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The reflection and transmission of plane *P*- and *S*-waves by a continuously stratified band: a new approach using invariant imbedding

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SUMMARY
The reflection and transmission of plane *P*- and *S*-waves by a laterally homogeneous band is discussed. A dyadic representation of a 'plane wave Green's tensor' is derived, which is used to describe the reflection and transmission of plane waves by a thin homogeneous layer in the first Born approximation. From this, the reflection and transmission by an arbitrarily thick continuously stratified band is derived using invariant imbedding. We derive an exact set of matrix Ricatti equations which describe the reflection and transmission of plane waves by the laterally homogeneous band. These equations remain regular at turning points, and incorporate both homogeneous and inhomogeneous waves within the heterogeneity. It is not necessary for the band to be stratified; the density and the elasticity tensor of the band may have an arbitrary depth dependence. It is shown that in case the band is a smooth heterogeneity without turning points, its only effect is a phase shift of the transmitted wave. In a numerical example for the analogue case of 1-D scattering in quantum mechanics the behaviour of homogeneous and inhomogeneous (tunneling) waves is illustrated.

Key words: anisotropy, body waves, invariant imbedding, scattering

1 INTRODUCTION

The reflection and transmission of plane waves in layered media is well understood. The propagator matrix formalism (Gilbert & Backus 1966; Aki & Richards 1980; Kennett 1983) gives an exact description of the reflected and transmitted plane wave field in stratified media. There are, however, three disadvantages of the propagator matrix method. First, since the medium under consideration has to consist of homogeneous layers, many layers are needed to model a strong velocity gradient. Second, for each layer the inverse of the fundamental matrix has to be calculated. A third disadvantage is that general anisotropic media can presently not be handled. This implies that coupling between the *P–SV* and the *SH* motion by anisotropy is not taken into account.

Here we present a theory which describes the reflection and transmission of plane waves by a laterally homogeneous band with an arbitrary variation of density and elasticity tensor as a function of depth. We demonstrate that it is possible to obtain exact first order differential equations which describe the reflection and transmission of plane waves, by employing the first Born approximation and an invariant imbedding technique (Budden 1955; Kennett 1984). These equations describe the propagation of both homogeneous and inhomogeneous waves within the heterogeneity. In contrast to other methods (e.g. Kennett 1974), the equations remain regular at turning points, so that both inhomogeneous and inhomogeneous waves are treated correctly.

In Section 2 the reflection and transmission of plane *P*- and *S*-waves by a thin homogeneous layer is treated in the first Born approximation. Among others, Hudson (1977), Malin (1980), Hudson & Heritage (1981), Malin & Phinney (1985) and Wu & Aki (1985) used this approximation to describe the scattering of body waves. Recently, Snieder (1986a) and Snieder & Nolet (1987) used a similar theory to describe surface wave scattering. In order to describe scattering in the Born approximation a Green's tensor for the background medium under consideration is needed. Hudson & Heritage (1981) and Wu & Aki (1985) used the Green's tensor for the excitation of seismic waves by a point force in a homogeneous background medium. Snieder (1986a) and Snieder & Nolet (1987) used a dyadic representation of the Green's tensor for the excitation of surface waves in a laterally homogeneous background medium. In order to describe reflection and transmission of plane waves in a homogeneous background media by a thin homogeneous layer a dyadic representation of a 'plane wave Green's tensor' is derived in the Appendix.

In Section 3 the reflection and transmission of plane waves by a continuously stratified band is discussed. We apply an invariant imbedding technique (Budden 1955; Kennett 1984) to the results of Section 2 for the thin homogeneous layer, in order to obtain a system of matrix Ricatti equations which describe the reflection and transmission of plane waves by a laterally homogeneous band. The connection between the reflection and transmission properties of a single isotropic layer and the propagator matrix method for multilayered isotropic media was shown by Kennett (1974). Here we demonstrate the connection between the scattering properties of a thin homogeneous layer and the reflection and transmission of plane waves by a laterally homogeneous band. It is shown that the only effect of a smooth laterally homogeneous band without turning points is a phase shift of the transmitted wave field. Finally, in Section 4, we illustrate the character of the reflection and transmission coefficients for the analogue case of 1-D scattering in quantum mechanics both for homogeneous and inhomogeneous (tunneling) waves. This confirms that the theory remains valid in the neighbourhood of turning points.

2 THE REFLECTION OF PLANE P- AND S-WAVES BY A THIN HOMOGENEOUS LAYER IN THE FIRST BORN APPROXIMATION

The use of the first Born approximation in seismic scattering problems was discussed in detail by Hudson & Heritage (1981). Following these authors we express the density and the elasticity tensor as $\rho = \rho_0 + \rho_s$ and $\mathbf{c} = \mathbf{c}^0 + \mathbf{c}^s$ respectively. In this paper, ρ_0 and $\mathbf{c}^0$ define a homogeneous, isotropic background medium in which $c^0_{ijkl} = \lambda_0\delta_{ij}\delta_{kl} + \mu_0[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$. ρ_s and $\mathbf{c}^s$ are perturbations of the background medium which define the scatterer. In our case the scatterer is a thin homogeneous layer. The displacement field $\mathbf{u}$ can be expressed as $\mathbf{u}^0 + \mathbf{u}^s$. Here $\mathbf{u}^0$ describes the displacement field of an unperturbed plane wave in a homogeneous background medium while $\mathbf{u}^s$ describes the displacement field of the scattered wave. The scattered wave $\mathbf{u}^s$ can be expressed as a Born series:

$$\mathbf{u}^s(\mathbf{r}) = \sum_{n=1}^{\infty} \mathbf{u}^n(\mathbf{r}), \tag{2.1}$$

in which $\mathbf{u}^1$ is the first Born approximation for the scattered wave. The unperturbed plane wave $\mathbf{u}^0$ satisfies the equation of motion in the background medium

$$L^0_{ij}u^0_j = 0, \tag{2.2}$$

where $L^0_{ij} = -\rho_0\omega^2\delta_{ij} - \lambda_0\,\partial_i\partial_j - \mu_0\,\partial_i\,\partial_j - \mu_0\delta_{ij}\,\partial_k\,\partial_k$. From now on we distinguish between downward and upward traveling plane waves respectively. The positive z-axis is chosen vertically downward. Downward traveling waves are denoted by the subscript D, while upward traveling waves are represented by the subscript U. The downward traveling plane wave solution of (2.2) can be expressed as

$$\mathbf{u}^0_D(x, z, t) = \mathbf{p}^\tau_D e^{i(kx + \nu_\tau z - \omega t)}, \tag{2.3}$$

where $\tau = 1$, 2 or 3, and $\nu_1 = \nu_\alpha = \sqrt{(\omega^2/\alpha^2) - k^2}$ while $\nu_2 = \nu_3 = \nu_\beta = \sqrt{(\omega^2/\beta^2) - k^2}$. τ is a polarization direction index; α and β are the P- and S-wave velocity for the homogeneous background medium. $\mathbf{p}^i_D (i = 1, 2, 3)$ are called polarization vectors because they describe the direction in which the wave oscillates. $\mathbf{p}^1_D$ and $\mathbf{p}^2_D$ describe the oscillation direction of the P–SV downgoing wave while $\mathbf{p}^3_D$ describes the oscillation direction of the SH downgoing wave. For a downgoing plane P-wave we have $\mathbf{p}^1_D = (\alpha/\omega)(k, 0, \nu_\alpha)$ while for a downgoing plane S-wave we have $\mathbf{p}^2_D = (\beta/\omega)(-\nu_\beta, 0, k)$ or $\mathbf{p}^3_D = (0, 1, 0)$.

