Variational Principle

\[ \psi (x) = \frac{1}{\sqrt{\int_{-\infty}^{\infty} \psi^2 (x) dx}} \]

\[ \frac{\delta^2 \psi [\phi]}{\delta \phi} = \frac{1}{2} \int \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 + V(x) \phi^2 \right) dx - \frac{1}{2} \int \rho^2 \phi^2 \ dx \]

\[ \int \] can result in Schrödinger eq. 5:

\[ 8\psi^2 = 0 \quad \text{subject to boundary conditions} \]

Uses - to find eigenfunctions

1. Guess trial function \( \phi(x, \lambda) \) with parameter \( \lambda \) to minimize
2. Minimize \( \psi^2 [\phi] \) in functional \( \Rightarrow 8\psi^2 = 0 \)
3. Once get lowest state find other orthogonal states, etc.

Test

\[ 8\psi^2 = \frac{8I_1}{I_1^2} + (-1) \frac{I_1 \delta I_2}{I_2^2} \]

\[ = \frac{I_1}{I_2} \left( 8I_1 - \frac{I_1}{I_2} 8I_2 \right) = 0 \]

allow \( p(x) \) to vary \( \psi = \frac{\phi}{p(x)} \)

\[ 8I_1 = \frac{1}{2} \int [2 \phi (x) \frac{dp}{dx} + V(x) \phi^2 + \frac{\delta}{\delta x} \frac{\delta}{\delta p} \phi^2] dx \]

\[ 8I_2 = \frac{1}{2} \int [\phi \frac{2 \phi^2}{dx} + V(x) \phi^2 + \frac{\delta}{\delta x} \frac{\delta}{\delta p} \phi^2] dx \]
\[ \int \frac{d}{dx} \left( \frac{1}{2} \frac{d}{dx} \right) \phi \, dx = \int \frac{d}{dx} \left[ \frac{1}{2} \phi^2 \right] \, dx - \int \frac{d}{dx} \left[ \frac{1}{2} \phi^2 \right] \, ds = 0 \text{ for all boundary conditions.} \]

\[ \Rightarrow \int \frac{d}{dx} \left[ \frac{1}{2} \phi^2 \right] + V(x) \phi = 0 \]

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In case where \( V = 0 \), \( \phi = \text{const} \)

\[ \text{lowest } \phi^2 \Rightarrow \text{lowest } \frac{d^2 \phi}{dx^2} / \phi \]

minimize curvature is in this simple case.

In general, minimize \( \phi^2 \).
Rayleigh Ritz

Given Sturm–Liouville ODE can find a complete and orthogonal set of eigenfunctions and an infinite set of eigenvalues $c^2$

$$\rho(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n(x) \cos(\omega_n t + \delta_n)$$

with $\phi_n$ initial conditions.

Look at functional $c^2 [\rho]$ not eigenvalues

$$c^2 [\rho] = \frac{1}{2} \int_a^b \left[ \frac{1}{2} (\sum_{n=1}^{\infty} A_n \phi_n'(x)^2 \right. + \left. u [\sum_{n=1}^{\infty} c_n \phi_n](x)^2 \right] dx$$

Integrate by parts

$$= c^2 [\rho] = \sum_{n=1}^{\infty} \frac{\omega_n a_n^2}{\omega_n a_n^2}$$

Functional is weighted sum - depends on initial conditions
If \( \omega_1 \) is lowest eigenvalue, can see:

\[
\mathcal{O}^2[\mathcal{E}] = \sum_{n=2}^\infty \frac{a_n^2}{\omega_n^2} + \omega_1^2 = \frac{\sum a_n^2}{\sum \frac{1}{\omega_n^2}} - \frac{\sum a_n^2}{\omega_1^2}.
\]

\[
= \sum_{n=2}^\infty a_n^2 \left( \omega_n^2 - \omega_1^2 \right) + \frac{\omega_1^2}{\sum a_n^2}.
\]

Since \( \omega_1 > \omega_n \) all terms are greater than zero \( \implies \mathcal{O}^2[\mathcal{E}] > \omega_1^2 \) always.

If \( p(x) \) is close to \( p_1(x) \) eigenvalue has close \( \implies \omega^2 \approx \omega_1^2 \)?

\[
p(x) = p_1(x) + \sum_{n=2}^{\infty} \varepsilon_n p_n(x)
\]

\[
\mathcal{O}^2[\mathcal{E}] = \omega_1^2 + \sum_{n=2}^{\infty} \varepsilon_n \left( \omega_n^2 - \omega_1^2 \right)
\]

\[
\implies \sum_{n=2}^{\infty} a_n^2 \left( \omega_n^2 - \omega_1^2 \right) = \sum_{n=2}^{\infty} \varepsilon_n (\omega_n^2 - \omega_1^2).
\]

1st order corrections in eigenfunction result in 2nd order in \( \varepsilon^2 \) - similar to QM perturbation theory.
Application: estimation of $\omega_n^2$

Notice that this is not as useful for higher terms since $(\omega_m^2 - \omega_n^2)$ for $n \neq 1$ can be negative.

Lagrange Ritz — guess trial function
Minimal relative to some parameter.

