Double Pendulum.

\[ l_1 = \text{const} \]
\[ l_2 = \text{const} \]

Generalized (coord \( \theta_1, \theta_2 \))

\[ T = ? \]

\[ x_1 = l_1 \sin \theta_1 \]
\[ x_2 = l_1 \sin \theta_1 + l_2 \sin \theta_2 \]
\[ y_1 = -l_1 \cos \theta_1 \]
\[ y_2 = -l_1 \cos \theta_1 - l_2 \cos \theta_2 \]

\[ \dot{x}_2 = \frac{l_1}{l_2} \cos \theta_2 \dot{\theta}_1 + \frac{l_1 \cos \theta_1 \cos \theta_2 + l_2 \sin \theta_2}{l_2} \dot{\theta}_2 \]

\[ \dot{x}_2 + \dot{y}_2 = \frac{l_1}{l_2} \dot{\theta}_1 + \frac{l_2}{l_2} \dot{\theta}_2 + 2 \frac{l_1}{l_2} \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \]

\[ V = -mg l_1 \cos \theta_1 - m_2 g (l_2 \cos \theta_2 + l_1 \cos \theta_1) \]

\[ L = \frac{1}{2} m_1 l_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left( l_1 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \right) + m_1 g l_1 \cos \theta_1 + m_2 g (l_2 \cos \theta_2 + l_1 \cos \theta_1) \]
To make eigenvalue problem, assume all \( \eta_j \) have same frequency \( \omega_j \)
This will give normal modes - should be \( n \) solutions (assume for now non-degenerate)
\[
\eta_i = \Re \left[ z = z_0 e^{i \omega t} \right] \quad z_0 \text{ complex no.}
\]

\[
E - L \sum_j (\nu_{ij} - \omega^2 m_{ij}) z_j = 0 \quad i = 1, \ldots, n
\]

\[
V = [\nu_{ij}] \quad M = [m_{ij}(q_i)]
\]

Eigenvalue problem \[
[V - \omega^2 M] \bar{z} = 0
\]

Why does require \( z_j \) to be complex?
2nd order 2nd Law \( \rightarrow \) need \( \eta(0) \) & \( \eta'(0) \)
to specify mohrin

What about \( n \) solutions \( \omega^2 \)
Because \( V \) and \( M \) are real and symmetric
\( \omega^2 \) is real, also expect \( \omega^2 \geq 0 \) for stable solutions
For one particular normal frequency $\omega_s$

$$[\nu - \omega_s^2 M] \ddot{z} = 0$$

all real

could have $\ddot{z}$ all real, but since require complex
must be common phase factor set by initial
condition

$\ddot{z}_j = \ddot{z}_j^0 e^{i \omega t}$

$z_j^0 = \overline{z}_j^0 e$

at $t = 0$\n
$\Re z_j^{(s)}(0) = \Re [z_j^{(s)}]^0$

$\Re [z_j^{(s)}]^0 = \omega \text{clm} [Z_j^{(s)}]^0$

gives initial velocity and position

Since common phase related to initial conditions
define eigenvectors that are real for a given
$\omega_s$, but $\ddot{z}_j^{(s)}$ has multiplying $e^{i \omega t}$

And normalize eigenvectors such that

$$\dot{p}_j^{(s) \top} M \ddot{p}_j^{(s) \top} = \text{diag} [1]$$

This will require a normalizing real. constant $C^{(s)}$

$$Z_j^{(s)} = C^{(s)} e^{i \omega t} p_j^{(s)} \quad j = \text{component of vector}$$

$\ddot{p}_j^{(s)}$ corresponds to oscillation
of one generalized co-ordinate
the vector $\hat{x}(t) = (C^{(s)} e^{i\Phi_s}) \hat{\rho}(s)$ is the normal mode - it describes the motion of the system.

The actual generalized co-ordinates $\hat{\eta}(t)$

$$\hat{\eta}(t) = \sum_s \rho_s \hat{x}$$

Since motion at each mass is linear combination of normal modes,

$$\hat{\eta}(t) = \sum_s C^{(s)} \rho_s(t) \cos(\omega_s t + \Phi_s)$$

have $2n$ constants $C^{(s)}$, $\Phi_s$ and $n$ constants $\omega^2$.

