, so that an expansion analogous to (18) can be made:

$$\tilde{m}(\varepsilon) = E[d(\varepsilon)] = E_1 d(\varepsilon) + E_2 d(\varepsilon)^2 + E_3 d(\varepsilon)^3 + \cdots. \tag{53}$$

This relation can be used in a derivation similar to the one shown in Section 3. Inserting (17) in the right-hand side of (53) gives

$$\tilde{m}(\varepsilon) = \varepsilon E_1 F_1 m + \varepsilon^2 (E_1 F_2 + E_2 F_1^2) m^2 + \varepsilon^3 (E_1 F_3 + E_2 F_1 F_2 + E_2 F_2 F_1 + E_3 F_1^3) m^3 + \cdots \tag{54}$$

Note that in general the mapping from the true model εm to the estimated model $\tilde{m}(\varepsilon)$ is non-linear. The quality of the estimator E depends on its ability to reproduce the correct model εm. In the ideal case, $\tilde{m}(\varepsilon) = \varepsilon m$ for every reasonable model m. This is achieved most accurately when the operators E_n satisfy

$$E_1 F_1 m \rightarrow \tilde{m}, \tag{55a}$$

$$(E_1 F_2 + E_2 F_1^2) m^2 \rightarrow 0, \tag{55b}$$

$$(E_1 F_3 + E_2 F_1 F_2 + E_2 F_2 F_1 + E_3 F_1^3) m^3 \rightarrow 0, \tag{55c}$$

$$\left(\sum_{j=1}^{n-1} E_j \sum_{i_1 + \cdots + i_j = n} F_{i_1} F_{i_2} \cdots F_{i_j} + E_n F_1^n \right) m^n \rightarrow 0. \tag{55d}$$

In these expressions, $\rightarrow$ means that the estimator E reproduces the model most accurately, whenever the right-hand side of (55) is more closely approximated. These relations simply are a restatement of the fact that the ideal relation $\tilde{m} = m$ implies a linear relation between m and $\tilde{m}$ (55a), and that spurious non-linear components in the mapping from m to $\tilde{m}$ should be as small as possible (55b, c, . . .)

For a 'good estimator' (in the sense that $\tilde{m} \approx m$ for every reasonable model m) the results of the preceding sections therefore pertain to the estimator E in the same way as for the exact inversion I. For example in Section 4, a quadratic tomographic inversion scheme was presented that corrects for ray bending. For the particular problem of Section 4, F_1 was the Radon transform, and I_1 was the inverse Radon transform, for which an explicit expression is known. For a general tomographic problem, F_1 constitutes slowness integrals over the (possibly curved) reference rays, and an operator I_1 cannot be formulated in closed form. However, the operator E_1 could in this case describe a least-squares fitting of the data, and the derivation of Section 4 can then be extended to incorporate curved reference rays instead of straight reference rays. In general, the conditions (55) for the estimator E can be used in the same way as the conditions (23) for I for the design of non-linear inversion schemes.

Similarly, the conclusions concerning the effects of errors apply both to the estimator E as well as to the exact inverse operator I. This means that in general non-linear inversion and estimation, one should be aware of the deteremental effects that the errors may have on the inversion.

7 DISCUSSION

A perturbative analysis of non-linear inversion sheds new light on the way non-linear inversion is operating. For forward and inverse problems which have both a regular perturbation expansion, only the linear component in the data contributes to the reconstruction of the the model. In the inversion, the non-linear components are removed from the data; this can happen either explicitly or implicitly. The implications for inverse scattering methods are further discussed in Snieder (1990).

The operator equations derived in Section 3 for the inverse mapping I_n can be used to design non-linear inversion schemes. If the non-linearity is strong this procedure may be cumbersome, and a few low-order terms may not be sufficient for an accurate reconstruction of the model. An example of an extremely non-linear inverse problem in resistivity sounding is shown by Parker (1984). For such a strongly non-linear problem, low-order approximations cannot be expected to give useful results. However, for problems that are weakly non-linear, the presented theory is useful to obtain the lowest order corrections to the resulting model due to the non-linearities. An application to a gravimetric non-linear inverse problem using the related theory of Schmidt (1908) (see also Rall 1974) is given by Granser (1987).

Up to a certain extent it is arbitrary whether one considers the non-linearities to be subtracted from the data, or whether the non-linear contributions are being subtracted from the reconstructed model. The latter is the case in iterative inversion of seismic reflection or quantum mechanical data using gradient methods (e.g. Tarantola 1984; Mora 1988; Snieder & Tarantola 1989). In such an inversion, the multiply reflected waves are in the first (linearized) gradient step mapped into spurious reflectors in the reconstructed model. In subsequent iterations these spurious reflectors are removed from the model, so that the non-linear components in the data do not give a net contribution to the reconstructed model. (This is also the case in the iterative solution shown in Fig. 6.) Therefore, the statement that the non-linear components in the data are removed during the non-linear inversion, really means that at some point in the inverse mapping from the data to the model, the non-linear components in the data are removed.

