THE SPECIAL THEORY OF RELATIVITY
AND ITS APPLICATIONS TO
CLASSICAL MECHANICS

by

Adam C Mahl
ABSTRACT

Chapter 15: Special Relativity

Problem 1:
Two spacecraft transmit messages to each other while passing at constant velocities of...

Meanwhile:

We observe your speed to be 38.5% c, and your time is passing at 92.3% the rate of ours. Does this mirror your observations?

Please help me. I think I'm lost.

Figure 0.1: Abstract
# TABLE OF CONTENTS

ABSTRACT ................................................................. iii

LIST OF FIGURES ....................................................... v

LIST OF SYMBOLS ......................................................... vi

LIST OF ABBREVIATIONS ................................................ vii

CHAPTER 1 THE SPECIAL THEORY OF RELATIVITY ............... 1
  1.1 The Parsimonious Postulates .................................... 1
  1.2 Lorentz and Galileo: Transformers in Disguise ............... 2
    1.2.1 A Brief Interval of Spacetime ............................ 4
  1.3 The Minkowski World Space ..................................... 7

CHAPTER 2 RELATIVISTIC CLASSICAL MECHANICS ............... 10
  2.1 Mass ↔ Energy .................................................. 10
  2.2 Lagrangian & Hamiltonian Mechanics .......................... 11
  2.3 Final Thoughts ................................................ 12

REFERENCES CITED ...................................................... 14
LIST OF FIGURES

Figure 0.1 Abstract .......................................................... iii

Figure 1.1 Reference frame $S'$ moves with velocity $\vec{v}$ relative to reference frame $S$. ........................................... 3

Figure 1.2 A Light Cone describing relative intervals. ......................... 6

Figure 2.1 Conclusion .......................................................... 13
LIST OF ABBREVIATIONS

Special Theory of Relativity . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . SToRy
Within the understanding and practice of Newtonian mechanics, time is an “absolute” concept with which there is a complete separability of space and time. For centuries this view remained constant and not only carried but progressed the study of Classical Mechanics. It leads to the idea of the laws of physics being invariant under a Galilean transformation, which in turn implies that the velocity of light would be different in two inertial frames with relative motion.

One of the widely accepted theories through the centuries was that there existed something known as a “luminiferous aether”. This aether is what filled the absence of everything else and allowed light to exist and move from one point to another. James Maxwell (of Maxwells equations) showed that light was an electromagnetic wave, and his research implied that waves all need a medium to propagate through. Maxwell’s work were widely accepted and validated at the turn of the century. Acceptance of the aether led Lorentz to develop his transformation theories, however at the same time as our understanding of nature became greater, science was becoming less compatible with the idea of the aether. This led to the famous Michelson-Morley experiment, which was the strong start towards disproving and eliminating the aether theory.

1.1 The Parsimonious Postulates

The Special Theory of Relativity is the theory credited to and detailed/proposed by Albert Einstein in his third 1905 paper titled “On the Electrodynamics of Moving Bodies”. Due to his work in the area of quantum mechanics and understanding wave-particle duality of the properties of light, Einstein was dissatisfied with the growing incompatibility surrounding the idea of the “luminiferous aether”. Beginning with a complete dismissal of that idea, and assuming only the most valid and successful
Einstein posited the following:[1]

The following reflexions are based on the Principle of Relativity and on the Principle of Constancy of the velocity of light. These two principles we define as follows:

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.

2. Any ray of light moves in the “stationary” system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body...

These can be described more concisely as saying that the fundamental laws of physics and nature are the same when expressed in any inertial reference frame, and the the speed of light in free space is a universal constant regardless of the sources motion.

Starting from these, Albert Einstein constructed The Special Theory of Relativity. This theory is widely held to be a piece of mathematical beauty and elegant logic. It’s implications and applications will be investigated in the following sections. It is important and relevant to note that this generalization of Newtonian Mechanics was still incomplete. 10 years later Einstein would publish his General Theory of Relativity; this theory pushes the generalization further to encompass non-inertial frames and gravitational effects using non-euclidean geometric descriptions of our universe.