The upward traveling plane wave solution of (2.2) can be expressed as

$$\mathbf{u}^0_U(x, z, t) = \mathbf{p}^\tau_U\, e^{i(kx - \nu_\tau z - \omega t)}, \tag{2.4}$$

where, again, $\tau = 1$, 2 or 3, and $\nu_1 = \nu_\alpha = \sqrt{(\omega^2/\alpha^2) - k^2}$ while $\nu_2 = \nu_3 = \nu_\beta = \sqrt{(\omega^2/\beta^2) - k^2}$. For an upgoing plane P-wave we have $\mathbf{p}^1_U = (\alpha/\omega)(k, 0, -\nu_\alpha)$ while for an upgoing plane S-wave we have $\mathbf{p}^2_U = (\beta/\omega)(\nu_\beta, 0, k)$ or $\mathbf{p}^3_U = (0, 1, 0)$. $\mathbf{p}^1_U$ and $\mathbf{p}^2_U$ describe the oscillation direction of the P–SV upgoing wave while $\mathbf{p}^3_U$ describes the oscillation direction of the SH upgoing wave.

In order to find the first Born approximation for the wave scattered by the thin layer, we need a plane wave Green's tensor G which satisfies

$$L^0_{ij}G_{jn} = \delta_{in}\delta(z - z_0)\, e^{i(kx - \omega t)}. \tag{2.5}$$

In the Appendix it is shown that this plane wave Green's tensor is given by

$$G(z, z_0, x, t) = \begin{cases} \left[\dfrac{ie^{i\nu_\alpha(z - z_0)}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}\mathbf{p}^1_D\mathbf{p}^1_D + \dfrac{ie^{i\nu_\beta(z - z_0)}}{2\mu_0\nu_\beta}(\mathbf{p}^2_D\mathbf{p}^2_D + \mathbf{p}^3_D\mathbf{p}^3_D)\right]e^{i(kx - \omega t)}, & z > z_0 \\ \left[\dfrac{ie^{i\nu_\alpha(z_0 - z)}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}\mathbf{p}^1_U\mathbf{p}^1_U + \dfrac{ie^{i\nu_\beta(z_0 - z)}}{2\mu_0\nu_\beta}(\mathbf{p}^2_U\mathbf{p}^2_U + \mathbf{p}^3_U\mathbf{p}^3_U)\right]e^{i(kx - \omega t)}, & z_0 > z \end{cases}$$
$$= \tilde{G}(z, z_0)e^{i(kx - \omega t)}, \tag{2.6}$$

thereby defining $\tilde{G}(z, z_0)$. The Born series for the scattered wave can be obtained from the 1-D analogue of expression (3) of Hudson & Heritage (1981) which is given by

$$u_i^{n+1}(x, z, t) = \int \Big\{ \rho_s(z_0)\omega^2 \tilde{G}_{ij}(z, z_0)\tilde{u}_j^n(z_0) - k^2 \tilde{G}_{ij}(z, z_0) c^s_{j11k}(z_0)\tilde{u}_k^n(z_0) - \left[\frac{d}{dz_0}\tilde{G}_{ij}(z, z_0)\right] c^s_{j33k}(z_0)\left[\frac{d}{dz_0}\tilde{u}_k^n(z_0)\right]$$
$$+ ik\tilde{G}_{ij}(z, z_0)c^s_{j13k}(z_0)\left[\frac{d}{dz_0}\tilde{u}_k^n(z_0)\right] - ik\left[\frac{d}{dz_0}\tilde{G}_{ij}(z, z_0)\right] c^s_{j31k}(z_0)\tilde{u}_k^n(z_0)\Big\} dz_0 e^{i(kx-\omega t)}, \tag{2.7}$$

where $\tilde{\mathbf{u}}^n(z_0)$ is defined by

$$\mathbf{u}^n(x, z, t) = \tilde{\mathbf{u}}^n(z)e^{i(kx-\omega t)}. \tag{2.8}$$

The first Born approximation for the wave scattered by the thin layer with width Δz_0 is given by

$$\mathbf{u}_i^1(x, z, t) = \Big\{ \rho_s(z_0)\omega^2 \tilde{G}_{ij}(z, z_0)\tilde{u}_j^0(z_0) - k^2 \tilde{G}_{ij}(z, z_0) c^s_{j11k}(z_0)\tilde{u}_k^0(z_0) - \left[\frac{d}{dz_0}\tilde{G}_{ij}(z, z_0)\right] c^s_{j33k}(z_0)\left[\frac{d}{dz_0}\tilde{u}_k^0(z_0)\right]$$
$$+ ik\tilde{G}_{ij}(z, z_0)c^s_{j13k}(z_0)\left[\frac{d}{dz_0}\tilde{u}_k^0(z_0)\right] - ik\left[\frac{d}{dz_0}\tilde{G}_{ij}(z, z_0)\right] c^s_{j31k}(z_0)\tilde{u}_k^0(z_0)\Big\} \Delta z_0 e^{i(kx-\omega t)}. \tag{2.9}$$

The argument z_0 denotes that the thin layer is located at depth z_0. Now we consider two types of unperturbed waves: downgoing and upgoing unperturbed waves respectively. In case of a downgoing unperturbed wave, $\mathbf{u}_\mathrm{D}^0$, we have

$$\tilde{\mathbf{u}}_\mathrm{D}^0(z) = \mathbf{p}_\mathrm{D}^\tau e^{i\nu_\tau z}. \tag{2.10}$$

Solving for the first Born approximation for the scattered wave, using (2.6), (2.10) and (2.9) yields

$$\mathbf{u}^1(x, z, t) = \begin{cases} \mathbf{p}_\mathrm{D}^1 t_\mathrm{D}^{1\tau}(z_0)\Delta z_0 e^{i(kx+\nu_\alpha z-\omega t)} + [\mathbf{p}_\mathrm{D}^2 t_\mathrm{D}^{2\tau}(z_0)\Delta z_0 + \mathbf{p}_\mathrm{D}^3 t_\mathrm{D}^{3\tau}(z_0)\Delta z_0] e^{i(kx+\nu_\beta z-\omega t)}, & z > z_0 \\ \mathbf{p}_\mathrm{U}^1 r_0^{1\tau}(z_0)\Delta z_0 e^{i(kx-\nu_\alpha z-\omega t)} + [\mathbf{p}_\mathrm{U}^2 r_\mathrm{U}^{2\tau}(z_0)\Delta z_0 + \mathbf{p}_\mathrm{U}^3 r_\mathrm{U}^{3\tau}(z_0)\Delta z_0] e^{i(kx-\nu_\beta z-\omega t)}, & z_0 > z, \end{cases} \tag{2.11}$$

where

$$t_\mathrm{D}^{1\tau}(z_0) = p_{i\mathrm{D}}^1[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) - \nu_\alpha\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) - k\nu_\alpha c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau-\nu_\alpha)z_0}}{2(\lambda_0+2\mu_0)\nu_\alpha}, \tag{2.12a}$$

$$t_\mathrm{D}^{2\tau}(z_0) = p_{i\mathrm{D}}^2[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) - \nu_\beta\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) - k\nu_\beta c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau-\nu_\beta)z_0}}{2\mu_0\nu_\beta}, \tag{2.12b}$$

$$t_\mathrm{D}^{3\tau}(z_0) = p_{i\mathrm{D}}^3[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) - \nu_\beta\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) - k\nu_\beta c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau-\nu_\beta)z_0}}{2\mu_0\nu_\beta}, \tag{2.12c}$$

$$r_\mathrm{U}^{1\tau}(z_0) = p_{i\mathrm{U}}^1[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) + \nu_\alpha\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) + k\nu_\alpha c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau+\nu_\alpha)z_0}}{2(\lambda_0+2\mu_0)\nu_\alpha}, \tag{2.12d}$$

$$r_\mathrm{U}^{2\tau}(z_0) = p_{i\mathrm{U}}^2[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) + \nu_\beta\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) + k\nu_\beta c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau+\nu_\beta)z_0}}{2\mu_0\nu_\beta}, \tag{2.12e}$$

$$r_\mathrm{U}^{3\tau}(z_0) = p_{i\mathrm{U}}^3[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) + \nu_\beta\nu_\tau c^s_{i33j}(z_0) - k\nu_\tau c^s_{i13j}(z_0) + k\nu_\beta c^s_{i31j}(z_0)] p_{j\mathrm{D}}^\tau \frac{ie^{i(\nu_\tau+\nu_\beta)z_0}}{2\mu_0\nu_\beta}. \tag{2.12f}$$

