Phase Argument for $S$

$$\frac{\partial S}{\partial t} = \sum \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t}$$

$$\implies \dot{q}_p - H = L$$

$$S = \int_{t_0}^{t_f} L(q, \dot{q}) \, dt' + S(t_0)$$

$$H \left( \frac{\partial S}{\partial q_i} - \frac{\partial S}{\partial q_j}, 9, \ldots, q_n \right) + \frac{\partial S}{\partial t} = 0$$

or

$$H \left( \frac{\partial S}{\partial q_i} - \frac{\partial S}{\partial q_n}, q_i, \ldots, q_n \right) = \lambda = E$$

In QM $p_k = \frac{\hbar}{i \partial q_k} \propto \frac{\lambda}{\hbar}$

In classical mechanics $\lambda \to 0$ or $\lambda$ small, since $p_i$ finite.

$$\psi \sim e^{-i S/\hbar} \quad \frac{\partial \psi}{\partial t} \approx \frac{i \hbar}{\psi} \frac{\partial \psi}{\partial t}$$

$5$ Equation TO

$$H \psi = \frac{i \hbar}{\psi} \frac{\partial \psi}{\partial t} = \frac{\lambda}{\hbar} \frac{\partial \psi}{\partial t}$$

$$= - \frac{\hbar}{\psi} \frac{\partial \psi}{\partial t}$$

$$\left( \frac{\hat{H} + \frac{\partial S}{\partial t}}{\partial t} \right) \psi = 0$$

$\text{Operator}$
Hamilton-Jacobi 2D anisotropic oscillator.

\[ E = H = \frac{1}{2m} \left[ p_x^2 + p_y^2 \right] + \frac{1}{2} (k_1 x^2 + k_2 y^2) \]

\[ S(x, y, \alpha_1, \alpha_2, t) = W_1(x, \alpha_1, \alpha_2) + W_2(y, \alpha_1, \alpha_2) - \alpha^2 t \]

\[ \alpha_1, \alpha_2, \alpha^2 \text{ must be related or only} \]

2 canonical momenta.

Canonical

\[ p_i = \frac{\partial S}{\partial \dot{\alpha}_i}, \quad q_i = \frac{\partial S}{\partial \ddot{\alpha}_i}, \quad H = 0 \]

\[ \dot{\alpha}_i = \text{const} \quad \ddot{\alpha}_i = \beta_i \]

\[ \frac{1}{2m} \left( \frac{\partial W_1}{\partial x} \right)^2 + \frac{1}{2m} \left( \frac{\partial W_2}{\partial y} \right)^2 + \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 = \alpha^2 \]

Since equation separable select \( \alpha_1, \alpha_2 \)

\[ \alpha^2 = \alpha_1 + \alpha_2 \]

\[ \frac{1}{2m} \left( \frac{\partial W_1}{\partial x} \right)^2 + \frac{1}{2} k_1 x^2 = \alpha_1 \quad \frac{1}{2m} \left( \frac{\partial W_2}{\partial y} \right)^2 + \frac{1}{2} k_2 y^2 = \alpha_2 \]

could find \( W_i \), but interest in \( p_i, q_i \).

\[ q_i (\alpha_i, \alpha_2) = \pm \int dq_i \sqrt{(\alpha_i - \frac{1}{2} k_i q_i) 2m} \]

\[ \beta_i = \frac{\partial S}{\partial \dot{\alpha}_i} = \frac{\partial W_i}{\partial \dot{\alpha}_i} - t = \pm \int dq_i \sqrt{\frac{2m}{\alpha_i - \frac{1}{2} k_i q_i}} \]

\[ q_i (t) = \left( \frac{2 \dot{\alpha}_i}{k_i} \right)^{1/2} \cos \omega_i (t + \beta_i) \quad \omega_i = \sqrt{\frac{k_i}{m}}. \]
\[ \alpha_i = \text{energy component} \]

In general, no obvious choice for \( P_i \)'s, but energy common.

Reminder that Hamilton-Jacobi is only if \( \Phi = 0 \Rightarrow H = \partial \Phi / \partial t \) guarantees that all \( P_i \) and \( Q_i \) are constants.

If \( \frac{\partial H}{\partial t} = \frac{\partial H}{\partial x_i} = 0 \Rightarrow H = \text{const.} \]

Can instead solve \( H = H = \alpha_i = \Phi(P_i) \)

\( \Phi(\alpha_i; P_i) \) only requires \( W(q_i; P_i) \)

In this approach \( Q_i = \frac{\partial \Phi}{\partial P_i} = \gamma_i \) not zero

\[ P_i = \alpha_i = \text{const.} \]

Since \( Q_i = \frac{\partial W}{\partial P_i} \quad P_i = \frac{\partial W}{\partial q_i} \]

\[ = \alpha_i \beta_i + \gamma_i \]

Can solve this as well
Action Angle Variables.