1.2 Lorentz and Galileo: Transformers in Disguise

To begin examining the implications of the SToRy postulates, we can start very simple and look at two inertial systems (reference frames) Figure 1.1.
Figure 1.1: Reference frame $S'$ moves with velocity $\vec{v}$ relative to reference frame $S$.

Suppose that we have the coordinates $\{x, y, z, t\}$ and $\{x', y', z', t'\}$ in the systems $S$ and $S'$ of Figure 1.1 respectively. Both systems were synchronized at time $t = 0$. The two systems are then shown to be related via the simple relations:

\[
\begin{align*}
    x' &= x - \vec{v}t \quad (a) \\
    y' &= y \quad (b) \\
    z' &= z \quad (c) \\
    t' &= t \quad (d)
\end{align*}
\]

(1.1)

This set of equations is known as the Galilean Transformation. They enable us to relate a measurement in one inertial reference frame to another. These equations describe relative distance (space) but keep a time that is consistent (absolute) across all inertial frames. Continuing with these equations we can see that $v'_x = v_x - \vec{v}$, which is a familiar equation relating the composite motion to the vector sum of the constituent velocities. However, looking at the case of $v'_x = c$, we see an immediate violation of postulate 2 where $v_x = c + \vec{v}$. The implied conclusion would be that time is not absolute, and that the Galilean transformation becomes invalid when the velocities involved approach magnitudes of the same order as $c$. A new transformation law must be found which will allow the speed of light to remain a constant in all inertial frames.
This transformation is known as the Lorentz Transformation after H.A. Lorentz. This transformation was “discovered” by several physicists between the years of 1887 and 1905. Most famously derived by Lorentz as a method of keeping Maxwell’s equations invariant between the accepted “luminiferous aether” and a moving frame, its mathematical properties were studied by Poincaré and Heaviside. Finally, Einstein rederived the transformation based only on the assumptions of his 2 postulates.

The simplest method of derivation is to assume that the transformation is a simple linear form of the Galilean Transformation; $x' = \gamma(x - \vec{v}t)$ and similarly by Eqn 1.a that $x = \gamma'(x' - \vec{v}t')$ where $\gamma = \gamma'$ due to the first postulate. The second postulate would imply that if $x' = ct'$for a given speed of $c$ then $x = ct$. Simple algebra yields a well known relation of $\gamma = \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}$ and the final set of relations known as the Lorentz Transformation:

\[
\begin{align*}
    x' &= \gamma(x - \vec{v}t) \\
    y' &= y \\
    z' &= z \\
    t' &= \gamma(t - \vec{v}x/c^2)
\end{align*}
\]  
\hspace{1cm} (1.2)

1.2.1 A Brief Interval of Spacetime

A slightly more mathematically rigorous method in deriving this transformation comes from beginning solely with the postulates from Section 1.1. If we define an event as something that occurs at a given time and place, than it can be defined by the four coordinates\{x, y, z, t\} within a space colloquially known as spacetime. It follows (geometrically) that an interval can be defined between any two events in spacetime as the following:

\[
s_{12} = \sqrt{c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2} \tag{1.3}
\]

Looking at infinitesimaly close events we have the given interval:

\[
ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1.4}
\]
For two events corresponding to the emission and reception of light (or an electromagnetic signal in general) it is seen that $ds^2 = 0$, and since this is based on the invariant nature of $c$, this is true for any frame which implies that $ds^2 = ds'^2$. Integrating these invariant intervals allows us to conclude that the interval between two spacetime events is the same in all inertial frames; $s = s'$.

This conclusion brings with it the concept of the light cone, graphically detailed in Figure 1.2. The origin or junction of the two halves of the cone represent the here/now interval. For intervals detailed in the “Future or Past” cone, mathematically the square of the spacetime interval is detailed by being greater than zero, and is said to be “time-like”, and a reference frame in which the two events occur at the same location can always be found, (watching someone texting on their cell phone while stopped at one light and then later again at another light, would to you, appear to take place in two different locations, however, to the texter, they are occurring in the same frame). Outside of the light cone the interval detailed is said to be “space-like”, these are mathematically imaginary, and there can always be found a reference frame in which the events occur simultaneously, (The separation of your office from your home is described by a space-like interval). When the interval is zero, as stated, the events are separated only by light rays. Within the time-like intervals is the concept of causality. Events occurring in the “Future” can be affected (receive a signal from) by the event at the origin, and conversely, events within the “Past” are able to affect (can send a signal towards) the event at the origin.

