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Linearized scattering of surface waves on a spherical Earth

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Abstract. Recently, a formalism for three-dimensional surface-wave scattering in a plane geometry was derived. Since teleseismic surface-wave data are generally recorded at epicentral distances large enough to be influenced by the sphericity of the Earth, it is necessary to find the effects of a spherical geometry on surface-wave scattering. The theory of surface-wave scattering relies heavily on a dyadic decomposition of the Green's function, and a new derivation is given for the (dyadic) Green's function of a spherically symmetric Earth. This new derivation employs Poisson's sum formula and is more rigorous than previous derivations. Using the dyadic Green's function, a relation is established with the scattering theory in a flat geometry. This finally leads to a linearized formalism for three-dimensional surface-wave scattering on a sphere. Even for shallow surface waves the effects of sphericity are important and necessitate a modification of the propagator terms in the expression for the scattered surface waves.

Key words: Seismology – Normal modes – Surface waves – Scattering – Inversion

Introduction

Mapping the lateral heterogeneities in the Earth is a major task of modern seismology. This problem has been attacked with two types of methods. The first method utilizes the great circle theorem for surface waves (Backus, 1964; Jordan, 1978; Dahlen, 1979a). This theorem states that if the heterogeneity varies smoothly in the horizontal direction, the surface wave is only influenced by the Earth's structure on the source-receiver great circle. By combining the information of many source-receiver great circles an image of the Earth can in principle be obtained (e.g. Woodhouse and Dziewonski, 1984; Montagner, 1986; Nataf et al., 1986). The second method consists of the tomographic inversion of large data sets of body-wave delay times. This can be done on a global scale (Dziewonski, 1984), on a continental scale (Spakman, 1986) or on a more local scale (Nercessian et al., 1984).

None of these methods is able to cope with true body-wave or surface-wave scattering, so that a large part of the seismic signal is not used. Scattering of body waves has been treated by several authors in the Born approximation (Hudson and Heritage, 1982; Malin and Phinney, 1985; Wu and Aki, 1985). However, up to this point none of these techniques could cope with a layered reference medium, and they have not yet been used for systematic inversions in global seismology.

Apart from scattering body waves, lateral heterogeneities also scatter surface waves and give rise to the coupling of normal modes of a laterally homogeneous Earth. That surface-wave scattering occurs in reality is shown by the observations of Levshin and Berteussen (1979) and Bungum and Capon (1974). Scattering of surface waves is caused by sharp lateral inhomogeneities, and therefore scattered surface waves can provide valuable information on these heterogeneities. These heterogeneities may be located far from the plane of the source-receiver great circle. Unlike other types of waves, scattered surface waves enable us to investigate upper mantle heterogeneities even in regions devoid of adequate seismic instrumentation such as oceans, continental margins and large parts of the continents. It is therefore important to develop a workable method for the interpretation of these waves which, so far, by necessity have been regarded as 'noise'.

Kennett (1984) devised an exact theory for the effects of lateral inhomogeneities on surface waves in two dimensions. This theory employs invariant embedding and therefore relies heavily on the fact that surface waves in two dimensions propagate in only one horizontal direction. At this point there is no exact theory for surface-wave scattering in three dimensions. Snieder (1986a) developed a perturbation theory for the scattering of surface waves in a flat geometry, for buried inhomogeneities. He showed how different modes are coupled, and how this gives rise to surface-wave scattering. As an example, a "great circle theorem" in a flat geometry was derived, and it was shown that Snell's law holds for the reflection of surface waves by a vertical interface between two media. Furthermore, an inversion procedure was presented for the reconstruction of the medium from scattered surface-wave data. In Snieder (1986b) a similar theory is presented for surface-wave scattering by surface topography, and it is shown there that the restriction that the inhomogeneity should be buried is not necessary.

One limitation of the theory presented by Snieder (1986a, b) is that the theory is formulated for a flat geometry. This paper serves to show how the theory for a flat geometry can be generalized for a spherical geometry. It is shown here that even for shallow surface waves the theory has to be modified, since the propagator terms are affected by the sphericity.

Paradoxically, the major part of this paper is devoted to a spherically symmetric Earth. The reason for this is that in order to give an efficient derivation of the scattering

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effects of lateral heterogeneities, it is necessary to have a simple dyadic representation of the Green's function of a laterally homogeneous Earth. In principle, this problem is already solved. Gilbert and Dziewonski (1975) and Vlaar (1976) present the response of a layered Earth, while Ben-Menahem and Singh (1968) give a dyadic representation of the Green's function of a homogeneous sphere. However, none of these theories provides an expression for the Green's function which is convenient for analytical work, and which also has a simple physical interpretation. It is for this reason that a new derivation is given in this paper, leading to a simpler dyadic representation of the Green's function.

In order to do this, the response of the Earth is written as a sum of normal modes. The far-field limit of the Green's function and its gradient is derived in the following two sections. It is shown in the Appendix how the sum of all normal modes can be reduced to a sum over radial mode numbers only. Then a theory is derived for the scattering by lateral heterogeneities. In the subsequent section it is shown that the scattering coefficients on the sphere are similar to the scattering coefficients in a flat geometry.

In order to be able to derive this theory, several restrictive assumptions have to be made. It is assumed throughout this paper that:

1. The heterogeneity is weak enough that a linearization in the heterogeneity can be performed.
2. The modes which are excited have a horizontal wavelength small compared to the circumference of the Earth.
3. The far-field limit can be used, i.e. the scatterer is several wavelengths removed from both the source and the receiver.

One word about the notation in this paper. The summation convention is used both for vector and tensor indices. Latin indices are used for vector components, while a Greek index is used for the radial mode number of surface waves. (For normal modes we retain the conventional "n".) The dot product in this paper is defined by

$$[\mathbf{A}\cdot\mathbf{B}]=A_i^* B_i \tag{1}$$

and the double contraction by

$$[\mathbf{C}:\mathbf{D}]=C_{ij}^* D_{ji}. \tag{2}$$

The response of a radially symmetric Earth in terms of its normal modes

The equation of motion for the excitation of an elastic inhomogeneous sphere by a point force $\mathbf{F}$ oscillating with frequency ω is given by:

$$L_{ij}\, s_j = F_i \tag{3}$$

where,

$$L_{ij} = -\delta_{ij}\rho\omega^2 - \partial_n (c_{inmj}\,\partial_m) \tag{4}$$

and c_{inmj} is the elasticity tensor.

In subsequent sections an expression is derived for the wave which is scattered by lateral heterogeneities. This expression contains the Green's function of a reference model, for which a spherically symmetric Earth is taken. For the moment we will restrict ourselves, therefore, to the excitation of a radially symmetric non-rotating Earth.