For periodic motion may have more natural constants (e.g., SHO 2D $E_x, E_y$ for toy).

2 types of periodic motion.

1) Closed - in phase space there is path called libration.

2) Not closed - have period and motion repeats rotation.

Eq: Simple Pendulum.

$$H = E = \frac{p_o^2}{2ml^2} - mgl \cos \theta$$

$$\Rightarrow p_o = \pm \sqrt{2ml^2(E + mgl \cos \theta)}$$

$\omega_0$ at bottom

$\omega_0^2 < 2g$ libration

$\omega_0^2 > 2g$ rotation

In some special case $H - T$ can be made more intuitive - action angle is when:

1) Conservative $H = \text{cont}$

2) Periodic

3) Separable.
\[ S = W_1(q_1, \alpha) + W_2(q_2, \alpha) + \ldots - \alpha_i t. \]

**Applications**
- determine freq. of oscillation without solving completely
- some perturbation problems
- QM transition

define \[ J_i = \int p_i dq_i \] over 1 period in phase space.

**Action Variables**
Since it depends only on constants \[ J_i(\alpha_1, \ldots, \alpha_n) \]
is not variable, depends only on initial conditions

\[ J_i = \int p_i dq_i = \frac{\partial W_i}{\partial q_i} = J(\alpha_1, \ldots, \alpha_n) \]
\( i \) no \( q_i \) integrated.

Now assume n \( J_i \)'s are invertible \( \Rightarrow \alpha_i(J_i) \)

\[ W(q_i, \alpha_i) \Rightarrow \overline{W}(q_i, J_i) \]
\[ S(q_i, \alpha_i, t) \Rightarrow \overline{S}(q_i, J_i, t) = \overline{W} - \alpha_i(J_i) t \]

\( \overline{p_i} \)
\[ W = \sum_{i} W_i(q_i, J_i, \ldots J_n) \]

for the new generating function \( \Sigma(q, J_i, t) \).

\[ p_i = \frac{\partial \Sigma}{\partial q_i} \quad \Omega_i = \frac{\partial \Sigma}{\partial J_i} = \beta_i = \text{const.} \]

new set of canonical co-ordinates \( \beta \): momenta.

\[ \beta_i = \frac{\partial W}{\partial J_i} = \frac{\partial W}{\partial J_i} + \frac{2\partial}{\partial J_i} \frac{\partial (J_i)}{\partial J_i} + \frac{\partial}{\partial J_i} \]

\( \beta_i \)

Angle Variable - does evolve with time.

recall that \( \chi_i = H \ln H - J_i \Rightarrow \frac{\partial H}{\partial J_i} \]

\[ \chi_i = \nu_i t + \beta_i \quad \text{linear increase} \]

\( t \) cover 1 period. What is \( \Delta \omega_i \):

\[ \Delta \omega_i = \int \partial \nu_i = \int \frac{\partial W}{\partial q_i} \partial q_i = \int \frac{2}{\partial J_i} \frac{\partial W}{\partial J_i} \partial J_i \]

\[ = \frac{2}{\partial J_i} \int \gamma_i \partial J_i = \frac{2 \gamma_i}{\partial J_i} = \frac{\partial \Sigma}{\partial J_i} \]

\( \omega_i \) over 1 period \( T \)\]

\[ \Delta \omega_i = \nu_i T \]

\[ \Rightarrow T = \frac{1}{\nu_i} \quad \nu_i \text{ is frequency} \]

We can find the period by calculating \( \frac{\partial H}{\partial J_i} \).
Example: Simple Harmonic Oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2} mc^2 x^2 = E \]

1. If so separable \( \frac{dt}{dt} = 0 \) \( E = \text{const.} \)

\[ \frac{1}{2m} \left[ \left( \frac{\partial W}{\partial q} \right)^2 + m^2 c^2 q^2 \right] = \alpha_1 \]

\[ W = \sqrt{2m\alpha_1} \int dq \sqrt{1 - \frac{mc^2 q^2}{2\alpha_1}} \]

\[ p = \frac{\partial W}{\partial q} = \pm \sqrt{2m\alpha_1 - mc^2 q^2} \]

We could find \( W(q, \alpha_1) \) but not needed.

\[ W = \frac{\alpha_1}{\omega} \left[ \sqrt{\frac{m}{2\alpha_1}} \sin^{-1} \left( \frac{mc^2 q^2}{2\alpha_1} \right) + \sin^{-1} \left( \sqrt{\frac{m}{2\alpha_1}} \right) \right] \]

\( \omega \) disappears due to integral.

More interested in \( J \)

\[ J = \int p \, dq = \int \sqrt{2m\alpha_1 - mc^2 q^2} \, dq \]

\( q = \sqrt{\frac{2\alpha_1}{mc^2}} \sin \theta \) substitution.

