

# Computation of Densities and Fluxes of Nonlinear Differential-Difference Equations

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Direct methods to find conserved densities and fluxes of differential-difference equations are presented and illustrated for the Kac-van Moerbeke and modified Volterra lattices. A Miura map which connects both lattices is explicitly constructed based on homotopic deformation. The map is used recursively to compute conserved densities of the Kac-van Moerbeke lattice. The algorithms presented could be implemented in computer algebra systems and could be used to investigate the integrability of semi-discrete lattices.

**Keywords:** Conserved Densities; DDEs, Semi-discrete; Lattice; Nonlinear Chains; Miura Map

## 1. Introduction

Dating back to the work of Fermi, Pasta, and Ulam in the 1950's (Fermi *et al.* 1965) differential-difference equations (DDEs) have been the focus of many nonlinear studies. A number of physically interesting problems can be modeled with nonlinear DDEs, including particle vibrations in lattices, currents in electrical networks, pulses in biological chains, etc. DDEs play an important role in queuing problems and discretizations in solid state and quantum physics. Last but not least, they are used in numerical simulations of nonlinear PDEs.

Recently, there has been a renewed interest in DDEs (see for example Teschl 2000 for a literature review). DDEs are *semi*-discretized because only the space variable is discretized while time is kept continuous. This is in contrast to their fully discretized counterparts, called difference equations, which are currently extensively studied (see for example Clarkson & Nijhoff 1999, Hietarinta *et al.* 2001, Levi & Ragnisco 2000, Levi *et al.* 1996). As shown in Hydon & Mansfield (2000) and Mansfield & Hydon (2002), the variational complex, viz. a geometric framework behind the computation of conservation laws for PDEs and DDEs, carries over to

nonlinear difference equations. Worked examples and applications can be found in Hydon (2001) and Mansfield & Hydon (2001).

In this paper we focus on one aspect of the integrability of DDEs, namely the computation of conserved densities and associated fluxes (of a given form) via direct methods which could be implemented in computer algebra systems like MAPLE, MATHEMATICA, and MUPAD.

The main result of this paper is three-fold.

- (1) Two novel methods to construct families of conserved densities are presented. The first method requires solving the integrability condition for the unknown density. Surprisingly, that integrability condition is quite amenable to analysis and a closed-form solution can be found. The solution method resembles the solution technique for solving the defining equations encountered in the computation of point symmetries of PDEs (Hereman 1996).
- (2) The second method is more algorithmic and primarily useful to find fluxes. It uses repeated decomposition of the identity operator  $I$  to find pieces in and outside the image (Im) of the operator  $\Delta = D - I$ , where  $D$  is the up-shift operator.
- (3) Third, a homotopic deformation of the KvM lattice is constructed explicitly. The Miura map (discrete analogue of the transformation (given in Miura 1968) which connects the KvM and the mV lattices follows as a special case. Similar to the strategy given in Miura *et al.* 1968, using the Miura map recursively one can generate densities of the KvM lattice order by order.

Miura maps, in connection with local conservation laws, allow one to construct master symmetries. In turn, based on master symmetries one can find conserved densities and build new integrable DDEs (Cherdantsev & Yamilov 1995, 1996).

The techniques described in this paper are also applicable to nonlinear systems of DDEs. Yet, to keep the ideas transparent and to avoid lengthy computations (which are best performed with computer algebra systems), we use the scalar Kacvan Moerbeke (KvM) (Kac & van Moerbeke 1975) and modified Volterra (mV) (Adler *et al.* 1999, Cherdantsev & Yamilov 1995) lattices as leading examples.

Strategies and MATHEMATICA code to automatically compute polynomial densities and generalized symmetries of DDEs can be found in Göktaş & Hereman 1998, 1999, Göktaş *et al.* 1997 and Hereman *et al.* 1998. In contrast, the methods in the present paper are no longer restricted to polynomial densities and fluxes. For computer algebra algorithms related to integrability testing based on master symmetries we refer to Oevel *et al.* 1989.