The response can conveniently be expressed as a sum over normal modes $\mathbf{s}^{nlm}$ (n, l and m are the conventional quantum numbers of the modes). According to Gilbert and Dziewonski (1975) or Vlaar (1976), the response to this point force is:

$$\mathbf{s}(\mathbf{r}) = \sum_{n,l,m} \frac{\omega_{nl}^2}{\omega_{nl}^2-\omega^2}\, \mathbf{s}^{nlm}(\mathbf{r})\,[\mathbf{s}^{nlm}(\mathbf{r}_s)\cdot\mathbf{F}]. \tag{5}$$

If a small amount of damping (α_{nl}) is introduced this can be written as:

$$\mathbf{s}(\mathbf{r}) = \sum_{n,l,m} \frac{-i}{\omega}\, \omega_{nl}^2\, C_{nl}(\omega)\, \mathbf{s}^{nlm}(\mathbf{r})\,[\mathbf{s}^{nlm}(\mathbf{r}_s)\cdot\mathbf{F}] \tag{6}$$

with

$$C_{nl}(\omega) = \tfrac{1}{2}(i(\omega-\omega_{nl})-\alpha_{nl})^{-1} + \tfrac{1}{2}(i(\omega+\omega_{nl})-\alpha_{nl})^{-1}. \tag{7}$$

For the moment we shall assume the source to be located at the pole. Furthermore, we shall restrict ourselves to the far-field response of the Earth. This means that the receiver is assumed to be located at such a colatitude that:

$$\sin\theta \gg \frac{m}{(l+\frac{1}{2})}. \tag{8}$$

Furthermore, we will only consider modes with a horizontal wavelength much smaller than the circumference of the Earth, i.e.

$$l \gg 1. \tag{9}$$

A point force or a point moment tensor only excites modes with

$$|m| \leqq 2 \tag{10}$$

so that (8) is satisfied several wavelengths from the source.

As shown by Dahlen (1979a), the toroidal (T) and spheroidal (S) modes in the far field, for $m \geqq 0$, behave as:

$$\mathbf{s}_T^{nlm}(\mathbf{r}) = \frac{(l+\frac{1}{2})}{\pi(\sin\theta)^{\frac{1}{2}}}\,\hat{\boldsymbol{\phi}}\, W_{nl}(r)\sin\left[(l+\tfrac{1}{2})\theta+\left(\frac{m}{2}-\frac{1}{4}\right)\pi\right] e^{im\phi}, \tag{11}$$

$$\mathbf{s}_S^{nlm}(\mathbf{r}) = \frac{1}{\pi(\sin\theta)^{\frac{1}{2}}}\left\{\hat{\mathbf{r}}\, U_{nl}(r)\cos\left[(l+\tfrac{1}{2})\theta+\left(\frac{m}{2}-\frac{1}{4}\right)\pi\right]\right.$$
$$\left. -\hat{\boldsymbol{\theta}}(l+\tfrac{1}{2})\, V_{nl}(r)\sin\left[(l+\tfrac{1}{2})\theta+\left(\frac{m}{2}-\frac{1}{4}\right)\pi\right]\right\} e^{im\phi}. \tag{12}$$

$\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors pointing in the direction of increasing r, θ and ϕ. W, U and V are the radial eigenfunctions defined in Dahlen (1979a). For negative m, the modes follow from the symmetry properties of the spherical harmonics, which leads to:

$$\mathbf{s}^{n,l,-m} = (-1)^m (\mathbf{s}^{n,l,m})^*. \tag{13}$$

The bilinear formula (6) also requires the normal modes at the source position (the pole). As shown by Ben-Menahem and Singh (1968), the normal modes close to the pole behave as:

$$\mathbf{s}_T^{nlm}(\mathbf{r}) = \left(\frac{l+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}} (l+\tfrac{1}{2})\, W_{nl}(r)\, \frac{\hat{\boldsymbol{\phi}} - im\hat{\boldsymbol{\theta}}}{2}\, (\delta_{m,1}-\delta_{m,-1})\, e^{im\phi}, \tag{14}$$

$$\mathbf{s}_S^{nlm}(\mathbf{r})=\left(\frac{l+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\Bigg(\hat{\mathbf{r}}\,U_{nl}(r)\,\delta_{m,0}-(l+\tfrac{1}{2})\,V_{nl}(r)\frac{(\hat{\boldsymbol{\theta}}+im\hat{\boldsymbol{\phi}})}{2}(\delta_{m,1}-\delta_{m,-1})\Bigg)e^{im\phi}. \quad (15)$$

The m-summation in the modal sum can now be performed analytically by inserting Eqs. (11), (12), (14) and (15) in Eq. (6). For spheroidal modes this leads to:

$$\mathbf{s}_S(r,\theta,\phi)=\sum_{n,l}\frac{-i}{\omega}\,C_{nl}(\omega)\,\frac{\omega_{nl}^2}{\pi(\sin\theta)^{\frac{1}{2}}}\cdot\left(\frac{l+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}$$
$$\cdot\Bigg\{\left[\hat{\mathbf{r}}\,U_{nl}(r)\cos\left[(l+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right]+\hat{\boldsymbol{\theta}}(l+\tfrac{1}{2})\,V_{nl}(r)\right.$$
$$\left.\cdot\cos\left[(l+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right]\right][\hat{\mathbf{r}}_s\cdot\mathbf{F}]\,U_{nl}(r_s)$$
$$+\left[-\hat{\mathbf{r}}\,U_{nl}(r)\cos\left[(l+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right]+\hat{\boldsymbol{\theta}}(l+\tfrac{1}{2})\,V_{nl}(r)\right.$$
$$\left.\cdot\cos\left[(l+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right]\right][\hat{\boldsymbol{\theta}}_s\cdot\mathbf{F}]\,(l+\tfrac{1}{2})\,V_{nl}(r_s)\Bigg\}. \quad (16)$$

The l-summation can be converted to an integral by means of Poisson's summation formula. This integral can be evaluated with a contour integration; this procedure is described in the Appendix. For the first orbit this yields, after some rearrangement, the following result for the sum of the spheroidal modes:

$$\mathbf{s}_S(r,\theta,\phi)=\sum_{\nu}\left(\frac{l_\nu+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\frac{\omega}{2u_g^\nu}\,[\hat{\boldsymbol{\theta}}(l_\nu+\tfrac{1}{2})\,V_\nu(r)-i\hat{\mathbf{r}}\,U_\nu(r)]$$
$$\cdot\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\theta+\frac{\pi}{4}\right]}{(\sin\theta)^{\frac{1}{2}}}\,[(\hat{\boldsymbol{\theta}}_s(l_\nu+\tfrac{1}{2})\,V_\nu(r_s)-i\hat{\mathbf{r}}_s\,U_\nu(r_s))\cdot\mathbf{F}]. \quad (17)$$

In this expression ν is the radial mode number, and u_g^ν is the angular group velocity of the ν-th mode. l_ν is related to the horizontal wavenumber (k_ν) of the surface-wave mode ν through the relation $k_\nu a=(l_\nu+\frac{1}{2})$, where a is the Earth's radius. The horizontal wavenumber (k_ν) depends continuously on frequency, therefore l_ν is not necessarily an integer.