The fact that the non-linearities are removed from the data (or the model) implies that non-linear inversion can be a very unstable process. In wave propagation problems timing errors can destroy the proper subtraction of multiply scattered waves, which can lead to an instability of the inversion. This can also be observed in the numerical experiments from Koehler & Taner (1977) for Goupillaud inversion of 1-D seismic reflection data. As shown by Berryman & Greene (1980), the Goupillaud inversion is formally equivalent to inversions using the Marchenko equation. The results of this paper imply that this instability is inherent to any non-linear inversion method for a particular non-linear inverse problem, including methods which minimize the misfit between data and synthetics with respect to the model parameters. The advantage of inversion using these optimization methods is that these methods allow *a priori* notions on the model to be incorporated in a natural way (Tarantola 1987), which makes it possible to regularize the inversion. It is, however, not evident that this regularization does not introduce unwanted artifacts in the inversion.

The results of Section 5 imply that if one does not take the effect of errors into account, one may just as well refrain from performing non-linear inversions. A similar conclusion was reached by Koltracht & Lancaster (1988) who suggest suppressing parts of the signal below a certain threshold level, the idea being that multiply scattered waves are weaker than single reflected waves. It will be clear that such an approach is not foolproof, and that valuable information may be lost. The only way to handle errors in the data, such as timing errors, amplitude errors (attenuation), or errors in the velocity model, is to include the physical factors that are uncertain in the inversion. In non-linear inversion using methods of optimization (Tarantola 1987), the inclusion of these parameters can be realized in a natural fashion.

In inverse wave propagation problems with a variable velocity, the uncertainty in the velocity of the reference medium is a substantial source of error in non-linear inversion, because it leads to uncertainties in the arrival times of multiply reflected waves. This will destroy the proper subtraction of the non-linearities. Exact inverse scattering methods have been developed only for media where the velocity is constant, such as the Schrödinger equation (Newton 1981; Chadan & Sabatier 1989), or relative to media with a higher velocity than the true medium (Rose, Cheney & DeFacio 1985; Cheney, Rose & De Facio 1989). The determination of the proper velocity is therefore a problem which deserves special attention in non-linear inversion.

In general, one cannot say that a physical problem is linear or non-linear, it is the formulation of a problem which is linear or non-linear. For example, consider the linear forward problem $d = Fm$. The non-linear transformation $m = e^{\hat{m}}$ leads to the non-linear forward problem $d = Fe^{\hat{m}}$. This is of course a trivial example, which turns a linear problem into a non-linear problem. For some non-linear problems one can reformulate the inverse problem in such a way that it can be recast in a more linear form. One could call such problems linearizable. (Not in the sense that the problem can be linearized locally, but in the sense that the problem can be linearized globally by some transformation of the model parameters or the data parameters.) However, for many inverse problems a more linear reformulation does not exist; such problems are intrinsically non-linear. It is advantageous to apply transformations to the data or the model which render the inversion more linear. After such a reformulation the non-linearities with their inherent complications are (partially) removed from the data. This simplifies the inversion, and reduces the contaminating effect of errors. A well-known example of a reformulation of an inverse problem is the use of traveltimes rather than waveforms, which leads for smooth media to an inverse problem which is only weakly non-linear. Similarly, fitting the envelope of oscillatory waveforms instead of the waveforms themselves renders the inverse problem more linear (Nolet, van Trier & Huisman 1986). A systematic search for reformulations of inverse problems might be crucial for the stable solution of non-linear inverse problems.

The derivations shown in this paper tacitly assume that both the forward and the inverse problem can be formulated as a regular perturbation series; see the equations (17) and (18). However, not every forward and inverse problem can be expressed in such a regular perturbation series. For example, suppose that the data are wave amplitudes, that the model consists of the wave velocity, and that dynamic ray theory is used to map the model into the data. At caustics, the amplitude is singular, and can therefore not be expressed in a regular perturbation series of the velocity. In this case, an inversion of the amplitude data is not possible with any inversion scheme, because the amplitude is not defined. Similarly, suppose the inverse mapping (18) is singular; in that case the inverse problem is ill-posed because a small change in the data leads to large changes in the resulting model. Therefore, inversion of data is in general only possible for problems where both the forward and the inverse problem are regular. To this kind of problem the results of this paper are applicable.