Note that $\mathbf{u}^1$ consists of transmitted downgoing waves and reflected upgoing waves (see Fig. 1, which shows an example for an incoming P-wave). $\mathbf{r}_\mathrm{U}$ and $\mathbf{t}_\mathrm{D}$ are reflection and transmission matrices, respectively. For instance, the reflection matrix element $r_\mathrm{U}^{1\tau}$ describes the reflection of a downgoing unperturbed wave polarized in the $\mathbf{p}_\mathrm{D}^\tau$ direction to an upgoing P-wave polarized in the $\mathbf{p}_U^1$ direction. Note that the superscript τ represents the polarization direction of the incoming wave, while the superscript 1 denotes the polarization direction of the reflected wave. The subscript U denotes that the reflected wave travels upward.

In case of an upgoing unperturbed wave, $\mathbf{u}_\mathrm{U}^0$, we have

$$\tilde{\mathbf{u}}_\mathrm{U}^0(z) = \mathbf{p}_\mathrm{U}^\tau e^{-i\nu_\tau z}. \tag{2.13}$$

Solving for the first Born approximation for the scattered wave, using (2.6), (2.13) and (2.9), yields

$$\mathbf{u}^1(x, z, t) = \begin{cases} \mathbf{p}_\mathrm{U}^1 t_\mathrm{U}^{1\tau}(z_0)\Delta z_0 e^{i(kx-\nu_\alpha z-\omega t)} + [\mathbf{p}_U^2 t_\mathrm{U}^{2\tau}(z_0)\Delta z_0 + \mathbf{p}_\mathrm{U}^3 t_\mathrm{U}^{3\tau}(z_0)\Delta z_0] e^{i(kx-\nu_\beta z-\omega t)}, & z > z_0 \\ \mathbf{p}_\mathrm{D}^1 r_\mathrm{D}^{1\tau}(z_0)\Delta z_0 e^{i(kx+\nu_\alpha z-\omega t)} + [\mathbf{p}_\mathrm{D}^2 r_\mathrm{D}^{2\tau}(z_0)\Delta z_0 + \mathbf{p}_\mathrm{D}^3 r_\mathrm{D}^{3\tau}(z_0)\Delta z_0] e^{i(kx+\nu_\beta z-\omega t)}, & z_0 > z, \end{cases} \tag{2.14}$$

where

$$t_\mathrm{U}^{1\tau}(z_0) = p_{i\mathrm{U}}^1[\rho_s(z_0)\omega^2\delta_{ij} - k^2 c^s_{i11j}(z_0) - \nu_\alpha\nu_\tau c^s_{i33j}(z_0) + k\nu_\tau c^s_{i13j}(z_0) + k\nu_\alpha c^s_{i31j}(z_0)] p_{j\mathrm{U}}^\tau \frac{ie^{-i(\nu_\tau-\nu_\alpha)z_0}}{2(\lambda_0+2\mu_0)\nu_\alpha}, \tag{2.15a}$$

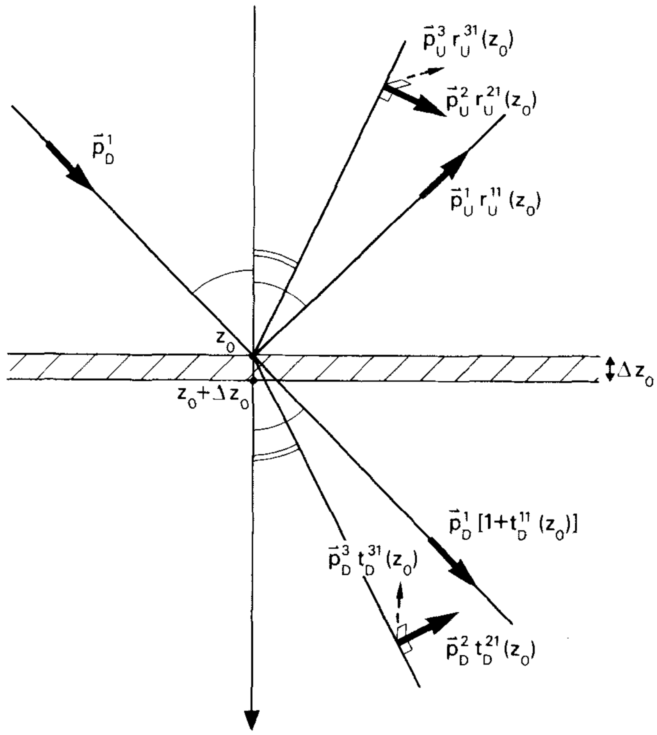


Figure 1. Transmission and reflection of an unperturbed P-wave which travels downward by a thin layer with width Δz_0 in the first Born approximation. The polarization directions and the amplitudes of the waves have been indicated.

$$t_U^{2\tau}(z_0) = p_{i_U}^2[\rho_s(z_0)\omega^2\delta_{ij} - k^2c_{i11j}^s(z_0) - \nu_\beta\nu_\tau c_{i33j}^s(z_0) + k\nu_\tau c_{i13j}^s(z_0) + k\nu_\beta c_{i31j}^s(z_0)]p_{j_U}^\tau \frac{ie^{-i(\nu_\tau-\nu_\beta)z_0}}{2\mu_0\nu_\beta}, \tag{2.15b}$$

$$t_U^{3\tau}(z_0) = p_{i_U}^3[\rho_s(z_0)\omega^2\delta_{ij} - k^2c_{i11j}^s(z_0) - \nu_\beta\nu_\tau c_{i33j}^s(z_0) + k\nu_\tau c_{i13j}^s(z_0) + k\nu_\beta c_{i31j}^s(z_0)]p_{j_U}^\tau \frac{ie^{-i(\nu_\tau-\nu_\beta)z_0}}{2\mu_0\nu_\beta}, \tag{2.15c}$$

$$r_D^{1\tau}(z_0) = p_{i_D}^1[\rho_s(z_0)\omega^2\delta_{ij} - k^2c_{i11j}^s(z_0) + \nu_\alpha\nu_\tau c_{i33j}^s(z_0) + k\nu_\tau c_{i13j}^s(z_0) - k\nu_\alpha c_{i31j}^s(z_0)]p_{j_D}^\tau \frac{ie^{-i(\nu_\tau+\nu_\alpha)z_0}}{2(\lambda_0+2\mu_0)\nu_\alpha}, \tag{2.15d}$$

$$r_D^{2\tau}(z_0) = p_{i_D}^2[\rho_s(z_0)\omega^2\delta_{ij} - k^2c_{i11j}^s(z_0) + \nu_\beta\nu_\tau c_{i33j}^s(z_0) + k\nu_\tau c_{i13j}^s(z_0) - k\nu_\beta c_{i31j}^s(z_0)]p_{j_D}^\tau \frac{ie^{-i(\nu_\tau+\nu_\beta)z_0}}{2\mu_0\nu_\beta}. \tag{2.15e}$$

$$r_D^{3\tau}(z_0) = p_{i_D}^3[\rho_s(z_0)\omega^2\delta_{ij} - k^2c_{i11j}^s(z_0) + \nu_\beta\nu_\tau c_{i33j}^s(z_0) + k\nu_\tau c_{i13j}^s(z_0) - k\nu_\beta c_{i31j}^s(z_0)]p_{j_D}^\tau \frac{ie^{-i(\nu_\tau+\nu_\beta)z_0}}{2\mu_0\nu_\beta}. \tag{2.15f}$$