For toroidal modes a similar result can be derived in the same way. These modes give the following contribution to the displacement:

$$\mathbf{s}_T(r,\theta,\phi)=\sum_{\nu}\left(\frac{l_\nu+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\frac{\omega}{2u_g^\nu}(l_\nu+\tfrac{1}{2})^2\,\hat{\boldsymbol{\phi}}\,W_\nu(r)$$
$$\cdot\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\theta+\frac{\pi}{4}\right]}{(\sin\theta)^{\frac{1}{2}}}\,W_\nu(r_s)\,[\hat{\boldsymbol{\phi}}_s\cdot\mathbf{F}]. \quad (18)$$

The expressions (17) and (18) for the spheroidal and toroidal mode displacements depend only on the epicentral distance and the source-receiver direction. This means that the choice of the pole position is irrelevant. In order to make this more explicit we shall denote the epicentral distance by Δ, the unit vector along the source-receiver great circle by $\hat{\boldsymbol{\Delta}}$, and the horizontal unit vector perpendicular to this great circle by $\hat{\boldsymbol{\phi}}$, see Fig. 1.

The spheroidal and toroidal mode contributions to the displacement can both be accommodated in the following expression:

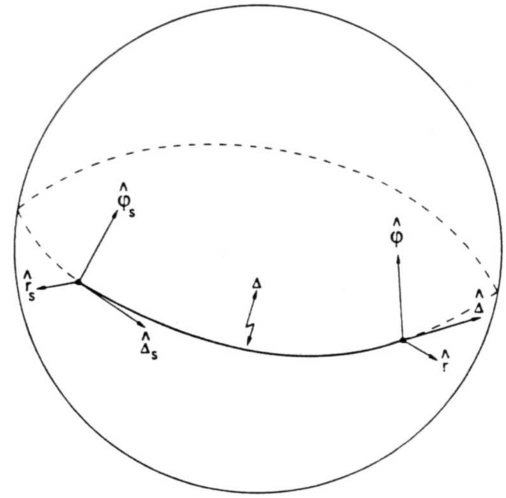

Fig. 1. Definition of the geometric variables for the direct wave

$$\mathbf{s}(\mathbf{r})=\sum_{\nu}\left(\frac{l_\nu+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\frac{\omega}{2u_g^\nu}\,\mathbf{p}^\nu(\mathbf{r},\Phi)$$
$$\cdot\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,[\mathbf{p}^\nu(r_s,\Phi)\cdot\mathbf{F}]. \quad (19)$$

The modal summation now includes both the toroidal and the spheroidal modes, and both types of modes are treated in an unified way. The **p** vectors are called the polarization vectors (Snieder, 1986a) since they describe the direction of oscillation of every mode. For spheroidal modes the polarization vector is:

$$\mathbf{p}^\nu(\mathbf{r},\Phi)=(l_\nu+\tfrac{1}{2})\,V_\nu(r)\,\hat{\boldsymbol{\Delta}}-i\,U_\nu(r)\,\hat{\mathbf{r}}. \quad (20)$$

While for toroidal modes:

$$\mathbf{p}^\nu(\mathbf{r},\Phi)=-(l_\nu+\tfrac{1}{2})\,W_\nu(r)\,\hat{\boldsymbol{\phi}}. \quad (21)$$

In these expressions Φ denotes the source-receiver great circle. Note that for toroidal modes the polarization vector is purely transverse, while for spheroidal modes the polarization vector has components both in the epicentral direction and the vertical direction which are 90° out of phase.

Up to this point the normalization of Dahlen (1979a) has been used implicitly, that is:

$$2\omega^2 I_1^\nu=1. \quad (22)$$

[In Dahlen (1979a) this expression is used with the normal-mode eigenfrequency (ω_ν) instead of the frequency of excitation (ω). However, as shown in the Appendix, the dominant contribution to the contour integral comes from the point $\omega=\omega_\nu$, so that ω and ω_ν can freely be exchanged after the surface-wave limit is taken.] The integral I_1^ν for spheroidal modes is:

$$I_1^\nu=\tfrac{1}{2}\int\rho(r)\,(U_\nu^2(r)+l_\nu(l_\nu+1)\,V_\nu^2(r))\,r^2\,dr. \quad (23\text{a})$$

While for toroidal modes:

$$I_1^\nu=\tfrac{1}{2}\int\rho(r)\,l_\nu(l_\nu+1)\,W_\nu^2(r)\,r^2\,dr. \quad (23\text{b})$$

Inserting Eq. (22) in Eq. (19) yields:

$$\mathbf{s}(\mathbf{r})=\sum_{\nu}\left(\frac{l_\nu+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\frac{1}{4\omega u_g^\nu I_1^\nu}\,\mathbf{p}^\nu(\mathbf{r},\Phi)\cdot\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,[\mathbf{p}^\nu(\mathbf{r}_s,\Phi)\cdot\mathbf{F}]. \quad (24)$$

The presence of the normalization integral I_1^ν in Eq. (24) makes it possible to renormalize the eigenfunctions U, V and W in the polarization vectors. For convenience we impose the following normalization:

$$I_1^\nu=\left(\frac{l_\nu+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}}\Big/4\omega u_g^\nu \quad (25)$$

which leads to

$$\mathbf{s}(\mathbf{r})=\sum_{\nu}\mathbf{p}^\nu(\mathbf{r},\Phi)\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,[\mathbf{p}^\nu(\mathbf{r}_s,\Phi)\cdot\mathbf{F}]. \quad (26)$$

So that the Green's function for the displacement at $\mathbf{r}_1$ due to a point force at $\mathbf{r}_2$ has a very simple form:

$$G_{ij}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{\nu}p_i^\nu(\mathbf{r}_1,\Phi)\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,p_j^{\nu*}(\mathbf{r}_2,\Phi). \quad (27)$$

This is a similar dyadic expansion of the Green's function to that in Snieder (1986a). Apart from the geometrical spreading factor, the Green's function has the same form on the sphere as in a flat geometry if the higher orbits are neglected. This can be seen by using the correspondence

$$\omega=k_\nu c_\nu,\quad k_\nu=(l_\nu+\tfrac{1}{2})/r,\quad U_g^\nu=u_g^\nu r,$$
$$l_1^\nu(z)\leftrightarrow-(l_\nu+\tfrac{1}{2})\,W_\nu(r),\quad r_1^\nu(z)\leftrightarrow(l_\nu+\tfrac{1}{2})\,V_\nu(r), \quad (28)$$
$$r_2^\nu(z)\leftrightarrow U_\nu(r),\quad \frac{\partial}{\partial z}\leftrightarrow-\frac{\partial}{\partial r},$$

so that

$$G_{ij}(\mathbf{r}_1,\mathbf{r}_2)=\left(\frac{\Delta}{\sin\Delta}\right)^{\frac{1}{2}}\sum_{\nu}\frac{1}{8c_\nu U_g^\nu I_1^\nu}\,p_i^\nu(\mathbf{r}_1,\Phi)\cdot\frac{\exp i\left(k_\nu r\Delta+\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}k_\nu r\Delta\right)^{\frac{1}{2}}}\,p_j^{\nu*}(\mathbf{r}_2,\Phi). \quad (29)$$

Apart from the $(\Delta/\sin\Delta)^{\frac{1}{2}}$ term, this is expression (3) of Snieder (1986a). It reflects the well-known travelling-wave character of the Earth's normal modes for large angular quantum number l.