Now referring back to the spacetime interval in Eqn. 1.4, let’s assume that this interval is taking place with two observers, one at rest and one in motion. If $dt$ is the time elapsed in the rest (lab) frame, then there is no relative motion to the observer in $s'$ and it can be shown that;

$$ds^2 = ds'^2 \implies c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2$$

(1.5)
where $dt'$ is known as the “proper time”. With simple algebra the relation between the two observed time intervals is shown to be;

$$dt' = \frac{dt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(1.6)

the familiar Lorentz Transformation factor! When detailed in this fashion, this phenomenon is known as time dilation and refers to the idea that moving clocks run slower (by a factor of $\gamma$). This feature of the SToRy has many well known implications such as the idea of the “twin paradox” (where one of two twins could fly quickly in a closed space path and return “younger” than his twin) and is integral to understanding important aspects of particle physics, where small particles with brief lifetimes can move at speeds close to c and appear long-lived in an observer’s lab frame. This effect has been experimentally verified, in various experiments such as the Hafele-Keating experiment.
The same idea could be applied to a space interval and results in the idea known as length contraction (or more formally the Lorentz-Fitzgerald contraction), where to an observer in motion (of order \(c\)) relative to an object, observes the object to be contracted in dimensions by a factor of \(\gamma\).

1.3 The Minkowski World Space

With the idea of the Lorentz Transformation in hand, a formalism is desired that requires the physical laws to have the same form in all uniformly moving systems. To test the invariance of a physical law under Lorentz, it is useful to write them in a similar “Four-vector” manner similar to how the Lorentz Transformation was detailed. The idea of this four dimensional space where the interval was examined and the laws now act and reside is often referred to as Minkowski World Space after Herman Minkowski a German physicist, who in addition to being one of Einstein’s professors contributed much towards furthering and understanding the Story from a geometrical standpoint.

Generalizing the Lorentz Transformation to transforming to any frame moving with an arbitrary relativistic velocity is shown to be the following:

\[
\begin{bmatrix}
    ct' \\
    x' \\
    y' \\
    z'
\end{bmatrix} =
\begin{bmatrix}
    \gamma & \gamma \beta_x & \gamma \beta_y & \gamma \beta_z \\
    \gamma \beta_x & 1 + (\gamma - 1) \frac{\beta_x^2}{\beta^2} & (\gamma - 1) \frac{\beta_x \beta_y}{\beta^2} & (\gamma - 1) \frac{\beta_x \beta_z}{\beta^2} \\
    \gamma \beta_y & (\gamma - 1) \frac{\beta_y \beta_x}{\beta^2} & 1 + (\gamma - 1) \frac{\beta_y^2}{\beta^2} & (\gamma - 1) \frac{\beta_y \beta_z}{\beta^2} \\
    \gamma \beta_z & (\gamma - 1) \frac{\beta_z \beta_x}{\beta^2} & (\gamma - 1) \frac{\beta_z \beta_y}{\beta^2} & 1 + (\gamma - 1) \frac{\beta_z^2}{\beta^2}
\end{bmatrix}
\begin{bmatrix}
    ct \\
    x \\
    y \\
    z
\end{bmatrix}
\]

(1.7)

Where \(\gamma = \frac{1}{\sqrt{1 - \beta^2}}\), and \(\beta = \frac{\vec{v}}{c}\), and \(\beta_i = \frac{v_i}{c}\). This general transformation matrix is known as a Lorentz Boost and corresponds to a rotation within our 4 dimensional space. Our definition of an “event” can now be shown to be defined by the 4 coordinates defining points known as “world points”. In our defined Minkowski World Space, each particle can be thought to correspond to a certain line called a “world line” made up of the world points. The points of this line determine the coordinates
of the particle at all moments of time. A particle in uniform rectilinear motion would have a straight world line.