How do we solve? What is normal co-ordinate?

$$\omega^2 \rightarrow \det \begin{vmatrix} V - \omega^2 M \end{vmatrix} = 0$$

$$\hat{\rho}^{(4)} M \hat{\rho}(0) = C^{(4)} \cos \Phi_t$$

$$\hat{\rho}^{(4)} M \hat{\rho}'(0) = -C^{(4)} \Omega_6 \sin \Phi_t$$
to get normal co-ordinates

define model matrix $A = \begin{bmatrix} \hat{\phi}^{(1)} & \hat{\phi}^{(2)} & \cdots & \hat{\phi}^{(n)} \end{bmatrix}$

$A^T M A = I$

$A^T V A = \omega^2 \rightarrow$ diagonal matrix

$\vec{3}(\hat{\eta}) = A^T M \vec{\eta}(t)$

Recall this simplifies Lagrangian.

$L = \frac{1}{2} \sum_j \left( \dot{\xi}_j^2 - \omega_j^2 \xi_j^2 \right)$
Return to double pendulum consider case

\[ l_1 = l_2 = L \]

eq motion \( \theta_1 = \theta_2 = 0 \)

\[ \cos(\theta_1 - \theta_2) = 1 \]

\[
\frac{\partial V}{\partial \theta_1} = m_1 l \sin \theta_1 + m_2 l \sin \theta_2
\]

\[
\frac{\partial^2 V}{\partial \theta_1^2} = m_1 g l \cos \theta_1 + m_2 g l \cos \theta_2,
\]

\[
= (m_1 + m_2) g l.
\]

\[
\frac{\partial^2 V}{\partial \theta_2^2} = m_2 g l
\]

\[
\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = 0
\]

\[
m_{ij} = \sum_k \left( \frac{\partial^2 V}{\partial q_i \partial q_k} \right) m_k \quad \text{sum over } x, y, z \quad \text{and } m_1, m_2
\]

By inspection of \( T \)

\[
m_{11} = (m_1 + m_2) l^2 \quad m_{22} = m_2 l^2
\]

\[
m_{21} = m_{12} = \frac{1}{2} (2m_2 l_1 l_2) = m_2 l^2
\]

\[
V = \begin{bmatrix}
(m_1 + m_2) g l & 0 \\
0 & m_2 g l
\end{bmatrix}
\]
\[ M = \begin{bmatrix} (m_1 + m_2) & m_2 \\ m_2 & m_2 \end{bmatrix} \]

Guess \( \Theta_i = \text{Re} \left[ C^{(s)} e^{i \omega t} \right] \)

\( C^{(s)} \) and \( \Phi \) related to mihel equations

\[ 0 = \begin{bmatrix} (m_1 + m_2) g \ell - \omega^2 (m_1 + m_2) \ell^2 - \omega^2 m_2 \ell^2 \\ -\omega^2 m_2 \ell^2 & m_2 g - \omega^2 m_2 \ell^2 \end{bmatrix} \]

\[ \left[ (m_1 + m_2) \frac{g}{\ell} - \omega^2 (m_1 + m_2) \right] \left[ m_2 \frac{g}{\ell} - \omega^2 m_2 \right] - \omega^4 m_2 \ell^2 = 0 \]

\[ (m_1 + m_2) m_2 \frac{g}{\ell^2} = \omega^2 \left[ (m_1 + m_2) \frac{g}{\ell} m_2 + (m_1 + m_2) m_2 \frac{g}{\ell} \right] + \omega^4 m_1 m_2 = 0 \]

\[ \omega^2 = \lambda \geq 0 \]

\[ \lambda^2 + \left( \frac{m_1 + m_2}{m_1} \right) \frac{g^2}{\ell^2} - 2 \lambda \left( \frac{m_1 + m_2}{m_1} \right) \frac{g}{\ell} = 0 \]

\[ \lambda = \frac{g}{\ell} \left( 1 \pm \sqrt{\frac{m_2}{m_1 + m_2}} \frac{m_1 + m_2}{m_2} \right) = \lambda_{1,2} \]