Example 1: mass $M$ on string

$$\frac{\omega}{2c} \tan \left( \frac{\omega L}{2c} \right) = \frac{GL}{M}$$

exact odd solution

"even" modes since symmetric about center.

$$V(x) = 0 \quad \tau(x) = 2 \quad \sigma(x) = \sigma(1 + \frac{M}{c} \delta(x - \frac{L}{2}))$$

$$\omega^2 [\rho] = \frac{L^2}{\alpha^2} \left[ \int_0^L \rho(x) x^2 dx \right]$$

$$= \frac{\int_0^L \rho(x) dx}{\sigma} \left( \frac{M \rho(L/2)^2}{\sigma} \right)$$

change will make $\omega^2$ smaller.

What guess could be made?

Text $\rho(x) = \begin{cases} A \alpha^2 & \alpha \text{ parameter} \\ A(L-x) & \end{cases}$

$$\Rightarrow \omega^2 = \frac{4L^2}{\sigma^2 L^2} \left[ \frac{\alpha^2 (2\alpha + 1)}{(2\alpha - 1)[1 + \frac{M}{GL} (2\alpha + 1)]} \right]$$

$$f(\alpha, \frac{M}{GL})$$
Solve for \( \alpha \) by \( \frac{df}{d\alpha} = 0 \)

\[
4\alpha^2 - 2\alpha - 1 + \frac{M}{UL} (2\alpha + 1)^2 (\alpha - 1) = 0
\]

when \( M = 0 \)

\( \alpha_{nn} = 0.9279 < 1 \)

products \( \omega_n^2 = \frac{2c}{L} (0.8632) \)

exact answer \( \omega_n = \frac{2c}{L} 0.8603 \checkmark \)

When \( M \gg 0 \) \( \alpha_{nn} = 1 \) as expected

In general guess

1. must satisfy boundary conditions.
2. “looks” like solution.
3. be easy to calculate
4. have variable parameter.

Last not necessary

Example 2 free string fixed endpoints

\( \rho = x \quad (L-x) \quad \text{no } \lambda^s \)

\[
\omega^2 = \frac{\int_0^L (L^2 - 4Lx + 4x^2) \, dx}{\int_0^L x^2 (L^2 + x^2 - 2x) \, dx}
\]

\[
= \frac{10}{L^2} \quad \text{exact } \frac{\Pi^2}{L^2} \quad 0.66 \text{ % error}
\]
better \( p_T = 4x \left[ x (L - x) \right]^x \).

can also get \( \omega_2 \)

\[ p_{2T} = x(x-L)(x-L/2) \]

could have found from

\[ p_{2T} \rightarrow \int p_T p_{2T} \delta dx - p_{2T} \]

guess but not necessarily orthogonal.

\[ \frac{\omega_2}{\omega} = \frac{\int_0^L (3x^2 - 3Lx + L^2) dx}{\int_0^L (x^3 - 3Lx^2 + L^2)^2 dx} = \frac{42}{L^2}. \]

\[ \omega_2 = \frac{6.4867}{L}, \text{ correct to } 3 \text{ significant figures.} \]
What solution for $u = \sigma^2 (L - x)$?

Recall for eigenfunctions $u(x)=0$
\[ u'(x=L) = 0 \]
\[ \rho(L) = 0 \]
\[ (x-L)^2 \quad \text{no parameter} \]
\[ (x-L)^2 \quad \text{1 parameter} \]

The above sections suggest possible perturbation treatments.

Short hand $L_0 \equiv -\frac{d}{dx}[\alpha(x)\frac{d}{dx}] + \nu(x)$.

Suppose solutions known for $L_0 \Rightarrow \rho_n, \omega_n$.

$L \Rightarrow L_0 + L_1 \quad L_1 = \text{perturbation term from } L_0$

Full development requires Green’s function approach – also text assumes $L_1 = \omega \chi(x)$

Can show solution to $L$ satisfies
\[ \rho(x) = \rho_n(x) + \sum_{q \neq n} \rho_q(x) \frac{1}{\omega^2 - \omega^2} \int \rho_q L_1 \rho_n \, dy \]

\[ -\rho_n(x) + \sum_{q \neq n} \rho_q(x) \int_{\omega^2 - \omega^2} \rho_q L_1 \rho_n \, dy \]

\[ L_1 = \omega \chi(x) \]

$\langle q | L_1 | n \rangle$
\[ p(x) = p_0(x) + 2 \sum_{\frac{q}{q+1}} \frac{\left< q_1 x \right>} {\omega^2 - \omega} \frac{A_0(x)} {2n} \]

\[ \omega^2 = \omega^2_n + 3 \left< n_1 v_1 n_1 \right> + 3^2 \sum \frac{\left< n_1 v_1 n_1 \right>}{2n} (\omega^2_n - \omega^2) \]

If can exactly solve one problem can solve a "clear" problem.

Mass on string example

\[ L_1 = M \delta(x - \frac{L}{2}) \] M "small" parameter.

C not quasie same since from \( \omega^2 \) term not \( \nu \) term.

\[ \Rightarrow \exists \nu_1(x) = \frac{\omega^2 M}{\delta} \delta(x - \frac{L}{2}) \]

\[ \omega^2 - \omega^2_n = -\omega^2 \frac{M}{\delta} \int p_n \delta(x - \frac{L}{2}) p_n \delta dx. \]

\[ p_n = \sqrt{\frac{2}{\lambda L}} \sin \left( \frac{n \pi x}{L} \right) \]

\[ \omega_n = \frac{n \pi c}{L}. \]

\[ \frac{\delta \omega_n}{\omega_n} = \begin{cases} -\frac{M}{\delta L} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases} \]