Note that $\mathbf{u}^1$ consists of transmitted upgoing waves and reflected downgoing waves (see Fig. 2, which shows an example for an incoming S-wave). Again, $\mathbf{r}_D$ and $\mathbf{t}_U$ are reflection and transmission matrices, respectively. For instance, the transmission matrix element $t_U^{2\tau}$ describes the transmission of an upgoing unperturbed wave polarized in the $\mathbf{p}_U^\tau$ direction as an upgoing S-wave polarized in the $\mathbf{p}_U^2$ direction. The subscript U denotes that the transmitted wave travels upward.

We stress that in equations (2.12a–f) and (2.15a–f) for the reflection and transmission matrix elements k denotes the horizontal wavenumber *in the background medium* and that ν_α, ν_β or ν_τ denote the vertical wave number *in the background medium*. Also μ_0 and λ_0 denote the elastic moduli of the background medium; ρ_s and $\mathbf{c}^s$ describe the properties of the laterally homogeneous layer.

It can be seen from (2.12a–f) and (2.15a–f) that in case of an isotropic layer, i.e. $c_{ijkl}^s = \lambda_s\delta_{ij}\delta_{kl} + \mu_s[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$, or a transversely isotropic layer with the z-axis as symmetry axis, we have

$$r_D^{13} = r_U^{13} = r_D^{23} = r_U^{23} = r_D^{31} = r_U^{31} = r_D^{32} = r_U^{32} = 0,$$

and

$$t_D^{13} = t_U^{13} = t_D^{23} = t_U^{23} = t_D^{31} = t_U^{31} = t_D^{32} = t_U^{32} = 0.$$

Therefore the P–SV reflection and transmission is decoupled from the SH reflection and transmission. If, furthermore, the angle of incidence is zero (i.e. $k = 0$), all the off-diagonal elements of the reflection and transmission matrices are zero: there is no conversion of the direct wave.

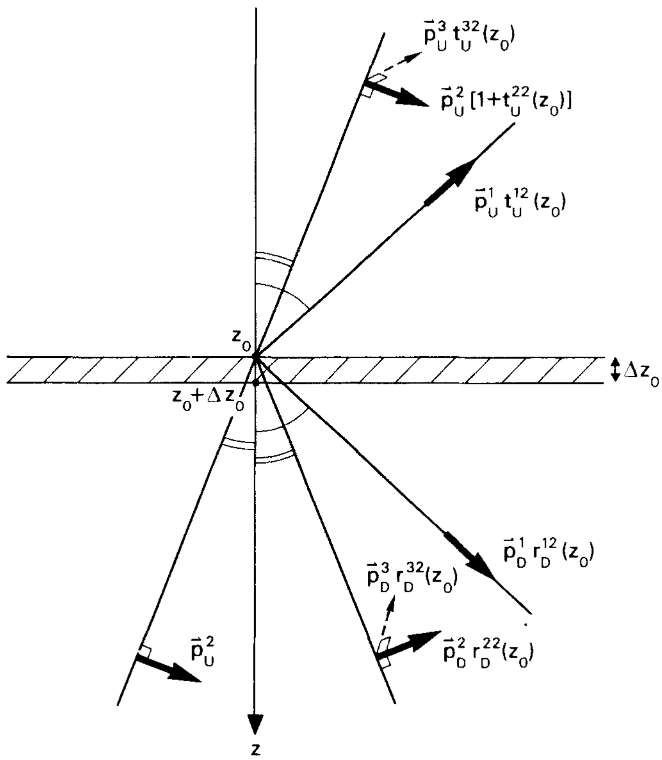


Figure 2. Transmission and reflection of an unperturbed S-wave which travels upward by a thin layer with width Δz_0 in the first Born approximation. The polarization directions and the amplitudes of the waves have been indicated.

It can be seen from expression (2.7) for the higher order Born approximations, that the n-th order Born approximation is of the order $(\Delta z_0)^n$.

3 THE REFLECTION AND TRANSMISSION OF PLANE P- AND S-WAVES BY A CONTINUOUSLY STRATIFIED BAND OF ARBITRARY THICKNESS

Using the results for the reflection and transmission of plane P- and S-waves by a thin layer in the first Born approximation, which are correct up to order Δz_0, we demonstrate how exact, first order, non-linear differential equations which describe the reflection and transmission of plane P- and S-waves by a laterally homogeneous band with an arbitrary thickness can be deduced. Suppose the thick band is confined between 0 and z_0. A downward traveling incident wave can be expressed as (2.3) while an upward traveling wave can be expressed as (2.4.). The superscript 0 in (2.3) and (2.4) can be omitted. In case of an incident plane wave traveling downward, we look for reflected and transmitted waves of the form

$$\mathbf{u}(x,z,t)=\begin{cases}\mathbf{p}_D^1 T_D^{1\tau}(z_0)e^{i(kx+\nu_\alpha z-\omega t)}+[\mathbf{p}_D^2 T_D^{2\tau}(z_0)+\mathbf{p}_D^3 T_D^{3\tau}(z_0)]e^{i(kx+\nu_\beta z-\omega t)}, & z>z_0\\ \mathbf{p}_U^1 R_U^{1\tau}(z_0)e^{i(kx-\nu_\alpha z-\omega t)}+[\mathbf{p}_U^2 R_U^{2\tau}(z_0)+\mathbf{p}_U^3 R_U^{3\tau}(z_0)]e^{i(kx-\nu_\beta z-\omega t)}, & z<0.\end{cases} \tag{3.1}$$

In case of an incident plane wave traveling upward we look for reflected and transmitted waves of the form

$$\mathbf{u}(x,z,t)=\begin{cases}\mathbf{p}_U^1 T_U^{1\tau}(z_0)e^{i(kx-\nu_\alpha z-\omega t)}+[\mathbf{p}_U^2 T_U^{2\tau}(z_0)+\mathbf{p}_U^3 T_U^{3\tau}(z_0)]e^{i(kx-\nu_\beta z-\omega t)}, & z<0\\ \mathbf{p}_D^1 R_D^{1\tau}(z_0)e^{i(kx+\nu_\alpha z-\omega t)}+[\mathbf{p}_D^2 R_D^{2\tau}(z_0)+\mathbf{p}_D^3 R_D^{3\tau}(z_0)]e^{i(kx+\nu_\beta z-\omega t)}, & z>z_0.\end{cases} \tag{3.2}$$

$\mathbf{T}_D(z_0)$, $\mathbf{T}_U(z_0)$, $\mathbf{R}_D(z_0)$ and $\mathbf{R}_U(z_0)$ are the transmission and reflection matrices for which we want to find differential equations. These matrices depend on the thickness of the band: z_0. Equations (3.1) and (3.2) describe the fact that we know the scattered wave field outside the laterally homogeneous band that consists of traveling P- and S-waves. Note that we are looking here for the solution for the *total* wave field for $z<0$ and $z>z_0$, while in the discussion of reflection and transmission by a thin homogeneous layer we were looking for the first Born approximation for the reflected and transmitted waves. Here we want to determine the exact transmitted and reflected wave field.