Expression (29) only takes the first orbit into account, but higher orbits can easily be included by adding similar terms to (29). The phase factors for the polar phase shift are given by Dahlen (1979a). For brevity we will neglect the contribution of the higher orbits.

The gradient of the Green's function and the excitation by a moment tensor

In the derivation of the scattered wave, the gradient of the Green's function is needed. For the moment let us once more assume that $\mathbf{r}_2$ in Eq. (27) is located at the pole. The expression for the gradient of the Green's function is somewhat more complicated in spherical coordinates than in Cartesian coordinates due to the affine terms in the derivative (Butkov, 1968). However, for the far-field Green's function the vertical derivatives and the derivative in the epicentral direction are of relative order $(l+\frac{1}{2})/r$, while the azimuthal derivative and the affine terms are of relative order $1/r$ and $\cot\theta/r$. This means that under the restrictions (8) and (9) the gradient tensor is given by:

$$\nabla\mathbf{f}=\hat{\mathbf{r}}\hat{\mathbf{r}}\partial_r f_r+\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\partial_r f_\theta+\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\partial_r f_\phi+\hat{\boldsymbol{\theta}}\hat{\mathbf{r}}\frac{1}{r}\partial_\theta f_r+\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta f_\theta+\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\frac{1}{r}\partial_\theta f_\phi. \quad (30)$$

If this expression is used, the far-field $\mathbf{r}_1$ gradient of the Green's function takes the following form if one resubstitutes $\hat{\boldsymbol{\Delta}}=\hat{\boldsymbol{\theta}}$:

$$\nabla^{(1)}G_{ij}(\hat{\mathbf{r}}_1,\mathbf{r}_2)=\sum_{\nu}\left(\hat{\mathbf{r}}\partial_r p_i^\nu+i\frac{(l_\nu+\frac{1}{2})}{r}\hat{\boldsymbol{\Delta}}p_i^\nu\right)(\mathbf{r}_1)\cdot\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,p_j^{\nu*}(\mathbf{r}_2). \quad (31\text{a})$$

The gradient with respect to the $\mathbf{r}_2$ coordinates follows by complex conjugation:

$$\nabla^{(2)}G_{ij}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{\nu}p_i^\nu(\mathbf{r}_1)\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\cdot\left(\hat{\mathbf{r}}\partial_r p_j^{\nu*}-i\frac{(l_\nu+\frac{1}{2})}{r}\hat{\boldsymbol{\Delta}}p_j^{\nu*}\right)(\mathbf{r}_2). \quad (31\text{b})$$

These expressions can be used to determine the response to an excitation by a moment tensor. The response to a single couple follows by adding the response to a point force $\mathbf{F}$ at $\mathbf{r}_s+\boldsymbol{\delta}$ to a point force $-\mathbf{F}$ at $\mathbf{r}_s-\boldsymbol{\delta}$ and Taylor-expanding the result in $\boldsymbol{\delta}$. If the directions of $\mathbf{F}$ and $\boldsymbol{\delta}$ are interchanged and the results are added, the response to a double couple couple is obtained. Taking the limit $\delta\to 0$ while keeping $F\delta$ constant and adding these results yields the following response to a moment tensor:

$$\mathbf{s}(\mathbf{r})=\sum_{\nu}\mathbf{p}^\nu(\mathbf{r})\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta+\frac{\pi}{4}\right]}{(\sin\Delta)^{\frac{1}{2}}}\,[\mathbf{E}^\nu:\mathbf{M}], \quad (32)$$

where the moment tensor is

$$\mathbf{M}=2(\boldsymbol{\delta}\mathbf{F}+\mathbf{F}\boldsymbol{\delta}) \quad (33)$$

and the excitation tensor $(\mathbf{E})$ is

$$\mathbf{E}^\nu=\left(\hat{\mathbf{r}}\partial_r+i\frac{(l_\nu+\frac{1}{2})}{r}\hat{\boldsymbol{\Delta}}\right)\mathbf{p}^\nu. \quad (34)$$

This means that the response to a moment tensor can, everywhere in this paper, be obtained by making the following substitution:

$$[\mathbf{p}\cdot\mathbf{F}]\to[\mathbf{E}:\mathbf{M}]. \quad (35)$$

In can be shown that apart from terms of relative order $1/l$, the excitation term $[\mathbf{E}:\mathbf{M}]$ is equivalent to the "Σ-expressions" of Dahlen (1979b). The excitation terms of Dahlen (1979b) can be written in the form $[\mathbf{E}:\mathbf{M}]$ by considering them in a coordinate system with the θ-*axis* along the source-receiver great circle. This particular choice of the coordinate system does not affect the excitation, since the double contraction is invariant under rotations. In this derivation, terms like $U(r)/r$ have been neglected because $|\partial_r U|$ is of the order $|(l+\frac{1}{2})\,U/r| \gg |U/r|$. This is consistent with the assumptions (8)–(10).

The response of a laterally inhomogeneous Earth

The previous sections dealt with a dyadic formulation for the response of a spherically symmetric Earth to a point force or moment tensor excitation. This section features a perturbation theory to treat the effect of lateral heterogeneities. Suppose that the density and the elasticity tensor have the following form:

$$\begin{aligned}\rho(r,\theta,\phi)&=\rho^0(r)+\varepsilon\,\rho^1(r,\theta,\phi)\\ \mathbf{c}(r,\theta,\phi)&=\mathbf{c}^0(r)+\varepsilon\,\mathbf{c}^1(r,\theta,\phi).\end{aligned} \tag{36}$$

The density ρ^0 and elasticity tensor $\mathbf{c}^0$ define a radially symmetric reference medium which is perturbed by the lateral heterogeneities ρ^1 and $\mathbf{c}^1$. The parameter ε is introduced to indicate that the perturbation is small, and facilitates a systematic perturbation approach.