There is a vast body of work on DDEs, including investigations of integrability criteria, the computation of densities, generalized and master symmetries, recursion operators, etc. Notably is work by Levi and colleagues (Levi & Ragnisco 1978, Levi & Yamilov 1997), Yamilov (Yamilov 1993, 1994) and co-workers (Adler *et al.* 1999, Cherdantsev & Yamilov 1995, 1996, Shabat & Yamilov 1991, 1997, Svinolupov & Yamilov 1991), where the classification of DDEs (into canonical forms), integrability tests, and connections between integrable PDEs and DDEs are analyzed in detail. To a large extent, the classification and integrability testing of discrete equations mimics that of continuous equations (for reviews and references consult Adler *et al.* 2000, Mikhailov *et al.* 1987, 1990, Sokolov & Shabat 1984).

A wealth of information about integrable DDEs can be found in papers by Suris (Suris 1997a, 1997b, 1999, 2001) and his book (Suris 2002) in progress. Suris and

others have shown that many lattices are closely related to the celebrated Toda lattice (Toda 1989), its relativistic counterpart due to Ruijsenaars (Ruijsenaars 1990), the KvM lattice, and the two-component Volterra system (Shabat & Yamilov 1991, 1997).

Our paper is organized as follows. Section 2 covers preliminary material about conservation laws of DDEs. The KvM lattice exemplifies the introductory notions and notation. In analogy to a result for PDEs, the discrete Euler operator (or variational derivative) is introduced as a valuable tool to test conserved densities. We prove the necessary and sufficient condition for a function of a discrete variable (and its shifts) to be the total difference of another function of discrete variables. The section concludes with an application of the Euler operator to verify a density of the KvM lattice.

Using the discrete Euler operator we outline a strategy to compute conserved densities of nonlinear lattices in Section 3. Details of the computation of a family of densities are given for the mV lattice.

In Section 4 we focus on the computation of fluxes. We give a direct procedure to compute densities as well as fluxes. We revisit the mV lattice to illustrate the method.

In Section 5 we construct the discrete version of the Miura map, which links the mV and the KvM lattices. In analogy with the continuous case, we use the map recursively to generate the densities for the KvM lattice.

We draw some conclusions in Section 6.

## 2. Differential-Difference Equations

Here we review preliminary material about densities and fluxes of differential-difference equations.

Consider a nonlinear (autonomous) DDE of the form

$$\frac{d}{dt} u_n = f(u_{n-l}, u_{n-l+1}, \dots, u_n, \dots, u_{n+m-1}, u_{n+m}) \quad (2.1)$$

with

$$\frac{\partial f}{\partial u_{n-l}} \frac{\partial f}{\partial u_{n+m}} \neq 0,$$

where  $n$  is an arbitrary integer. In general,  $f$  is a vector-valued function of a finite number of dynamical variables and each  $u_k$  is a vector-valued function of  $t$ .

The index  $n$  may lie in  $\mathbf{Z}$ , or the  $u_k$  may be periodic,  $u_k = u_{k+N}$ . The integers  $l$  and  $m$  measure the degree of non-locality in (2.1). If  $l = m = 0$  then the equation is local and reduces to a system of ordinary differential equations.

The (up-)shift operator  $D$  is defined by

$$D u_k = u_{k+1}.$$

Its inverse, called the down-shift operator, is given by  $D^{-1} u_k = u_{k-1}$ . Obviously,  $u_k = D^k u_0$ . The action of  $D$  and  $D^{-1}$  is extended to functions by acting on their arguments. For example,

$$D g(u_p, u_{p+1}, \dots, u_q) = g(D u_p, D u_{p+1}, \dots, D u_q) = g(u_{p+1}, u_{p+2}, \dots, u_{q+1}).$$

In particular,

$$D \left( \frac{\partial}{\partial u_k} g(u_p, u_{p+1}, \dots, u_q) \right) = \frac{\partial}{\partial u_{k+1}} g(u_{p+1}, u_{p+2}, \dots, u_{q+1}).$$

Moreover, for equations of type (2.1), the shift operator commutes with the time derivative

$$\begin{aligned} D \left( \frac{d}{dt} u_n \right) &= D f(u_{n-l}, u_{n-l+1}, \dots, u_n, \dots, u_{n+m-1}, u_{n+m}) \\ &= f(u_{n-l+1}, u_{n-l+2}, \dots, u_{n+1}, \dots, u_{n+m}, u_{n+m+1}) \\ &= \frac{d}{dt} u_{n+1} \\ &= \frac{d}{dt} (D u_n). \end{aligned}$$

