Applying

Sound waves = perturbation of uniform stationary fluid = assumptions: no external force, such as gravity, fluid compressible ($\nabla \cdot \mathbf{v} \neq 0$).

\[ \rho = \rho_0 + \rho' \quad \mathbf{v} = \mathbf{v}' \quad P = P_0 + P' \]

1) \( \frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho \mathbf{v}') = 0 \) continuity equation

\[ \Rightarrow \frac{\partial \rho'}{\partial t} + \nabla \cdot \mathbf{v}' = 0 \]

2) \( \frac{\partial \mathbf{v}'}{\partial t} + \mathbf{v} \cdot \nabla (\rho \mathbf{v}') = \frac{1}{\rho_0} \nabla P' + \frac{\eta}{\rho_0} \nabla \cdot \mathbf{v}' + \left( \frac{e + \eta}{2} \right) \nabla (\nabla \cdot \mathbf{v}') \]

\[ \Rightarrow \frac{\partial \mathbf{v}'}{\partial t} = -\frac{1}{\rho_0} \nabla P' \]

3) Assume no heat loss (isentropic)

\[ P_0 + P' = P \left( s, \rho_0 + \rho' \right) \]

\[ = P \left( s, \rho_0 \right) + \rho' \frac{\partial P}{\partial \rho'} |_{s} \]

\[ \Rightarrow P' = c^2 \rho' \] where \( c^2 = \frac{\partial P}{\partial \rho'} |_{s} \)
At this point have 3 equations with three unknowns \( p', \mathbf{v}', \mathbf{v}' \).

Look at linearized Euler result and take curl:
\[
\nabla \times \frac{\partial \mathbf{v}'}{\partial t} = \frac{2}{\rho} \mathbf{\nabla} \times (\mathbf{v} \times \mathbf{v}') = -\frac{1}{\rho} \mathbf{\nabla} \times \mathbf{v} \mathbf{v}' = 0
\]

So \( \nabla \times \mathbf{v}' \) time independent.

If disturbing forces conservative \( \mathbf{\nabla} \times \mathbf{v} = 0 \),
so \( \mathbf{\nabla} \times \mathbf{v}' = 0 \) (Thomson's theorem).

We can now use the scalar velocity potential \( \phi \equiv \mathbf{v}' = -\mathbf{\nabla} \phi \)

2) \[-\frac{\partial}{\partial t} \mathbf{\nabla} \phi = -\frac{1}{\rho_0} \mathbf{\nabla} \rho' = -\frac{1}{\rho_0} \mathbf{\nabla} (c^2 \rho') \]

1) \[\frac{\partial \rho'}{\partial t} + \nabla^2 \phi = 0\]

2) \[\frac{\partial \rho'}{\partial t} = c^2 \frac{\partial \phi}{\partial t}\]

Setting gradient equal in 2) \(- c^2 \phi = \rho_0 \frac{\partial \phi}{\partial t} \).

Now from 1) \[\nabla^2 \phi = \frac{1}{\rho_0} \frac{\partial}{\partial t} \left( \frac{\rho_0}{c^2} \frac{\partial \phi}{\partial t} \right)\]

or \[\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}\]

Remindere: isentropic, compressible, non-viscous, no external force.
Solution of wave equation similar to before. No boundary conditions.
\[ \hat{n} \cdot \vec{v} = 0 \text{ at well fixed no velocity at well}\]
or
\[ \frac{\partial \Phi}{\partial t} = 0 \text{ free surface, } P' = 0 \]
\[ \Rightarrow \rho' = 0 \Rightarrow \frac{\partial \Phi}{\partial t} = 0 \]

In above have used continuity equation (cons. of mass), Enthalpy equation (2nd Law or cons. of momentum) \( \Rightarrow \) cons. of energy (isentropic case).

Changes for viscous fluid.
\[ \rho = \rho_0 + \rho' \quad \vec{v} = \vec{v}' \quad P = P_0 + P' \]
\[ \rho' = \rho_0 + \rho' \quad T = T_0 + T' \]

have effects due to thermal conductivity and damping forces.

1) \[ \frac{\partial \vec{v}'}{\partial t} = \frac{1}{\rho_0} \nabla P' + \frac{1}{\rho_0} \nabla^2 \vec{v}' + \frac{1}{\rho_0} \left( \frac{4}{3} \frac{\partial}{\partial t} \right) \nabla \left( \rho \cdot \vec{v}' \right) \]
2) \[ T \rho_0 \frac{\partial s'}{\partial t} = k \vec{u} \nabla^2 \vec{v}' \]
3) \[ \frac{\partial P'}{\partial t} + \rho_0 \nabla . \vec{v}' = 0 \leftarrow \text{as before} \]

5 variables 3 equations.
$T', s', \rho', P'$ thermodynamic quantities that are not independent. In linearized relationships ($s, \rho$ independent variables)

\[
P' = \left( \frac{\partial P}{\partial s} \right)_\rho s' + \left( \frac{\partial P}{\partial \rho} \right)_s \rho'
\]

\[
T' = \left( \frac{\partial T}{\partial s} \right)_\rho s' + \left( \frac{\partial T}{\partial \rho} \right)_s \rho'
\]

\[
\frac{\partial P}{\partial \rho} \bigg|_s = c^2 \quad \frac{\partial T}{\partial s} \bigg|_\rho = \frac{T_0}{\rho C_v} \
\]

\[
\frac{\partial T}{\partial \rho} \bigg|_s = \frac{1}{\rho^2} \left( \frac{\partial P}{\partial s} \right)_\rho
\]

\Rightarrow 3 equations in $s', \rho', \vec{v}'$

notice that the system is not irrotational so cannot use Φ

Solution assume

\[
\rho' = \rho_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}
\]

\[
\vec{v}' = \vec{v}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}
\]

\[
s' = s_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}
\]

Normal modes have transverse $\vec{v} \cdot \vec{h} = 0$ and longitudinal modes

Transverse $\omega = -i \left( \frac{\mu}{\rho_0} \right) k^2$

damped viscous wave
Longitudinal waves. \( \hat{k} \cdot \vec{v} = 0 \)

\[
\left[ \omega^2 - c^2 k^2 + i \frac{\omega \kappa}{\rho_o} \left( \frac{4}{3} \eta + \xi \right) \right] \rho' - \rho_o \kappa \left( \frac{2i}{3} \rho \right) \xi' = 0
\]

\[
\frac{i \kappa \rho_0 \epsilon \rho \left( \frac{4i}{3} \right)}{\rho_0} \rho' + (\omega + i \xi - \kappa c^2 k^2) \xi' = 0
\]

2 coupled equations in \( \rho' \) and \( \xi' \)

\[\tau = \frac{C_p}{C_v} K \text{ thermal diffusivity. \( \tau = \frac{k_{th}}{\rho_o c_p} \)}\]

\[\rho'(z,t) = \rho_o e^{-\tau z} \exp \left[ i \frac{\omega}{c} (z - ct) \right]\]

\[\alpha = \frac{\omega^2}{2 \rho_0 c^2} \left[ \frac{4}{3} \eta + \xi + k_{th} \frac{C_v}{C_p} \right]\]

\( \text{loss due to viscosity} \)

\( \text{loss due to thermal conduction} \)