The equation of motion is given by Eqs. (3) and (4). If the decomposition (36) is used, the differential operator L can be written as:

$$L=L^0+\varepsilon L^1. \tag{37}$$

The displacement can be expressed as a perturbation series in ε:

$$\mathbf{s}=\mathbf{s}^0+\varepsilon\mathbf{s}^1+O(\varepsilon^2). \tag{38}$$

In this way the displacement field is divided into a direct wave ($\mathbf{s}^0$) and a scattered wave [the $O(\varepsilon)$ terms of $\mathbf{s}$]. Inserting Eqs. (37) and (38) in Eq. (3), and taking the terms proportional to ε^0 and ε^1 together gives:

$$L^0\mathbf{s}^0=\mathbf{F} \tag{39}$$

$$L^0\mathbf{s}^1=-L^1\mathbf{s}^0. \tag{40}$$

The direct wave can be expressed in the Green's function of the spherically symmetric reference medium. For a point force excitation at $\mathbf{r}_s$ one finds:

$$s_i^0(\mathbf{r})=G_{ij}(\mathbf{r},\mathbf{r}_s)\,F_j(\mathbf{r}_s). \tag{41}$$

Hudson (1977) showed that in the absence of topography variations Eq. (40) is solved by:

$$\begin{aligned}s_i^1(\mathbf{r})=\{&\textstyle\int G_{ij}(\mathbf{r},\mathbf{r}')\,\rho^1(\mathbf{r}')\,\omega^2\,G_{jl}(\mathbf{r}',\mathbf{r}_s)\,d^3r'\\ &-\textstyle\int[\partial_m G_{ij}(\mathbf{r},\mathbf{r}')]\,c^1_{jmnk}(\mathbf{r}')\,[\partial_n G_{kl}(\mathbf{r}',\mathbf{r}_s)]\,d^3r'\}\,F_l(\mathbf{r}_s).\end{aligned} \tag{42}$$

This expression is hard to interpret due to the presence of the gradient of the Green's function. If the dyadic form of the Green's function (27) and its gradient (31) is inserted in Eq. (42), and if an isotropic medium is assumed, the scattered wave takes after quite a bit of algebra, the following form:

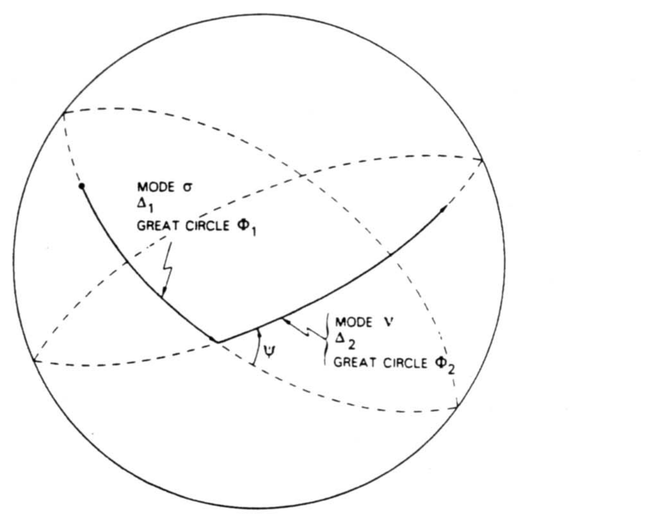

Fig. 2. Definition of the geometric variables for the scattered wave

$$\mathbf{s}^1(\mathbf{r})=\sum_{\nu,\sigma}\int\!\!\int \mathbf{p}^{\nu}(\mathbf{r},\Phi_2)\,\frac{\exp i\left[(l_\nu+\frac{1}{2})\,\Delta_2+\frac{\pi}{4}\right]}{(\sin\Delta_2)^{\frac{1}{2}}}\,V^{\nu\sigma}(\theta',\phi')\cdot\frac{\exp i\left[(l_\sigma+\frac{1}{2})\,\Delta_1+\frac{\pi}{4}\right]}{(\sin\Delta_1)^{\frac{1}{2}}}\,[\mathbf{p}^{\sigma}(\mathbf{r}_s,\Phi_1)\cdot\mathbf{F}]\,\sin\theta'\,d\theta'\,d\phi'. \tag{43}$$

In this expression Δ_1 is the epicentral distance between the source and the scatterer, while Φ_1 denotes the source-scatterer great circle. Δ_2 is the epicentral distance between the scatterer and the reçeiver, and Φ_2 denotes the scatterer-receiver great circle. See Fig. 2 for the definition of variables. Note that the scattered wave is expressed as an integral over the horizontal extent of the heterogeneity. The depth dependence of the inhomogeneity is contained in the $V^{\nu\sigma}$ term. The coefficients $V^{\nu\sigma}$ are related to the perturbations in the density (ρ^1) and in the Lamé parameters (λ^1 and μ^1):

$$\begin{aligned}V^{\nu\sigma}=&\int\rho^1\omega^2\,[\mathbf{p}^{\nu}(\Phi_2)\cdot\mathbf{p}^{\sigma}(\Phi_1)]\,r^2\,dr\\ &-\int\lambda^1\left((i[\partial_r\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\mathbf{r}}]+\frac{(l_\nu+\frac{1}{2})}{r}[\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\Delta}_2])\right.\\ &\left.\cdot\,(-i[\hat{\mathbf{r}}\cdot\partial_r\mathbf{p}^{\sigma}(\Phi_1)]+\frac{(l_\sigma+\frac{1}{2})}{r}[\hat{\Delta}_1\cdot\mathbf{p}^{\sigma}(\Phi_1)])\right)r^2\,dr\\ &-\int\mu^1\left[[\partial_r\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\mathbf{r}}]\,[\hat{\mathbf{r}}\cdot\partial_r\mathbf{p}^{\sigma}(\Phi_1)]\right.\\ &-i\frac{(l_\nu+\frac{1}{2})}{r}[\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\mathbf{r}}]\,[\hat{\Delta}_2\cdot\partial_r\mathbf{p}^{\sigma}(\Phi_1)]\\ &+i\frac{(l_\sigma+\frac{1}{2})}{r}[\partial_r\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\Delta}_1]\,[\hat{\mathbf{r}}\cdot\mathbf{p}^{\sigma}(\Phi_1)]\\ &\left.+\frac{(l_\nu+\frac{1}{2})(l_\sigma+\frac{1}{2})}{r^2}[\mathbf{p}^{\nu}(\Phi_2)\cdot\hat{\Delta}_1]\,[\hat{\Delta}_2\cdot\mathbf{p}^{\sigma}(\Phi_1)]\right]r^2\,dr\\ &-\int\mu^1\left([\partial_r\mathbf{p}^{\nu}(\Phi_2)\cdot\partial_r\mathbf{p}^{\sigma}(\Phi_1)]\right.\\ &\left.+\frac{(l_\nu+\frac{1}{2})(l_\sigma+\frac{1}{2})}{r^2}[\hat{\Delta}_2\cdot\hat{\Delta}_1]\,[\mathbf{p}^{\nu}(\Phi_2)\cdot\mathbf{p}^{\sigma}(\Phi_1)]\right)r^2\,dr.\end{aligned} \tag{44}$$