Thus, with the use of the shift operator, the entire system (2.1) which may be an infinite set of ordinary differential equations is generated from a single equation

$$\frac{d}{dt} u_0 = f(u_{-l}, u_{-l+1}, \dots, u_0, \dots, u_{m-1}, u_m) \quad (2.2)$$

with

$$\frac{\partial f}{\partial u_{-l}} \frac{\partial f}{\partial u_m} \neq 0.$$

Next, we define the (forward) difference operator,  $\Delta = D - I$ , by

$$\Delta u_k = (D - I) u_k = u_{k+1} - u_k,$$

where  $I$  is the identity operator. This operator takes the role of a spatial derivative on the shifted variables as many examples of DDEs arise from discretization of a PDE in  $(1+1)$  variables (Shabat & Yamilov 1988). The difference operator clearly extends to functions, that is

$$\Delta g(u_p, u_{p+1}, \dots, u_q) = g(u_{p+1}, u_{p+2}, \dots, u_{q+1}) - g(u_p, u_{p+1}, \dots, u_q).$$

For any function  $g = g(u_p, u_{p+1}, \dots, u_q)$ , the total time derivative  $D_t g$  is computed as

$$\begin{aligned} D_t g &= \sum_{k=p}^q \frac{\partial g}{\partial u_k} \frac{d}{dt} u_k \\ &= \left( \sum_{k=p}^q \frac{\partial g}{\partial u_k} D^k \right) \frac{d}{dt} u_0 \\ &= \left( \sum_{k=p}^q \frac{\partial g}{\partial u_k} D^k \right) f(u_{-l}, u_{-l+1}, \dots, u_m) \end{aligned}$$

on solutions of (2.1). A simple calculation shows that the shift operator  $D$  commutes with  $D_t$ , and so do  $D$  and  $\Delta$ .

A function  $\rho = \rho(u_p, u_{p+1}, \dots, u_q)$  is a (conserved) *density* of (2.2) if there exists a function  $J = J(u_r, u_{r+1}, \dots, u_s)$ , called the (associated) *flux*, such that

$$D_t \rho + \Delta J = 0 \quad (2.3)$$

is satisfied on the solutions of (2.2). Eq. (2.3) is called a *local conservation law*.

Any shift of a density is trivially a density since

$$D_t D^k \rho + \Delta D^k J = D^k (D_t \rho + \Delta J) = 0,$$

with associated flux  $D^k J$ .

Constants of motion for (2.2) are easily obtained from a densities and their shifts. Indeed, for any density  $\rho$  with corresponding flux  $J$ , consider

$$\Omega = \sum_{k=p}^q D^k \rho.$$

The total time derivative of  $\Omega$  is

$$\begin{aligned} D_t \Omega &= - \sum_{k=p}^q \Delta D^k J \\ &= - \sum_{k=p}^q (D^{k+1} - D^k) J \\ &= - (D^{q+1} - D^p) J. \end{aligned}$$

Applying appropriate boundary conditions (e.g. all  $u_k \rightarrow 0$ , as  $k \rightarrow \pm\infty$ ) one gets the conservation law

$$D_t \left( \sum_{k=-\infty}^{\infty} D^k \rho \right) = \lim_{p \rightarrow -\infty} D^p J - \lim_{q \rightarrow \infty} D^{q+1} J = 0.$$

For a periodic chain where  $u_k = u_{k+N}$ , after summing over a period, one obtains

$$D_t \left( \sum_{k=0}^{N-1} D^k \rho \right) = D^0 J - D^N J = J - J = 0.$$

In either case,  $\Omega$  is a constant of motion of (2.2) since  $\Omega$  does not change with time.

A density which is a *total difference*,

$$\rho = \Delta F \tag{2.4}$$

(so that  $D_t \rho = \Delta D_t F$  and therefore  $J = -D_t F$  is an associated flux), is called *trivial*. These densities lead to trivial conservation laws since

$$\Omega = \sum_{k=p}^q D^k \Delta F = D^{q+1} F - D^p F$$

holds identically (not just on solutions of (2.2)).