ACKNOWLEDGMENTS

The constructive remarks of Professor D. Gubbins and two anonymous referees are greatly appreciated. This research was partially supported by the European Communities (DG XII).

REFERENCES

Agranovich, Z. S. & Marchenko, V. A., 1963. *The Inverse Problem of Scattering Theory*, Gordon & Breach, New York.

Backus, G. E., 1983. Application of mantle filter theory to the magnetic jerk of 1969, *Geophys J. R. astr. Soc.*, **74**, 713–746.

Balanis, G. N., 1972. The plasma inverse problem, *J. Math. Phys.*, **13**, 1001–1005.

Berryman, J. G. & Greene, R. R., 1980. Discrete inverse methods for elastic waves in layered media, *Geophysics*, **45**, 213–233.

Burridge, R., 1980. The Gel'fand–Levitan, the Marchenko and the Gopinath–Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems, *Wave Motion*, **2**, 305–323.

Chadan, K. & Sabatier, P. C., 1989. *Inverse Problems in Quantum Scattering Theory*, 2nd edn, Springer–Verlag, New York.

Chapman, C. H., 1987. The Radon transform and seismic tomography, in *Seismic Tomography, with Applications in Global Seismology and Exploration Geophysics*, pp. 24–48, ed. Nolet, G., Reidel, Dordrecht.

Chapman, C. H. & Cary, P. W., 1986. The circular harmonic Radon transform, *Inverse Problems*, **2**, 23–49.

Cheney, M., Rose, J. H. & DeFacio, B., 1989. A new equation of scattering theory and its use in inverse scattering, *Wave Motion*, **11**, 175–184.

Ge, D. B., 1987. An iterative technique in one-dimensional profile inversion, *Inverse Problems*, **3**, 399–406.

Granser, H., 1987. Nonlinear inversion of gravity data using the Schmidt–Lichtenstein approach, *Geophysics*, **52**, 88–93.

Jannane, H., *et al.*, 1989. Wavelengths of earth structures that can be resolved from seismic reflection data, *Geophysics*, **54**, 906–910.

Koehler, F. & Taner, M. T., 1977. Direct and inverse problems relating reflection coefficients and reflection response for horizontally layered media, *Geophysics*, **42**, 1199–1206.

Koltracht, I. & Lancaster, P., 1988. Threshold algorithms for the prediction of reflection coefficients in a layered medium, *Geophysics*, **53**, 908–919.

Mora, P., 1988. Elastic wavefield inversion of reflection and transmission data, *Geophysics*, **53**, 750–759.

Moses, H. E., 1956. Calculation of the scattering potential from reflection coefficients, *Phys. Rev.*, **102**, 559–567.

Newton, R. G., 1981. Inverse scattering III, three dimensions continued, *J. Math. Phys.*, **22**, 2191–2200.

Nolet, G., van Trier, J. & Huisman, R., 1986. A formalism for the nonlinear inversion of seismic surface waves, *Geophys. Res. Lett.* **13**, 26–29.

Parker, R. R. L., 1984. The inverse problem of resistivity sounding, *Geophysics*, **49**, 2143–2158.

Prosser, R. T., 1969. Formal solutions of inverse scattering problems, *J. Math. Phys.*, **10**, 1819–1822.

Rall, L. B., 1974. Theory of nonlinear integral equations, in *Numerical Solution of Integral Equations*, ed. Delves, L. M. & Walsh, J., Clarendon Press, Oxford.

Rose, J. H., Cheney, M. & DeFacio, B., 1985. Three-dimensional inverse scattering: Plasma and variable velocity wave equations, *J. Math. Phys.*, **26**, 2803–2813.

Schmidt, E., 1908. Über die Auflösung der nichtlinearen Integralgleichung und die Verzweigung ihrer Lösungen, *Mathematische Annalen*, **65**, 370–399.

Snieder, R., 1990. The role of the Born-approximation in nonlinear inversion, *Inverse Problems*, **6**, 246–266.

Snieder, R. & Tarantola, A., 1989. Imaging of quantum mechanical potentials, *Phys. Rev. A*, **39**, 3303–3309.

Tarantola, A., 1984. Inversion of seismic reflection data in the acoustic approximation, *Geophysics*, **49**, 1259–1266.

Tarantola, A., 1987. *Inverse Problem Theory, Methods for Data Fitting and Model Parameter Estimation*, Elsevier, Amsterdam.

Tromp, J. & Snieder, R., 1989. The reflection and transmission of plane P- and S-waves by a continuously stratified band: a new approach using invariant embedding, *Geophys. J.*, **96**, 447–456.