The expressions (43) and (44) can be interpreted in a simple way, despite their complicated form. Reading Eq. (43) from right to left one follows the "life history" of the scattered wave. At the source, mode σ is excited. The excitation is described by the projection of the force on the polarization vector of mode σ. The wave then travels from the source to the scatterer, the phase shift and the geometrical spreading being described by the term

$$\exp i\left[(l_\sigma+\tfrac{1}{2})\,\Delta_1+\frac{\pi}{4}\right]\Big/(\sin \Delta_1)^{\frac{1}{2}}.$$

After this, scattering occurs at (θ', ϕ'). This is described by the term $V^{\nu\sigma}$, which will be called the interaction matrix. The scattering also involves mode conversion to mode ν since a summation over all modes ν and σ is performed in Eq. (43). The wave then travels to the receiver, which is described by another propagator term. Finally, at the receiver the direction of the displacement oscillation is given by the polarization vector $\mathbf{p}^\nu$.

The expression for the scattered wave (43) closely resembles the expression given by Woodhouse and Girnius (1982) for elastic waves on a laterally inhomogeneous elastic sphere. Both their results and Eq. (43) express the scattered wave as an integral over the horizontal extent of the heterogeneity. However, Woodhouse and Girnius present their result in the time domain which, in the Born approximation, leads to a divergence for large times. The formalism presented here does not have this problem, thus making it possible to consider scattered waves with shorther periods. Furthermore, their formalism does not handle interactions between modes, which are fully taken care of in the theory presented here.

Equations (43) and (44) are obtained for a point force. Since the expressions are linear in the excitation, a more general excitation can be treated by integrating over the source coordinate $\mathbf{r}_s$. Excitation by a moment tensor can be incorporated with the substitution (35). It is shown in Snieder (1986b) how topography variations can be treated within the same formalism.

Analysis of the interaction matrix

The most interesting part of Eq. (43) is of course the interaction matrix $V^{\nu\sigma}$, because this matrix determines how the modes interact with each other. Unfortunately, Eq. (44) is not easy to interpret because this expression is extremely complicated. However, comparing Eq. (44) with the interaction matrix in a flat geometry [see (27) of Snieder (1986a)], using the correspondence (28), one finds that these expressions are equivalent. (The only difference is that here the depth integral is absorbed in the interaction terms.) The interaction terms are analysed in great detail in Snieder (1986a). It is shown there that even though $V^{\nu\sigma}$ depends on the polarization vectors of the incoming and the outgoing waves, $V^{\nu\sigma}$ depends in a very simple way on the scattering angle ψ defined by:

$$\cos\psi=[\Delta_2\cdot\Delta_1]_{\text{scatterer}} \tag{45}$$

(see Fig. 2). As in Snieder (1986a), the interaction matrix takes a simple form if analysed for toroidal and spheroidal modes separately:

$$V_{TT}^{\nu\sigma}=(l_\nu+\tfrac{1}{2})(l_\sigma+\tfrac{1}{2})\int\left(W^\nu W^\sigma\rho^1\omega^2-(\partial_r W^\nu)(\partial_r W^\sigma)\mu^1\right)r^2\,dr\cos\Psi-\frac{(l_\nu+\tfrac{1}{2})^2(l_\sigma+\tfrac{1}{2})^2}{r^2}\int W^\nu W^\sigma\mu^1 r^2\,dr\cos 2\psi, \tag{46}$$

$$V_{ST}^{\nu\sigma}=(l_\nu+\tfrac{1}{2})(l_\sigma+\tfrac{1}{2})\int\left(-V^\nu W^\sigma\rho^1\omega^2+\left(\frac{1}{r}U^\nu+\partial_r V^\nu\right)(\partial_r W^\sigma)\mu^1\right)r^2\,dr\sin\Psi+\frac{(l_\nu+\tfrac{1}{2})^2(l_\sigma+\tfrac{1}{2})^2}{r^2}\int V^\nu W^\sigma\mu^1 r^2\,dr\sin 2\psi, \tag{47}$$

$$V_{TS}^{\nu\sigma}=-V_{ST}^{\nu\sigma}, \tag{48}$$

$$\begin{aligned}V_{SS}^{\nu\sigma}=&\int\Bigg\{U^\nu U^\sigma\rho^1\omega^2-\left(\frac{(l_\nu+\tfrac{1}{2})^2}{r}V^\nu-\partial_r U^\nu\right)\left(\frac{(l_\sigma+\tfrac{1}{2})^2}{r}V^\sigma-\partial_r U^\sigma\right)\lambda^1\\&-\left(\frac{(l_\nu+\tfrac{1}{2})^2(l_\sigma+\tfrac{1}{2})^2}{r^2}V^\nu V^\sigma+2(\partial_r U^\nu)(\partial_r U^\sigma)\right)\mu^1\Bigg\}r^2\,dr\\&+(l_\nu+\tfrac{1}{2})(l_\sigma+\tfrac{1}{2})\int\left(V^\nu V^\sigma\rho^1\omega^2-\left(\frac{U^\nu}{r}+\partial_r V^\nu\right)\left(\frac{U^\sigma}{r}+\partial_r V^\sigma\right)\mu^1\right)r^2\,dr\cos\Psi\\&-\frac{(l_\nu+\tfrac{1}{2})^2(l_\sigma+\tfrac{1}{2})^2}{r^2}\int V^\nu V^\sigma\mu^1 r^2\,dr\cos 2\psi.\end{aligned} \tag{49}$$

$V_{TT}^{\nu\sigma}$ describes the coupling between the toroidal mode ν and the toroidal mode σ, $V_{TS}^{\nu\sigma}$ describes the conversion from the spheroidal mode σ to the toroidal mode ν, etc. Snieder (1986a) gives calculations of these terms for a flat Earth structure. It is shown there that the interaction terms are, in general, a strong function of frequency. Since for high l the modes of a spherical Earth are not dramatically different from the modes in a flat Earth structure, this conclusion remains valid in the spherical case.

Discussion, general inversion with surface waves

The scattering theory developed in the previous sections makes it possible to calculate the surface waves scattered by lateral inhomogeneities in a spherical earth. It is shown in Snieder (1986a) how this theory can be modified for the situation that the scatterers are not embedded in a laterally homogeneous medium, but in a reference medium with smooth lateral heterogeneities. The effect of surface perturbations on surface waves (Snieder, 1986b) can be taken into account in the same fashion as in the previous derivation.