Two densities  $\rho, \tilde{\rho}$  are called *equivalent* if  $\rho - \tilde{\rho} \in \text{Im } \Delta$ , i.e.  $\rho - \tilde{\rho} = \Delta F$  for some  $F$ . Equivalent densities, denoted by  $\rho \sim \tilde{\rho}$ , differ by a trivial density and yield the same conservation law. Also,  $\rho \sim D^k \rho$ , and (2.4) expresses that  $\rho \sim 0$ .

To illustrate the above notions we now turn to a simple example.

*Example: The Kac-van Moerbeke lattice*

Consider the Kac-van Moerbeke (KvM) lattice (Kac & van Moerbeke 1975, Manakov 1975, Moser 1975),

$$\frac{d u_n}{d t} = u_n(u_{n+1} - u_{n-1}), \quad (2.5)$$

or, equivalently,

$$\frac{d u_0}{d t} = u_0(u_1 - u_{-1}). \quad (2.6)$$

Eq. (2.6) is often referred to as a Volterra lattice (Volterra 1931, Yamilov 1994), although it is a special case of the two-component Volterra system (Hirota & Satsuma 1976, Shabat & Yamilov 1991).

Lattice (2.6) arises in the study of Langmuir oscillations in plasmas, population dynamics, quantum field theory, polymer science, and appears in the context of matrix factorizations (see references in Teschl 2000).

The KvM lattice (2.5) appears in the literature in various forms, including

$$\frac{d R_n}{d t} = \frac{1}{2} (e^{-R_{n-1}} - e^{-R_{n+1}}), \quad (2.7)$$

and

$$\frac{d w_n}{d t} = w_n(w_{n+1}^2 - w_{n-1}^2). \quad (2.8)$$

Both are related to (2.5) by simple transformations (Teschl 2000).

The function

$$\rho = u_0$$

is a density for (2.6) since

$$\begin{aligned} D_t \rho &= \left( \frac{\partial}{\partial u_0} \rho \right) \frac{d}{d t} u_0 \\ &= u_0(u_1 - u_{-1}) \\ &= \Delta(u_0 u_{-1}) \end{aligned}$$

with associated flux  $J = -u_0 u_{-1}$ . Thus  $\rho_k = D^k \rho = u_k$  are also densities with associated fluxes  $J_k = D^k J = -u_k u_{k-1}$ . Therefore,

$$\Omega = \sum_{k=p}^q D^k \rho = \sum_{k=p}^q u_k,$$

and

$$\begin{aligned} D_t \Omega &= \sum_{k=p}^q \Delta(u_k u_{k-1}) \\ &= \sum_{k=p}^q u_{k+1} u_k - u_k u_{k-1} \\ &= u_{q+1} u_q - u_p u_{p-1} \\ &= - (D^{q+1} - D^p) J. \end{aligned}$$

Applying appropriate boundary conditions we get the conservation law

$$D_t \left( \sum_{k=-\infty}^{\infty} u_k \right) = \lim_{q \rightarrow \infty} u_{q+1} u_q - \lim_{p \rightarrow -\infty} u_p u_{p-1} = 0.$$

Eq. (2.6) has infinitely many couples  $\rho$  and  $J$ . The above computations could be repeated for other conserved densities, such as  $\rho = u_0(u_0 + 2u_1)$  and  $\rho = u_0(u_0^2 + 3u_1(u_0 + u_1 + u_2))$  listed in Göktaş *et al.* 1997.

We now introduce a tool to test whether or not a function is a total difference.

*The discrete Euler operator (variational derivative)*

It is well-known that a (continuous) function  $g(u(x), u'(x), u''(x), \dots, u^{(n)}(x))$  can be integrated with respect to  $x$  if and only if (iff) the variational derivative of  $g$  vanishes. Formally,  $g = D_x h$  for some function  $h(u(x), u'(x), u''(x), \dots, u^{(m)}(x))$  iff  $E(g) = 0$ . Here,  $D_x$  refers to total differentiation with respect to  $x$  and  $E$  is the continuous Euler operator (variational derivative).

In particular, a function  $\rho$  is a conserved density of a PDE iff  $D_t \rho$  is in the kernel (Ker) of the Euler operator (see for example Göktaş & Hereman 1997). We now present a discrete analogue of this important result.

A function  $g = g(u_p, u_{p+1}, \dots, u_q)$ ,  $p, q \in \mathbf{Z}$ , is a *total difference* if there exists another function  $h = h(u_p, u_{p+1}, \dots, u_{q-1})$ , such that  $g = \Delta h = (D - I)h$ .

