Reason for Strings

1) Derivation wave equation
2) Lagrangian density and Lagrange equation(s)
3) Boundary value example
4) \Rightarrow Field theory

Review Transverse Wave String

\[ M \dddot{u}_0 = -T \left( \sin \theta - \sin \Phi \right) \]
\[ = -\frac{T}{a} \left( \mu_{in} - \mu_i \right) - \left( \mu_i - \mu_{in} \right) \]
\[ \subset \text{spring-like} \]

Wave equation apparent \( \lim_{a \to 0} \frac{M}{a} \to \rho \)

Solution by assuming \( \Phi_s \equiv \rho_s \cos (ct + \Phi_s) \)
\( \rho_s \) eigenvector \( \lambda_s \) eigenvalue
(Chapter 4)
\( \Phi_s, \rho_s \) from initial conditions
For N masses find N $\omega_n$ from determinant

$$\omega_n^2 = \frac{4\pi}{ma} \sin^2 \left( \frac{n\pi}{2(N+1)} \right), \quad n=1 \ldots N$$

determinant also gives $p_{n-1} + p_{n+1} = \lambda p_n$ where

$$\lambda = 2\cos \left( \frac{n\pi}{N+1} \right)$$

Could find eigenvalues but easier to assume travelling wave solution

$$u(x,t) = A \exp \left[i(kx - \omega t)\right]$$

$$m \ddot{u}_n + \frac{2\pi}{a} u_n - \frac{\pi}{a} (u_{n+1} + u_{n-1}) = 0$$

$$\Rightarrow -m \omega_n^2 + \frac{\pi}{a} (2 - \frac{e^{k\pi} + e^{-k\pi}}{2 \cosh k}) = 0$$

$$\omega_n^2 = \frac{4\pi}{ma} \sin^2 \left( \frac{k\pi}{2} \right), \quad k_n = \frac{n\pi}{(N+1)a} = \frac{2\pi}{a} \frac{n}{N+1}$$

Notes - only small displacements ID
- No assumption at boundary conditions
- $\omega, k$ are assumed constant.
- $k$ and $-k$ degenerate in $\omega$ same wave travelling left & right?
Imposing fixed endpoints at $x=0$, $x=L=(N+1)a$

$$U(x,t) = \sum A_n \sin \left( \frac{n\pi x}{(N+1)a} \right) e^{-\lambda n^2 t}$$

$x_i = (N+1)a$, $x_j = ja$

An complex $= 2$ initial conditions.

Look at $N \to \infty$ continuum limit.

$(1+N)a = L = \text{const} \Rightarrow a \to 0$

$m/a = \delta = \text{const} \Rightarrow m \to 0$

$$\omega_n = \frac{4\pi^2 \sin^2 \left( \frac{n\pi}{2(N+1)} \right)}{ma} \sim \frac{4\pi^2}{me} \left( \frac{n\pi}{2(N+1)} \right)^2$$

$$\omega_n = \frac{L}{\delta} \left[ \frac{1}{(N+1)a} \right]^2 \frac{2^2}{n\pi^2} = \frac{L}{\delta} \left( \frac{n\pi}{2} \right)^2.$$

On continuous variable. $U = \sqrt{\frac{L}{\delta}}$ wave speed.

In general $\frac{L}{\delta}$ not necessarily constant.
Field theory insight simplified (and more general) if start with Lagrangian density.

\[ L = \frac{1}{2} m \sum_{i} \dot{u}_i^2 - \frac{\hbar}{2} \sum_{i} (\mu_i - \mu) \text{ discretized.} \]

\[ \sum_{i} a_i = \sum_{i} dx_i \quad \lim_{\Delta \to 0} \sum_{i} \frac{a_i}{\Delta} \to \int_0^L dx. \]

\[ L \to \int_{0}^{L} \left( \frac{1}{2} \left( \sigma(x) \left( \frac{\partial u}{\partial t} \right)^2 - \tau(x) \left( \frac{\partial u}{\partial x} \right)^2 \right) \right) dx \]

\[ = L \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, u \right) \approx u_0 + \int_0^L \]

\( \sigma \) and \( \tau \) special dependence.

Hamilton's principle: fixed endpoints.

\[ \delta \int_{t_1}^{t_2} \int_0^L dx \left( \frac{\partial L}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial L}{\partial x} \frac{\partial u}{\partial x} \right) = 0 \]

BC. \( u(x, t_1) = u(x, t_2) = 0 \quad \forall x \).

What about special endpoints? \( 0, L \)

Recall why needed time BC.