In general, the (unknown) heterogeneities will have a wide range of horizontal spatial scales. Inhomogeneities with a horizontal scale of the order of the horizontal wavelength are efficient scatterers. This can be described with the theory of the two previous sections. Heterogeneities which vary on a horizontal scale much larger than the horizontal wavelength do not give rise to scattering, but they do affect the propagation of the surface waves. The great circle theorem (Jordan, 1978; Dahlen, 1979a) can be used for this type of heterogeneity either with linearized inversions using dispersion data (Nolet, 1977), or with a waveform fitting technique which can be either linearized (Lerner-Lam and Jordan, 1983) or nonlinear (Nolet et al., 1986).

It would be desirable to have an inversion scheme with can cope with all the scales of the heterogeneity. This algorithm should be able to handle both the scattering effects of the small-scale inhomogeneities and the effects of the large-scale heterogeneity on the propagation of surface waves. This can, in principle, be achieved along the following lines.

Let us designate the data (which consist of a large set of seismograms) by "d". The lateral heterogeneity can be expanded in a set of basis functions (which might be functions defining a cell model), so that the heterogeneity can be represented by a model vector "m" of expansion coefficients. The lateral heterogeneity "m" is superposed on a laterally homogeneous background model "M". Furthermore, we shall use "s" to designate the synthetic seismograms for this model:

$$s=s(m). \tag{50}$$

The relation between the model perturbation and the changes in the synthetic seismograms is, in general, strongly nonlinear because small perturbations in the wavenumber k_v are multiplied by the epicentral distance $r\Delta$. However, this nonlinearity is only important in modelling the propagation effects on surface waves. We can hopefully treat the scattering amplitudes in a linearized way with the single scattering theory presented in this paper. In that case the synthetic seismograms can be written [using Eqs. (38), (41) and (43)] symbolically as:

$$s=g_0 F+g_{\text{out}} V g_{\text{in}} F. \tag{51}$$

In this expression F denotes the excitation, while g_0, g_{in} and g_{out} denote the propagator terms and polarization vectors for the direct surface wave, the surface wave propagating to the scatterer and the scattered surface wave, respectively. The interaction terms V are given in Eq. (49). Since we assumed that the scattering is linear, the interaction terms can be written as:

$$V(m)=\frac{\partial V}{\partial m} m. \tag{52}$$

The synthetic seismograms then depend on the model in the following way:

$$s(m)=g_0(m) F+g_{\text{out}}(m) \frac{\partial V}{\partial m} m\, g_{\text{in}}(m) F. \tag{53}$$

The inversion can now proceed by fitting the synthetic seismograms to the data. This can be done by minimizing the misfit (S):

$$S=\|s(m)-d\|+\gamma\|m\|. \tag{54}$$

A regularization parameter γ is added to ensure stability, $\|\cdot\|$ denotes a suitable measure of the misfit. The inversion can therefore be treated as a (nonlinear) optimization problem. These problems can be solved iteratively.

However, these iterative schemes need the gradient of the synthetic seismograms with respect to the model parameters. This gradient can be determined from Eq. (53) by varying the model by a small amount δm, and linearizing the change δs in δm:

$$\delta s(m)=\left(\frac{\partial g_0}{\partial m} F+g_{\text{out}} \frac{\partial V}{\partial m} g_{\text{in}} F\right) \delta m. \tag{55}$$

(Here we tacitly assumed that terms of the order $m\delta m$ in the scattering term can be ignored, this is consistent with the Born approximation.) The derivatives $\partial V/\partial m$ can be obtained from Eq. (49) analytically, so that only the derivatives of the propagator $\partial g_0/\partial m$ of the direct-wave term have to be determined. These derivatives can be obtained by direct calculation of the Frechet derivatives using ray tracing (Babich et al., 1976) or Gaussian beams (Yomogida and Aki, 1985). A faster, but less accurate way to estimate the derivatives is to combine the great circle theorem with results from WKB theory, as in Nolet et al. (1986).

In principle, it should therefore be possible to invert for heterogeneities with a large range of horizontal spatial scales. The price one has to pay is that the number of unknowns is extremely large. The cell size (or the minimum wavelength of the basis functions in which the heterogeneity is expanded) has to be smaller than the wavelength of the scattered waves. This means that several thousands of unknowns have to be determined for an inversion on a continental scale, requiring a huge data set. With the continuing growth in power of even moderate machines, this is no computational problem. However, if insufficient data are available, widely different models may give an equally reasonable fit to the data. A broad-band digital seismic network with a density that matches the length scale of the lithospheric heterogeneities, as proposed in the ORFEUS (Nolet et al., 1985) and PASSCAL (1984) proposals, is necessary to make this type of inversion feasible.

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Appendix
The evaluation of the sum of normal modes

The sum of normal modes (16) can be evaluated by simple summation over l, but for high frequencies this summation becomes rather expensive. A modified approach to the calculation of the Green's function for surface waves in a spherical Earth was given by Nolet (1976), using the FFT, and by Lerner-Lam and Jordan (1983) using the Filon quadrature algorithm. Here we describe the FFT method.

This philosophy of the FFT method is to extend the l-summation from $l=0$ to ∞ to a summation from $l=-\infty$ to ∞, after which Poisson's sum formula and a contour integration make it possible to evaluate this sum. Now first consider the sum

$$S_+(\theta)=\sum_{l=0}^{\infty} b_+(l)\cos\left[(l+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right] \tag{56}$$

and assume that under the transformation

$$l\rightarrow -l-1 \tag{57}$$

the b_+ coefficients behave as follows:

$$b_+(-l-1)=ib_+(l) \qquad (l<0). \tag{58}$$

By expanding the cosine in Eq. (56) in two exponentials, and making the substitution (57) for l in the term with the negative exponent, one finds with Eq. (58) that

$$S_+(\theta)=\sum_{l=-\infty}^{\infty} \tfrac{1}{2} b_+(l)\exp i\left[(l+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right]. \tag{59}$$

Likewise, if S_- is defined by

$$S_-(\theta)=\sum_{l=0}^{\infty} b_-(l)\cos\left[(l+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right] \tag{60}$$

and if b_- has the following symmetry property

$$b_-(-l-1)=-ib_-(l) \qquad (l<0), \tag{61}$$

then S_- satisfies:

$$S_-(\theta)=\sum_{l=-\infty}^{\infty} \tfrac{1}{2} b_-(l)\exp i\left[(l+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right]. \tag{62}$$

These results can be used to evaluate the modal sum (16). In order to do this, the symmetry properties of the l-dependent coefficients in Eq. (16) under the transformation (57) have to be determined. It is shown in Aki and Richards (1980) that the spheroidal modes depend only on $l(l+1)$. This quantity is invariant under the substitution (57), and therefore C_{nl}, ω_{nl}, U_{nl} and V_{nl} are invariant under this transformation. [A similar result holds for toroidal modes, which depend on l only through the combination $(l-1)(l+2)$. This quantity also does not change under (57).] Apart from terms which are invariant under (57), the coefficients of the $\cos[(l+\frac{1}{2})-\pi/4]$ terms in Eq. (16) are proportional to $(l+\frac{1}{2})^{\frac{1}{2}}$ or $(l+\frac{1}{2})^{\frac{5}{2}}$. Likewise, the $\cos[(l+\frac{1}{2})+\pi/4]$ coefficients in Eq. (16) are proportional to $(l+\frac{1}{2})^{\frac{3}{2}}$.