In order for our physical laws to remain invariant under this transformation, all terms in the equation describing the physical law would have to be described as tensors of the same rank, which are known as four-vectors. In the previous discussions we have already been describing our position in this manner, where the positions four vector was given by $X = (x, y, z, ct)$. Very often it is common to represent the 4th time component as $ict$ (and correspondingly changing the Lorentz Boost matrix). This is not implying that some quantity is imaginary/complex. It is simply introduced as a factor that helps to simplify the rigerous math often required in the relativistic transformation equations.

All such quantities and thus terms leading to equations, and ultimately finishing with physical laws, can be described by four vectors and four scalars. The methodology of arriving at the invariant four vector versions of each is fairly intuitive, even if at first glance, the result is not. In the limit of small velocities not of order $c$, the relativistic generalizations should reduce back to the well known Newtonian version. For example deriving the four vector velocity is as straightforward as comparing the differential of the four vector position to the differential of the proper time (previously discussed). This results in a four vector velocity of the form $\mathbf{V} = \gamma(v_x, v_y, v_z, c)$. Seeing the form of $X$, not only is the method intuitive, the result makes sense as well. This four vector velocity of a particle lays tangent to the world line of the particle. The four vector acceleration would be derived by looking at $\frac{d\mathbf{V}}{dt}$.

One of the most important quantities in physics is the momentum. Most any aspect of mechanics (ie force, energy) can be found to be derived from momentum and the conservative quality of it. The four vector momentum is derived by looking at the mass of the particle moving with a four vector velocity. Mass is by definition said to be invariant within any transformation. Earlier derivations and discussions
of the SToRy made reference to relativistic mass, but a clearer picture is understood by modern references by redefining certain quantities and understanding that mass is invariant. \( \mathbf{P} = m \mathbf{V} = \gamma (p_x, p_y, p_z, mc) \). Looking ahead and recalling how momentum is integrally related to energy (as especially seen in Hamiltonian mechanics), a very famous and familiar relation is beginning to appear.
CHAPTER 2
RELATIVISTIC CLASSICAL MECHANICS

With the basic four vector quantities defined, and a good grasp of the theory behind the Theory, we can start to look at how we can apply this knowledge to the basics of what has been studied in Classical Mechanics. It is important to be aware that there are different ways of deriving the same result, sometimes with one way revealing different useful information along the way.

2.1 Mass $\iff$ Energy

With a defined four momentum, let’s look at Newton’s second law and what a four force will look like; $\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$. Working through the relations we arrive at

$$\mathbf{F} = \gamma (F_x, F_y, F_z, \frac{\mathbf{F} \cdot \mathbf{v}}{c})$$

$$F_i = \frac{d(\gamma m v_i)}{dt}$$

$$\mathbf{F} \cdot \mathbf{v} = \frac{d(\gamma mc^2)}{dt}$$

(2.1)

We know that $\mathbf{F} \cdot \mathbf{v}$ is the rate at which the force does work on a particle, and we can set that equal to the time rate of change of the particles kinetic energy $\frac{dT}{dt}$. This yields the particles relativistic kinetic energy as $T = \gamma mc^2 - mc^2$. While this doesn’t seem to mean anything on first glance, let’s look at the first order expansion of $\gamma$ and the limit as $\beta \ll 1$ (non relativistic speeds). $T \approx (1 + \frac{1}{2} \beta^2)mc^2 - mc^2 = \frac{1}{2}m\mathbf{v}^2$ our very familiar classical Newtonian Kinematics expression of kinetic energy. Rearranging the formula and interpreting the quantity $\gamma mc^2$ as the total energy, we arrive at $E = T + mc^2$. The quantity $mc^2$ is known as the rest energy or rest mass, and is the root of Einstien’s famous revalation that mass and energy are equivalent. This is experimentally seen all the time in nuclear physics and engineering with respect to the binding energy of the nucleus. It is the very core concept behind nuclear power and nuclear weapons. The relation also shows mathematically that as the particles speed approaches the
speed of light, the kinetic energy approaches infinity, verifying the idea that particles with rest mass, cannot be accelerated past the speed of light.

The total energy may also be related back to the momentum in a way that results in the expression \( E^2 = (pc)^2 + (mc^2)^2 \). This is a useful relation since you can see for a particle with no rest mass such as a photon, the total energy is simply \( pc \).