In order to do this we use invariant imbedding (Budden 1955; Kennett 1984): we add a thin homogeneous layer with width Δz_0, perturbation in density $\rho_s(z_0)$ and perturbation in elasticity tensor $\mathbf{c}^s(z_0)$, to the bottom of the thick band of thickness z_0 as

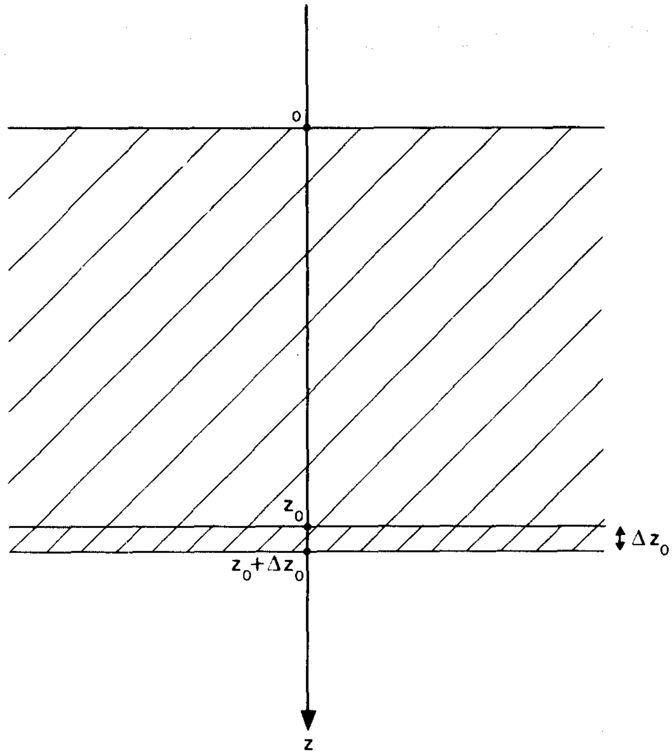


Figure 3. Laterally homogeneous band confined between $z = 0$ and $z = z_0$. The adding of the thin layer with width Δz_0 to the bottom of the band makes it possible to find exact, first order differential equations for the reflection and transmission coefficients.

shown in Fig. 3, in order to obtain exact differential equations for the reflection and transmission matrices as a function of the extension z_0 of this band. This is achieved by taking the limit $\Delta z_0 \rightarrow 0$. The reflection and transmission by the thin layer is described by the reflection and transmission matrices $\mathbf{t}_{\mathrm{D}}(z_0)$, $\mathbf{t}_{\mathrm{U}}(z_0)$, $\mathbf{r}_{\mathrm{D}}(z_0)$ and $\mathbf{r}_U(z_0)$ as defined in the discussion of the scattering of plane waves by a thin layer in the previous section. The effect of adding the thin layer is treated for the reflection matrix element $R^{12}_{\mathrm{U}}(z_0 + \Delta z_0)$, that is the reflection of a S-wave traveling downward, $\mathbf{p}^2_{\mathrm{D}} e^{i(kx + \nu_\beta z - \omega t)}$, to the P-wave traveling upward, $\mathbf{p}^1_{\mathrm{U}} R^{12}_{\mathrm{U}}(z_0 + \Delta z_0) e^{i(kx - \nu_\alpha z - \omega t)}$. We find that

$$\begin{aligned} R^{12}_{\mathrm{U}}(z_0 + \Delta z_0) = R^{12}_{\mathrm{U}}(z_0) &+ [T^{11}_{\mathrm{U}}(z_0) r^{11}_{\mathrm{U}}(z_0) T^{12}_{\mathrm{D}}(z_0) + T^{11}_{\mathrm{U}}(z_0) r^{12}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0) + T^{11}_{\mathrm{U}}(z_0) r^{13}_{\mathrm{U}}(z_0) T^{32}_{\mathrm{D}}(z_0) \\ &+ T^{12}_{\mathrm{U}}(z_0) r^{21}_{\mathrm{U}}(z_0) T^{12}_{\mathrm{D}}(z_0) + T^{12}_{\mathrm{U}}(z_0) r^{22}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0) + T^{12}_{\mathrm{U}}(z_0) r^{23}_{\mathrm{U}}(z_0) T^{32}_{\mathrm{D}}(z_0) + T^{13}_{\mathrm{U}}(z_0) r^{31}_{\mathrm{U}}(z_0) T^{12}_{\mathrm{D}}(z_0) \\ &+ T^{13}_{\mathrm{U}}(z_0) r^{32}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0) + T^{13}_{\mathrm{U}}(z_0) r^{33}_{\mathrm{U}}(z_0) T^{32}_{\mathrm{D}}(z_0)] \Delta z_0 + O[(\Delta z_0)^2]. \end{aligned} \tag{3.3}$$

Note that we take into account all possible combinations of reflection and transmission by the thin layer and the thick band that result in a reflected P-wave. For instance $T^{13}_{\mathrm{U}}(z_0) r^{32}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0)$ should be read from right to left as follows: the incident S-wave $\mathbf{p}^2_{\mathrm{D}} e^{i(kx + \nu_\beta z - \omega t)}$ is transmitted by the thick band as a S-wave traveling downward $\mathbf{p}^2_{\mathrm{D}} T^{22}_{\mathrm{D}}(z_0) e^{i(kx + \nu_\beta z - \omega t)}$, this wave is reflected by the thin layer as a S-wave traveling upward $\mathbf{p}^3_{\mathrm{U}} r^{32}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0) e^{i(kx - \nu_\beta z - \omega t)}$; finally this wave is reflected by the thick band as the P-wave traveling upward $\mathbf{p}^1_{\mathrm{U}} T^{13}_{\mathrm{U}}(z_0) r^{32}_{\mathrm{U}}(z_0) T^{22}_{\mathrm{D}}(z_0) e^{i(kx - \nu_\alpha z - \omega t)}$.

Equation (3.3) can be rewritten as

$$\frac{R^{12}_{\mathrm{U}}(z_0 + \Delta z_0) - R^{12}_{\mathrm{U}}(z_0)}{\Delta z_0} = T^{1\sigma}_{\mathrm{U}}(z_0) r^{\sigma\tau}_{\mathrm{U}}(z_0) T^{\tau 2}_{\mathrm{D}}(z_0) + O(\Delta z_0), \tag{3.4}$$

where a sum over σ and τ should be performed. Taking the limit $\Delta z_0 \rightarrow 0$ yields

$$\frac{d}{dz_0} R^{12}_{\mathrm{U}}(z_0) = T^{1\sigma}_{\mathrm{U}}(z_0) r^{\sigma\nu}_{\mathrm{U}}(z_0) T^{\nu 2}_{\mathrm{D}}(z_0). \tag{3.5}$$