The square root in these expressions has to defined with some care. In the subsequent derivation we want to do a contour integration with the variable $\xi=l+\frac{1}{2}$. We want to avoid a branch cut in the complex upper plane, so that we take the branch cut for the square root in the lower plane. This means that for $(l+\frac{1}{2})<0$

$$(l+\tfrac{1}{2})^{\frac{1}{2}}=+i[-(l+\tfrac{1}{2})]^{\frac{1}{2}}$$

and therefore

$$[-(l+\tfrac{1}{2})]^{\frac{1}{2}} = -i(l+\tfrac{1}{2})^{\frac{1}{2}} \qquad [(l+\tfrac{1}{2})<0]. \tag{63}$$

This means that the l-dependent coefficients of the $\cos[(l+\frac{1}{2})-\pi/4]$ term in Eq. (16) satisfy Eq. (61), while the coefficients of the $\cos[(l+\frac{1}{2})+\pi/4]$ term satisfy Eq. (58). Using Eqs. (59) and (62) we can write Eq. (16) then as:

$$\mathbf{s}_S = \tfrac{1}{2}\sum_n \sum_{l=-\infty}^{\infty} A_{nl}(r,\theta)\, C_{nl}(\omega)\, e^{i[(l+\frac{1}{2})\theta+\frac{\pi}{4}]} \tag{64}$$

with

$$A_{nl}(r,\theta) = \frac{-1}{\omega}\frac{\omega_{nl}^2}{\pi(\sin\theta)^{\frac{1}{2}}}\left(\frac{l+\frac{1}{2}}{2\pi}\right)^{\frac{1}{2}} (\hat{\mathbf{r}}\,U_{nl}(r) + i\hat{\boldsymbol{\theta}}(l+\tfrac{1}{2})\,V_{nl}(r))$$

$$\cdot[(-i\hat{\mathbf{r}}\,U_{nl}(r_s) + \hat{\boldsymbol{\theta}}(l+\tfrac{1}{2})\,V_{nl}(r_s))\cdot\mathbf{F}]. \tag{65}$$

Application of Poisson's sum formula leads to:

$$\mathbf{s}_S = \tfrac{1}{2}\sum_{j,n}(-1)^j \int_{-\infty}^{\infty} A_n(r,\theta,\xi)\, C_n(\omega,\xi)\, e^{i(\xi\theta+\frac{\pi}{4}+2\pi\xi j)}\, d\xi. \tag{66}$$

If we restrict ourselves to the direct arriving wave ($j=0$) this reduces to:

$$\mathbf{s}_S = \tfrac{1}{2}\sum_n \int_{-\infty}^{\infty} A_n(r,\theta,\xi)\, C_n(\omega,\xi)\, e^{i(\xi\theta+\frac{\pi}{4})}\, d\xi. \tag{67}$$

For one value of ω, say ω_0, the function $C_n(\omega_0,\xi)$ is sharply peaked around ξ_n, where $\omega_n(\xi_n)=\omega_0$. Thus, the integral may be approximated by:

$$\mathbf{s}_S(\omega_0) = \tfrac{1}{2}\sum_n A_n(r,\theta,\xi_n)\, D_n(\omega_0) \tag{68}$$

where

$$D_n(\omega_0) = \int_{-\infty}^{\infty} e^{i(\xi\theta+\frac{\pi}{4})}\, C_n(\omega_0,\xi)\, d\xi. \tag{69}$$

Because of Eq. (7), the integrand in Eq. (69) has two poles, one in the first quadrant and one in the third quadrant. Since the integral (69) is only needed for $\theta>0$, the contour should be closed in the upper half plane so that only the pole in the first quadrant gives a contribution. This contribution can easily be evaluated by a Taylor expansion around $\xi_n>0$:

$$\omega(\xi) = \omega_0 + (\xi-\xi_n)\, u_g^n + \dots$$

where $u_g^n = \dfrac{d\omega}{d\xi}$ is the angular group velocity of mode n (in radians per second), evaluated in ξ_n. The pole is located in

$$\xi_n' = \xi_n + i\alpha(\xi_n)/u_g^n, \tag{70}$$

which gives a residue

$$2\pi i\,\mathrm{Res}(\xi=\xi_n') = \frac{-\pi}{u_g^n}\exp[i\xi_n - \alpha(\xi_n)/u_g^n]\,\theta. \tag{71}$$

We then find

$$D_n(\omega_0) = \frac{-\pi}{\omega}\exp[i\xi_n - \alpha(\xi_n)/u_g^n]\,\theta, \tag{72}$$

which gives for the contribution of the spheroidal modes

$$\mathbf{s}_S(\mathbf{r},\omega) = -\sum_n A_n(r,\theta,\xi_n)\frac{\pi}{2u_g^n}\exp[i\xi_n - \alpha(\xi_n)/u_g^n]\,\theta. \tag{73}$$

This finally proves Eq. (17).

For toroidal modes, the derivation is completely analogous. The derivation can also be applied directly to the excitation by a moment tensor given by the "Σ-expressions" of Dahlen (1979b). In the normal-mode sum of Dahlen, two types of terms can be seen to occur after using relations like $\sin(x+\pi/4)=\cos(x-\pi/4)$. The first kind of term is proportional to

$$(l+\tfrac{1}{2})^{\frac{1}{2}}(l+\tfrac{1}{2})^{\text{odd number}}\cos\left[(l+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right],$$

while the second type of term is proportional to

$$(l+\tfrac{1}{2})^{\frac{1}{2}}(l+\tfrac{1}{2})^{\text{even number}}\cos\left[(l+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right].$$

Therefore, the coefficients of the cosine terms satisfy Eqs. (58) and (61) and the same derivation can be used to evaluate the l-summation.

The evaluation of the l-summation, as it is presented here, leads to the same results as in Dahlen (1979a). However, Dahlen makes three approximations which are not needed. Firstly, Dahlen ignores the pole in the third quadrant. Secondly, he extends the lower bound of the ξ-integration from 0 to $-\infty$. Thirdly, he ignores the incoming wave term. This incoming wave term could only be ignored because Dahlen also ignored the pole in the third quadrant. The derivation presented here gives a more rigorous proof of his result.