### 2.2 Lagrangian & Hamiltonian Mechanics

Rederiving the Lagrangian formalism under relativistic conditions can be approached a few ways. One possibility is to start with Hamilton’s principle and treat the action as a Lorentz Scalar (it has to be the same in various inertial frames), solve the action integral of a free particle \((U=0)\) by integrating along the world line of the particle between two events (an interval, which we essentially solved for in Eqn 1.5 and Eqn 1.6) and arrive at the Lagrangian for a free particle.

\[
S = -mc \int_1^2 ds' ds'^2 = c^2 dt'^2 \\
\frac{dt}{dt' \gamma} \\
S = -mc^2 \int_1^2 dt \sqrt{1 - \beta^2} \\
S = \int_1^2 dt \mathcal{L}
\]  

Thus for a free particle our Lagrangian is given by Eqn 2.3 and can also be expanded in the same manner our relativistic kinetic energy was.

\[
\mathcal{L} = -mc^2 \sqrt{1 - \beta^2} \\
\mathcal{L} = -mc^2 + \frac{1}{2}mv^2...
\]  

Inspection of this makes the mass energy equivalence relation even clearer as a free particle at rest will have the only \(-mc^2\) the defined “rest mass” left in its Lagrangian which while it is no longer given exactly by our new relativistic \( T - U \), it still carries the particles energy information.

Another possibly less obtuse way is to start by assuming what we’ve learned in non-relativistic Lagrangian Mechanics still holds under the transformation (which is our hopeful goal). Since we have just derived the four momentum we are going to go
back to our relationship of $P = \frac{d\mathcal{L}}{dt}$. This yields a Lagrangian of the form:

$$\mathcal{L} = -mc^2 \sqrt{1 - \beta^2} - U(x_i) \quad (2.4)$$

where the integration constant has been set to the potential for assumed reasons. Either method can in turn follow to the derivation of the other. Looking at the Lagrange Equations yields further confirmation. We see several familiar relations and identical comparisons to our four force Eqn 2.1

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial U}{\partial x_i}$$

$$\frac{d}{dt} (\gamma mv_i) = -\frac{\partial U}{\partial x_i} = F_i = \frac{d}{dt} p_i \quad (2.5)$$

With our relativistic Lagrangian confirmed its a small matter to extend it towards Hamiltonian Mechanics, where the Hamiltonian is given by $H = \sum \dot{x}_i p_i - \mathcal{L}$. Lets start with a single particle moving through a conservative field.

$$H = \sum \dot{x}_i p_i - \mathcal{L}$$

$$H = \sum v_i (\gamma mv_i) - \mathcal{L}$$

$$H = \gamma mv^2 + mc^2 \sqrt{1 - \beta^2} + U$$

$$H = \gamma mc^2 + U$$

$$H = T + mc^2 + U = E \quad (2.6)$$

So even though our Lagrangian is not given exactly by T-U in a relativistic environment, the Hamiltonian is still the total energy for a conservative potential, with the definitions of the canonical momentum holding!

### 2.3 Final Thoughts

The Special Theory of Relativity has recently been coming to the forefront of the news and the laypersons thoughts recently due to the controversy surrounding the OPERA neutrino anomaly. Neutrinos were measured in a high precision, well funded experiment and were calculated to be travelling faster than the speed of light. Due to their quantum flavor oscillations, we know they have a finite mass, which would put them directly in violation of the STOry’s postulates and kinetic energy implications.
After researching and writing this report, it is hard to not gain an even deeper appreciation for the SToRy and its implications. It is both elegant and rigorous at the same time, and to truly learn and appreciate the techniques in Classical Mechanics throughout the year, it is even more incredible to see them all integrate so flushly and cleanly with the SToRy. While it still is and will remain detailed only as a theory, it is too impressive to discount. Just as Newtonian Mechanics is valid at so many levels and common applications, so is the Theory. When it isn’t, science turns to the General Theory of Relativity. Past that there is more science that will simply need to be discovered, derived, studied, understood, and applied. It doesn’t seem like it can invalidate the SToRy, only build around it, as the SToRy does to our Classical Mechanics techniques we have used for centuries.

Figure 2.1: Conclusion
REFERENCES CITED