This equation is exact because the terms of the order Δz_0 in expression (3.4) go to zero in the limit $\Delta z_0 \rightarrow 0$. The first order differential equation has to be supplemented with a boundary condition. For $z_0 = 0$, the inhomogeneous band has a vanishing thickness, so that there are no reflected waves: $R^{12}_{\mathrm{U}}(z_0 = 0) = 0$. Differential equations for the other reflection and transmission coefficients can be obtained in a similar way. This gives the following Ricatti equations for the reflection and transmission matrices:

$$\frac{d}{dz_0}\mathbf{R}_U(z_0)=\mathbf{T}_U(z_0)\cdot\mathbf{r}_U(z_0)\cdot\mathbf{T}_D(z_0), \tag{3.6a}$$

$$\frac{d}{dz_0}\mathbf{R}_D(z_0)=\mathbf{r}_D(z_0)+\mathbf{t}_D(z_0)\cdot\mathbf{R}_D(z_0)+\mathbf{R}_D(z_0)\cdot\mathbf{t}_U(z_0)+\mathbf{R}_D(z_0)\cdot\mathbf{r}_U(z_0)\cdot\mathbf{R}_D(z_0), \tag{3.6b}$$

$$\frac{d}{dz_0}\mathbf{T}_D(z_0)=\mathbf{t}_D(z_0)\cdot\mathbf{T}_D(z_0)+\mathbf{R}_D(z_0)\cdot\mathbf{r}_U(z_0)\cdot\mathbf{T}_D(z_0), \tag{3.6c}$$

$$\frac{d}{dz_0}\mathbf{T}_U(z_0)=\mathbf{T}_U(z_0)\cdot\mathbf{t}_U(z_0)+\mathbf{T}_U(z_0)\cdot\mathbf{r}_U(z_0)\cdot\mathbf{R}_D(z_0), \tag{3.6d}$$

with the boundary conditions $\mathbf{T}_D(0)=\mathbf{T}_U(0)=\mathbf{I}$ and $\mathbf{R}_D(0)=\mathbf{R}_U(0)=\mathbf{O}$, where $\mathbf{I}$ is the unit matrix and $\mathbf{O}$ is the nil matrix. The former boundary condition describes the unperturbed transmission of plane waves in the absence of a band heterogeneity while the latter boundary condition denotes that there are no reflected waves in that case. The $\cdot$ represents a product of matrices: $(\mathbf{B}\cdot\mathbf{C})^{\nu\sigma}=\mathbf{B}^{\nu\tau}\mathbf{C}^{\tau\sigma}$. Note that the reflection and transmission matrices for the thick band are dimensionless, while the reflection and transmission matrices for the thin layer have dimension *length*$^{-1}$.

We thus have 36 coupled, complex, first order, non-linear differential equations that describe the reflection and transmission of plane *P*- and *S*-waves by a band heterogeneity. These equations contain no singularities because $\mathbf{t}_D$, $\mathbf{t}_U$, $\mathbf{r}_D$ and $\mathbf{r}_U$ are regular throughout the heterogeneity. The differential equations have the same form as the equation derived by Kennett (1984), which describe the propagation of surface waves in a 2-D laterally heterogeneous medium. These equations can be solved numerically in the same way as done by Kennett (1984) and Kennett & Mykkelveit (1984). As pointed out by Kennett (1984), these Ricatti equations transform a two point boundary value problem into an initial value problem which is much easier to solve numerically.

The reflection and transmission matrices $\mathbf{R}$ and $\mathbf{T}$ in this paper are different from the reflection and transmission matrices used in the reflectivity method (e.g. Kennett 1974). For example, in Kennett (1974) $\mathbf{R}_j$ and $\mathbf{T}_j$ are the local coefficients for upgoing and downgoing waves within the inhomogeneous band at layer *j*. In this paper $\mathbf{R}(z_0)$ and $\mathbf{T}(z_0)$ denote the reflected and transmitted waves outside a band of thickness z_0. This implies that we do not make any statements of the wave field inside the inhomogeneous band. Specifically, the propagation characteristics of the inhomogeneous band do not enter the equations explicitly. (It is for this reason that the equations remain regular in the presence of turning points.) However, implicitly the propagation characteristics of the inhomogeneous band are contained in the Ricatti equations (3.6).

Consider a smooth medium without turning points. In that case we can approximate equations (3.6c and d)

$$\frac{d}{dz_0}\mathbf{T}_D(z_0)=\mathbf{t}_D(z_0)\cdot\mathbf{T}_D(z_0), \tag{3.7a}$$

$$\frac{d}{dz_0}\mathbf{T}_U(z_0)=\mathbf{T}_U(z_0)\cdot\mathbf{t}_U(z_0). \tag{3.7b}$$

Let us consider normal incident waves (i.e. $k=0$). As mentioned in the discussion of the reflection and transmission of plane waves by a thin isotropic layer, all the off-diagonal elements of $\mathbf{t}_D$ and $\mathbf{t}_U$ are zero in case of normal incidence. Because of the boundary conditions $\mathbf{T}_D(0)=\mathbf{T}_U(0)=\mathbf{I}$ it follows that all the off-diagonal elements of $\mathbf{T}_D(z_0)$ and $\mathbf{T}_U(z_0)$ are zero. For the remaining diagonal elements we find

$$T_U^{\nu\nu}(z_0)=\exp\left[\int_0^{z_0}t_U^{\nu\nu}(\zeta)d\zeta\right] \tag{3.8a}$$

$$T_D^{\nu\nu}(z_0)=\exp\left[\int_0^{z_0}t_D^{\nu\nu}(\zeta)d\zeta\right]. \tag{3.8b}$$

Because $t_D^{\nu\nu}=t_U^{\nu\nu}$ we have $T_D^{\nu\nu}=T_U^{\nu\nu}$. Equations (3.8a) and (3.8b) are the justification of the approximations (8.5) and (3.12) made by Snieder (1986b) and Snieder (1987) respectively. All the elements of $\mathbf{t}_D$ and $\mathbf{t}_U$ are purely imaginary in this case. We have for example

$$t_D^{11}=t_U^{11}=\frac{ik_\alpha}{2\rho_0\omega^2}[\rho_s\omega^2-k_\alpha^2(\lambda_s+2\mu_s)], \tag{3.9}$$

so the only effect of the band is a phase shift of the transmitted wave. Here $k_\alpha=\omega/\alpha$. This phase shift corresponds to a wavenumber perturbation

$$\delta k_\alpha = \frac{k_\alpha}{2\rho_0\alpha^2}[\rho_s\alpha^2 - \lambda_s - 2\mu_s], \tag{3.10}$$

so that the phase shift induced by the inhomogeneous band is correctly taken care of.

4 A NUMERICAL EXAMPLE FOR 1-D SCATTERING IN QUANTUM MECHANICS

In this section we illustrate the behaviour of homogeneous and inhomogeneous waves in a 1-D quantum mechanical example, and show explicitly that inhomogeneous waves are handled correctly by the Ricatti equations for the reflection and transmission coefficients. Up to this point the theory is presented for the reflection and transmission of plane seismic waves. Using a strategy similar to the one used in Sections 2 and 3 for the seismic case, a similar set of scalar Ricatti equations can be derived which describes the 1-D reflection and transmission of plane waves in quantum mechanics. The 1-D Schroedinger equation can be expressed as

$$\frac{d^2}{dx^2}\psi(x) + [k^2 - V(x)]\psi(x) = 0, \tag{4.1}$$

where ψ is the wave function, k is the wavenumber and V is a potential. If we define $\psi = \psi^0 + \psi^s$, with

$$\psi^0(x) = e^{\pm ikx}, \tag{4.2}$$

we find that

$$\frac{d^2}{dx^2}\psi^s(x) + k^2\psi^s(x) = V(x)[\psi^0(x) + \psi^s(x)]. \tag{4.3}$$