Assume \( u(x, t) = u_0(x, t) + \epsilon \eta(x, t) \)

\[ \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} + \epsilon \frac{\partial \eta}{\partial t} \]

\[ \delta u = \eta \]
Now look at various path - path expand how \( L \) changes

\[
L = L \text{true} + \frac{\partial L}{\partial \eta} \eta' + \frac{\partial L}{\partial \eta'} \eta'' + \frac{\partial L}{\partial \xi} \xi' + \frac{\partial L}{\partial \xi'} \xi''
\]

\[
\eta' = \frac{\partial \eta}{\partial x} \quad \eta = \frac{\partial \eta}{\partial t} \quad \xi, x, t
\]

\[
\eta' \frac{\partial L}{\partial \eta'} = \frac{d}{dx} \left( \frac{\partial L}{\partial \eta} \right) - \eta \frac{d}{dt} \left( \frac{\partial L}{\partial \eta'} \right)
\]

total.

\[
\eta \frac{\partial L}{\partial \eta} = \frac{d}{dt} \left( \frac{\partial L}{\partial \eta} \right) - \eta \frac{d}{dt} \left( \frac{\partial L}{\partial \eta'} \right)
\]

In our specific case

\[
\frac{\partial \eta}{\partial \eta'} = \varepsilon(u, x) \frac{du}{dx}
\]

\[
\frac{\partial L}{\partial \eta'} = \sigma(x) \eta
\]

more general BC spatially

\[
\frac{\partial u}{\partial x} \bigg|_{x=0} = 0 \quad \text{or} \quad u(0) = u(L) = 0
\]

fixed endpoint

E-L equation

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \eta'} \right) + \frac{\partial}{\partial \xi} \left( \frac{\partial L}{\partial \xi'} \right) - \frac{\partial L}{\partial \eta} = 0
\]

\[
\Rightarrow \sigma(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ \tau(x) \frac{\partial u}{\partial x} \right]
\]
Special case: $\delta$, $\gamma$ constants

\[ C^2 = \frac{\delta}{\gamma} \quad \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{Wave Equation} \]

As before, guess normal modes.

\[ u(x, t) = C \phi(x) \cos(\omega t + \phi) \]

- $C$ complex, real, $\omega$, $\phi$ initial cond.
- $\phi$ continuous, $\phi(x)$ eigenfunction
- multiplicative sep of variables
  since linear

\[ \phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} = \sqrt{\frac{2}{L}} \sin k_n x \]

for fixed endpoints, as well small oscillations imposed normalization condition

BC @ L: \[ \int_0^L \! \phi_n(x) \phi_m(x) \, dx = \delta_{nm} \]

ORTTHONORMALITY

To see relationship to initial conditions

\[ u(x, 0) = f(x) \quad U(x, 0) = g(x) \]

Separate into SINE 3 COSINE SERIES
\[ u(x, t) = \sum_{n=1}^{\infty} \left( \frac{2}{\pi n} \right)^{1/2} \sin \frac{n \pi x}{L} \times \left[ a_n \cos \frac{n \pi c}{L} t + b_n \sin \frac{n \pi c}{L} t \right] \]

**Fourier coefficients**

\[ a_n = \sqrt{\frac{2}{L}} \int_{0}^{L} f(x) \cos \frac{n \pi x}{L} \, dx \]

\[ b_n = \frac{L}{n \pi c} \sqrt{\frac{2}{L}} \int_{0}^{L} g(x) \sin \frac{n \pi x}{L} \, dx \]

This relates to initial guess

\[ a_n = \zeta_n \cos \varphi_n \quad b_n = -\zeta_n \sin \varphi_n \]

If \( f(x) \), \( g(x) \) continuous can find

fourier coefficients for all \( \Rightarrow \) complete set.

looking at \( H = E \)

\[ = \frac{1}{2} \sum_{n=1}^{\infty} c_n \frac{c_n^2}{c_n} \]

\[ = \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) c_n^2 \]

\( 2 \) each mode contributes to \( E \).
An alternative to standing wave solution
(Benoulli) is travelling wave. (Joseph)

define \( r = x - ct \) \( s = x + ct \).

\[ u(x,t) = u(r,s) \text{ change of variables.} \]

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}. \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s^2} + \frac{2}{c^2} \frac{\partial^2 u}{\partial r \partial s}. \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} \right]. \]

\[ \frac{\partial^2 u}{\partial s \partial r} = 0. \]

What separates out variables?

\[ u(r,s) = \psi(r) + \phi(s). \]

\[ u(x,t) = \psi(x-ct) + \phi(x+ct). \]

It again has same boundary conditions

\[ u_0 = f(x) \text{ i.e. } u_0 = g(x). \text{ can show} \]

\[ u(x,t) = \frac{1}{2} \left[ f(x-ct) + f(x+ct) \right] + \frac{1}{2c} \int_{x+ct}^{x+ct} g(y) \, dy. \]
can show that d'Alambert's Bernoulli are equivalent.

If $f(x)$ continuous and $f'(x)$ continuous and differentiable can expand it sensibly.
- Further sense complete

General String Equation

\[ L = \frac{1}{2} \frac{\partial}{\partial t} (\frac{\partial u}{\partial t})^2 - \frac{1}{2} \frac{\partial}{\partial x} (\frac{\partial u}{\partial x})^2 - \frac{1}{2} v(x) u^2 \]

referring for wave $V(x)$.

\[ E - L \text{ equation.} \]

\[ + \frac{d}{dx} \left[ R(x) \frac{\partial u}{\partial x} \right] + v(x) u - \Delta u \frac{\partial^2 u}{\partial x^2} = 0 \]

If by normal mode solution $U = x(x) \cos(\omega t + \varphi)$

\[ - \frac{d}{dx} \left[ R(x) \frac{\partial x}{\partial x} \right] + v(x) p(x) = \omega^2 \cos(x) p(x) \]

Helmholtz-Liouville Equation...