Integrate in 2 parts.

\[ J = \frac{2\alpha_1}{mc^2} \int_{0}^{\pi} \cos \theta \, d\theta = \frac{2\pi \alpha_1}{mc^2} \]

\[ J = \frac{2\pi E}{\omega} \]
\[ \nu = \frac{ IH }{ J } = \frac{2}{ \pi } \left( \frac{Iw}{2\pi} \right) = \frac{\omega}{2\pi} = f \]

Suppose have 3-D H0 ansolope.

\[ W(q_1 q_2 q_3 q_4 q_5 q_6) = \sum_{n=0}^{2} W(q_e T_1 T_2 T_3) \]
Kepler Problem. 

\[ T = \frac{m}{r} \left( r\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \]

\[ p_r = mr \dot{r} \quad p_\theta = mr^2 \dot{\theta} \quad p_\phi = mr^2 \sin^2 \theta \dot{\phi} \]

\[ \Rightarrow H = \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) - \frac{k}{r} = \text{const} = E \]

\( \phi \) is cyclic \( \Rightarrow p_\phi = \text{const} = \alpha \phi \)

1 particle 3 DPs \( \Rightarrow 6 \) constants of motion.

Since \( p_\phi = \alpha \phi \), \( \alpha \phi \phi = \alpha \phi \phi \) term.

\[ S = W_r (r, \alpha_1, \alpha_2, \alpha_3) + W_\theta (\theta, \alpha_1, \alpha_2, \alpha_3) \]

\[ + \alpha_3 \phi - \alpha_1 \phi \]

Notice we have taken the 3 PIs and assigned \( \alpha_1 = E \), \( \alpha_3 \) related to \( \phi \).

As expected because of cyclic variables \( \phi, r \).

\[ E = \alpha_1 = \frac{1}{2} \left[ \left( \frac{\partial W_r}{\partial \dot{r}} \right)^2 + \frac{\alpha_3^2}{\sin^2 \theta} \right] - \frac{k}{r} \]

Separation of variables

\( \frac{(\partial W_r)^2}{\partial \dot{r}^2} + \frac{\alpha_3^2}{\sin^2 \theta} = \alpha_2^2 \) => assign 3 \( r^0 \) PIs

\( \frac{(\partial W_\theta)^2}{\partial \dot{\theta}^2} + \frac{\alpha_2^2}{\sin^2 \theta} = 2m (\alpha_1 + \frac{k}{r}) \)
from $H-J$ could try to solve $U_r, W_0$

$\Rightarrow \alpha_i = \beta_i$ other three constants.

But know the system is periodic so use action-angle approach.

$$\frac{d\phi}{dp} = p$$

$$J_\phi = \int p \, q \, dq = \alpha_3 \int dq = 2\pi \alpha_3$$

$$J_\theta = \int p_\theta \, d\theta = \int \sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\sin^2 \theta}} \, d\theta$$

This is messy integration—require $p_\theta$ to be real, which limits allowed values for $\alpha_2, \alpha_3$—see Goldstein

result

$$J_\theta = 2\pi (\alpha_2 - \alpha_3)$$

$$J_r = \int p_r \, dr = \int \left[ 2m \alpha_1^2 + \frac{2m k}{r} \left( \frac{J_\theta - J_\phi}{r^2} \right)^2 \right] \, dr$$

$$p_r$$

limits of integration

set by $r_{min}, r_{max}$

double valued for closed loop.

$$J_r = - (J_\theta + J_\phi) + \pi k \sqrt{\frac{2m}{-E}}$$

solve for $E$
\[ E = \frac{-2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2} = H(J_r, J_\theta, J_\phi) \]

To find the period,

\[ \frac{1}{\tau} = \dot{\Psi} = \frac{\partial H}{\partial \Psi} = \frac{-2\pi^2(-2) mk^2}{(J_r + J_\theta + J_\phi)^3} \]

\[ \Rightarrow \tau = \pi k \sqrt{\frac{m}{-2E^2}} = 2\pi \sqrt{\frac{m}{\hbar}} \left( \frac{k}{-2E} \right)^{\frac{3}{2}} \]

\[ a = \frac{2\pi k}{21E} \Rightarrow \tau = 2\pi \sqrt{\frac{m}{\hbar}} \frac{a^{\frac{3}{2}}}{k} \]

Notice this is true for all \( J_1 \sim 3 \) resulting in closed orbit.

To relate to Quantum Mechanics define

\[ J_n = J_\phi + J_r + J_\theta \]

\[ J_\theta = J_\phi + J_\theta \]

\[ J_m = J_\phi \]

If \( J_n = nh \), \( J_\theta = \ell h \), \( J_m = mh \)

\( n, \ell, m \) integer. Get Sommerfeld-Wilson conditions - QM from quantization of \( J_i \)