Using the first Born approximation, the reflection and transmission of a plane wave $\psi^0(x)$ by a rectangular potential with height $V(x)$ and width Δx can be described in the same way as in Section 2. From this, a set of scalar Ricatti equations can be derived in the same way as in Section 3. If we replace the subscript D (down) by R (right) and the subscript U (up) by L (left) and replace z_0 by x_0, we find the following scalar Ricatti equations for the 1-D quantum mechanical case

$$\frac{d}{dx_0}R_L(x_0) = T^2(x_0)r_L(x_0), \tag{4.4a}$$

$$\frac{d}{dx_0}R_R(x_0) = r_R(x_0) + 2t(x_0)R_R(x_0) + r_L(x_0)R_R^2(x_0), \tag{4.4b}$$

$$\frac{d}{dx_0}T(x_0) = t(x_0)T(x_0) + r_L(x_0)T(x_0)R_R(x_0), \tag{4.4c}$$

in which $T = T_L = T_R$ and $t = t_L = t_R$. In these equations we have

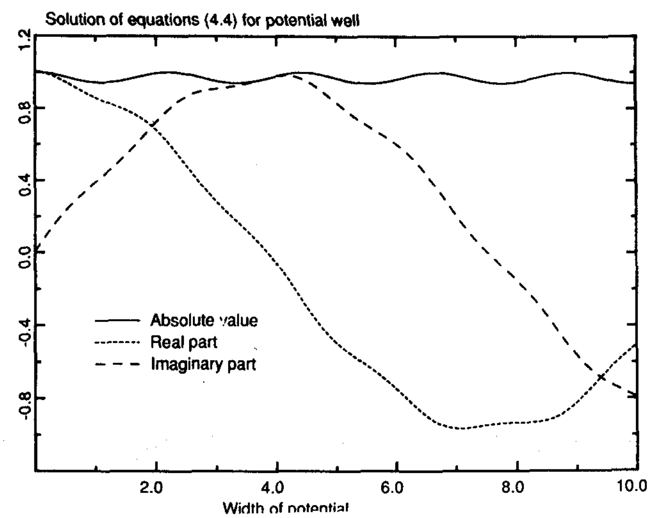
$$t(x_0) = -\frac{i}{2k}V(x_0), \tag{4.5a}$$

$$r_L(x_0) = -\frac{i}{2k}V(x_0)e^{2ikx_0}, \tag{4.5b}$$

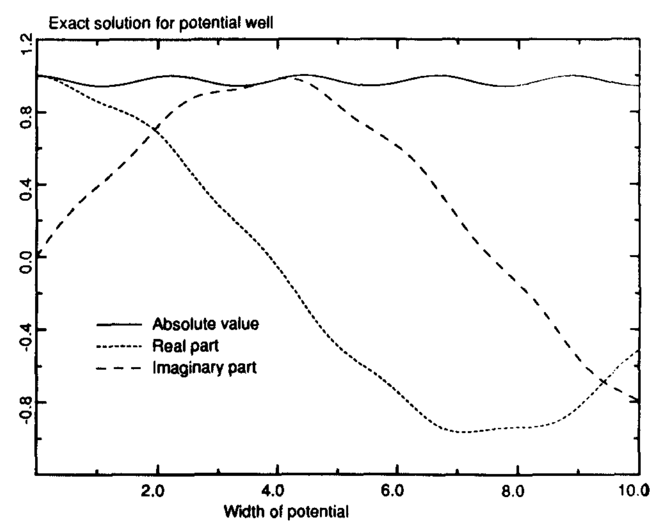
$$r_R(x_0) = -\frac{i}{2k}V(x_0)e^{-2ikx_0}. \tag{4.5c}$$

In the last expression k is the wavenumber for the incident plane wave *in the background medium*. Note again that the Ricatti equations are non-singular because t, r_L and r_R are regular within the potential. The boundary equations for the Ricatti equations are $T(0) = 1$ and $R_L(0) = R_R(0) = 0$.

In order to show that the differential equations (4.4a–c) describe the propagation of both homogeneous and inhomogeneous waves within the potential, we solve the equations (4.4a–c) numerically for a rectangular potential for which analytical solutions are known (Mertzbacher



(a)



(b)

Figure 4.(a) Analytical solution for the transmission coefficient as a function of the width of the potential in case $V < k^2$. Here $k = 1$ and $V = -1$. (b). Numerical solution for the transmission coefficient as a function of the width of the potential as obtained from equations (4.4) in case $V < k^2$. Here $k = 1$ and $V = -1$.

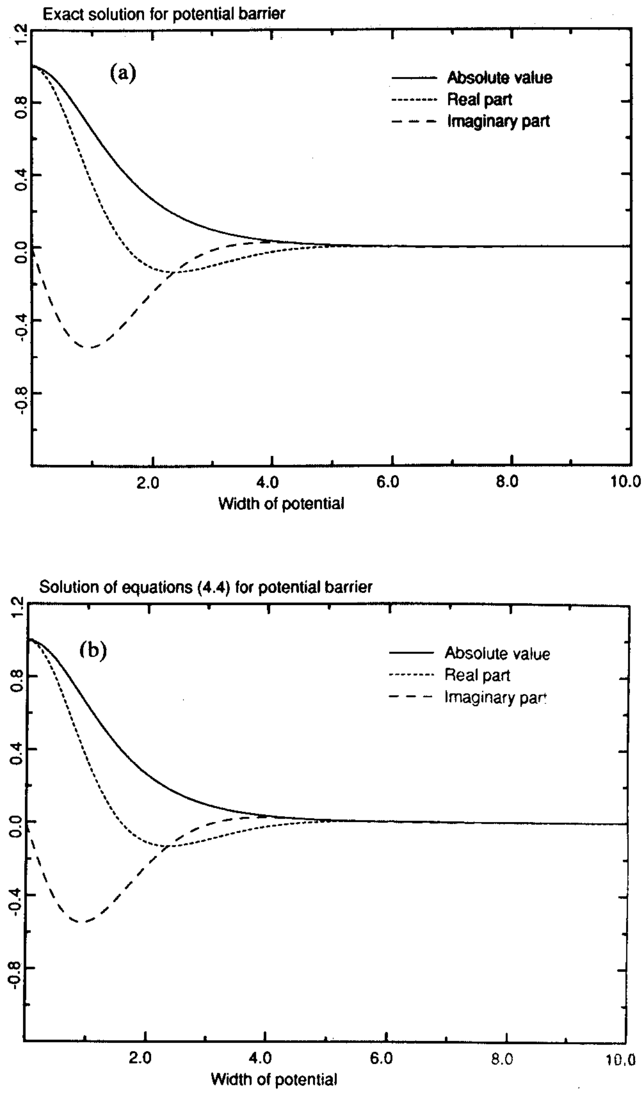


Figure 5. (a) Analytical solution for the transmission coefficient as a function of the width of the potential in case $V > k^2$. Here $k = 1$ and $V = 2$. (b) Numerical solution for the transmission coefficient as a function of the width of the potential as obtained from equations (4.4) in case $V > k^2$. Here $k = 1$ and $V = 2$.

1970), and consider both homogeneous ($V < k^2$) and inhomogeneous ($V > k^2$) solutions. In the latter case tunneling occurs. We compare the numerical results with the analytical solutions. In Fig. 4(a) the analytical solution for the transmission coefficient, T, is shown as a function of the width of the potential in case $V < k^2$. Figure 4(b) shows the solution for the transmission coefficient for the same problem as obtained from the Ricatti equations. Figure 5(a) and (b) shows the transmission coefficient as a function of the width of the barrier in case $V > k^2$ for the analytical solution and the numerical solution respectively. It is clear from Figs 4 and 5 that the Ricatti equations (4.4a–c) correctly describe the propagation of both homogeneous and inhomogeneous waves within the potential, despite the fact that the propagators within the potential do not enter the theory explicitly.

5 CONCLUSIONS

Exact, first-order, non-linear differential equations which describe the reflection and transmission of plane waves by a laterally homogeneous band are derived by employing the first Born approximation and an invariant imbedding technique. The basic concept is the fact that the contribution to the total wave field of the higher order Born approximations for the waves scattered by the thin layer with width Δz goes to zero in the limit $\Delta z \rightarrow 0$. Invariant imbedding extends the Born approximation in a bootstrap fashion to the full non-linear response. The differential equations can be solved numerically, in the same way as was done by Kennett (1984) and Kennett & Mykkeltveit (1984). The advantages of this approach are, apart from numerical efficiency, that the density and the elasticity tensor may have an arbitrary depth dependence, and that the band may be anisotropic; P–SV and SH coupling is incorporated in the method. Furthermore the method incorporates the propagation of both homogeneous and inhomogeneous waves within the heterogeneity as we demonstrated numerically in Section 4; the theory remains valid in the neighbourhood of turning points. Finally we show that in the absence of turning points the only effect of a smooth laterally homogeneous band is a phase shift of the transmitted wave field. A present limitation is that we do not incorporate the effects of a free surface.

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APPENDIX

In order to find the plane wave Green's tensor G which satisfies

$$L^0_{ij}G_{jn} = \delta_{in}\delta(z - z_0)e^{i(kx-\omega t)}, \tag{A.1}$$

where $L^0_{ij} = -\rho_0\omega^2\delta_{ij} - \lambda_0\partial_i\partial_j - \mu_0\partial_i\partial_j - \mu_0\delta_{ij}\,\partial_k\partial_k$, we first try to find the solution $\mathbf{u}$ of

$$L^0_{ij}u_j = F_i\delta(z - z_0)e^{i(kx-\omega t)}, \tag{A.2}$$

where $\mathbf{F}$ is a constant vector. As an appropriate ansatz we look for solutions

$$\mathbf{u}(x, z, t) = \begin{cases} \mathbf{p}^1_{\mathrm{D}}A_{\mathrm{D}}e^{i(kx+\nu_\alpha z-\omega t)} + [\mathbf{p}^2_{\mathrm{D}}B_{\mathrm{D}} + \mathbf{p}^3_{\mathrm{D}}C_{\mathrm{D}}]e^{i(kx+\nu_\beta z-\omega t)}, & z > z_0 \\ \mathbf{p}^1_{\mathrm{U}}A_{\mathrm{U}}e^{i(kx-\nu_\alpha z-\omega t)} + [\mathbf{p}^2_{\mathrm{U}}B_{\mathrm{U}} + \mathbf{p}^3_{\mathrm{U}}C_{\mathrm{U}}]e^{i(kx-\nu_\beta z-\omega t)}, & z_0 > z. \end{cases} \tag{A.3}$$

$\mathbf{p}^i_{\mathrm{D}}$ and $\mathbf{p}^i_{\mathrm{U}}$ $(i = 1, 2, 3)$ are defined in Section 2. The boundary conditions at $z = z_0$ are (Aki & Richards 1980, Ben-Menahem & Singh 1981)

$$\mathbf{u}_{\mathrm{D}}(x, z_0, t) = \mathbf{u}_{\mathrm{U}}(x, z_0, t), \tag{A.4a}$$

and

$$[\boldsymbol{\sigma}_{\mathrm{D}}(x, z_0, t) - \boldsymbol{\sigma}_{\mathrm{U}}(x, z_0, t)]\cdot\hat{z} = -\mathbf{F}e^{i(kx-\omega t)}, \tag{A.4b}$$

where $\hat{z} = (0, 0, 1)$ and $\boldsymbol{\sigma}_{\mathrm{D}} = \lambda_0(\nabla\cdot\mathbf{u}_{\mathrm{D}})\mathbf{I} + \mu_0[\nabla\mathbf{u}_{\mathrm{D}} + (\nabla\mathbf{u}_{\mathrm{D}})^{\mathrm{T}}]$, with a similar expression for $\boldsymbol{\sigma}_{\mathrm{U}}$. Here $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{I}$ is the unit matrix and the superscript T denotes transpose. Now we have six linear equations (A.4a,b) and six unknowns A_{D}, B_{D}, C_{D}, A_{U}, B_{U} and C_{U}. Solving for the unknown yields

$$A_{\mathrm{D}} = \frac{ie^{-i\nu_\alpha z_0}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}[\mathbf{p}^1_{\mathrm{D}}\cdot\mathbf{F}], \tag{A.5a}$$

$$B_{\mathrm{D}} = \frac{ie^{-i\nu_\beta z_0}}{2\mu_0\nu_\beta}[\mathbf{p}^2_{\mathrm{D}}\cdot\mathbf{F}], \tag{A.5b}$$

$$C_{\mathrm{D}} = \frac{ie^{-i\nu_\beta z_0}}{2\mu_0\nu_\beta}[\mathbf{p}^3_{\mathrm{D}}\cdot\mathbf{F}], \tag{A.5c}$$

$$A_{\mathrm{U}} = \frac{ie^{i\nu_\alpha z_0}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}[\mathbf{p}^1_{\mathrm{U}}\cdot\mathbf{F}], \tag{A.5d}$$

$$B_{\mathrm{U}} = \frac{ie^{i\nu_\beta z_0}}{2\mu_0\nu_\beta}[\mathbf{p}^2_{\mathrm{U}}\cdot\mathbf{F}], \tag{A.5e}$$

$$C_{\mathrm{U}} = \frac{ie^{i\nu_\beta z_0}}{2\mu_0\nu_\beta}[\mathbf{p}^3_{\mathrm{U}}\cdot\mathbf{F}]. \tag{A.5f}$$

The plane wave Green's tensor is therefore given by

$$\begin{aligned} G(z, z_0, x, t) &= \begin{cases} \left[\dfrac{ie^{i\nu_\alpha(z-z_0)}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}\mathbf{p}^1_{\mathrm{D}}\mathbf{p}^1_{\mathrm{D}} + \dfrac{ie^{i\nu_\beta(z-z_0)}}{2\mu_0\nu_\beta}(\mathbf{p}^2_{\mathrm{D}}\mathbf{p}^2_{\mathrm{D}} + \mathbf{p}^3_{\mathrm{D}}\mathbf{p}^3_{\mathrm{D}})\right]e^{i(kx-\omega t)}, & z > z_0 \\ \left[\dfrac{ie^{i\nu_\alpha(z_0-z)}}{2(\lambda_0 + 2\mu_0)\nu_\alpha}\mathbf{p}^1_{\mathrm{U}}\mathbf{p}^1_{\mathrm{U}} + \dfrac{ie^{i\nu_\beta(z_0-z)}}{2\mu_0\nu_\beta}(\mathbf{p}^2_{\mathrm{U}}\mathbf{p}^2_{\mathrm{U}} + \mathbf{p}^3_{\mathrm{U}}\mathbf{p}^3_{\mathrm{U}})\right]e^{i(kx-\omega t)}, & z_0 > z \end{cases} \\ &= \tilde{G}(z, z_0)e^{i(kx-\omega t)}, \end{aligned} \tag{A.6}$$

thereby defining $\tilde{G}(z, z_0)$.