A necessary and sufficient condition for a function  $g$  to be a total difference is that

$$E(g) = 0, \quad (2.9)$$

where  $E$  is the discrete Euler operator (variational derivative) (Adler *et al.* 1999, Shabat & Yamilov 1991) defined by

$$E(g) = \sum_{k=p}^q D^{-k} \left( \frac{\partial}{\partial u_k} g \right). \quad (2.10)$$

Note that we can rewrite the Euler operator as

$$E(g) = \frac{\partial}{\partial u_0} \left( \sum_{k=p}^q D^{-k} g \right), \quad (2.11)$$

and that

$$\sum_{k=p}^q D^k \Delta = D^{q+1} - D^p.$$

A sketch of the proof of (2.9) appears in Levi & Yamilov 1997. We first show that the condition (2.9) is *necessary*.

Suppose that  $g = \Delta h$  with  $h = h(u_p, u_{p+1}, \dots, u_{q-1})$  then

$$\begin{aligned} E(\Delta h) &= \frac{\partial}{\partial u_0} \left( \sum_{k=p}^q D^{-k} \Delta h \right) \\ &= \frac{\partial}{\partial u_0} (D^{-p+1} h - D^{-q} h) \\ &= 0 \end{aligned} \quad (2.12)$$

since neither  $D^{-p+1}h = h(u_1, u_2, \dots, u_{q-p})$  nor  $D^{-q}h = h(u_{p-q}, u_{p-q+1}, \dots, u_{-1})$  depend on  $u_0$ . Thus, if  $g$  is a total difference then  $E(g) = 0$ .

Conversely, to prove that (2.9) is *sufficient*, suppose that  $E(g) = 0$  holds. Without loss of generality, we can set  $p = 0$  since we can replace  $g$  by  $D^{-p}g$  (and relabel  $q - p$  by  $q$ ). Thus, for  $g = g(u_0, u_1, \dots, u_q)$  we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial u_q} (E(g)) \\ &= \frac{\partial^2}{\partial u_q \partial u_0} \left( \sum_{k=0}^q D^{-k} g \right) \\ &= \frac{\partial^2}{\partial u_0 \partial u_q} g \end{aligned} \tag{2.13}$$

since no other term will depend on  $u_q$ . Thus,

$$\begin{aligned} g(u_0, u_1, \dots, u_q) &= \tilde{g}^{(1)}(u_0, u_1, \dots, u_{q-1}) + g^{(2)}(u_1, u_2, \dots, u_q) \\ &= g^{(1)}(u_0, u_1, \dots, u_{q-1}) + \Delta g^{(2)}(u_0, u_1, \dots, u_{q-1}) \end{aligned} \tag{2.14}$$

with  $g^{(1)}(u_0, u_1, \dots, u_{q-1}) = \tilde{g}^{(1)}(u_0, u_1, \dots, u_{q-1}) - g^{(2)}(u_0, u_1, \dots, u_{q-1})$ . Therefore,

$$0 = E(g) = E(g^{(1)}) + E(\Delta g^{(2)}) = E(g^{(1)})$$

by (2.12). Repeating this procedure on  $g^{(1)}$ , we can write  $g^{(1)}$  as a function of the  $q - 1$  variables  $u_0, u_1, \dots, u_{q-2}$  plus a total difference. We eventually obtain

$$g(u_0, u_1, \dots, u_q) = K(u_0) + \Delta G(u_0, u_1, \dots, u_{q-1}),$$

where  $\Delta G$  is the sum of all the total difference terms and with

$$0 = E(g) = E(K) = \frac{\partial K}{\partial u_0}.$$

Hence,  $K$  is a constant. Let  $\varphi$  be the unit ‘‘constant’’ shifted function; that is  $D^k \varphi = k$  and so  $\Delta \varphi = 1$ . Thus  $K = K \Delta \varphi$  and so  $g$  is a total difference,

$$g = \Delta (G + K\varphi).$$

The integration of (2.13), resulting in (2.14) is important. Indeed, if (2.13) is satisfied then separation of variables leads to a total difference. This result will be used on several occasions later in the paper.

#### *Application of the Euler operator*

The discrete Euler operator (2.10) is a most valuable tool to test conserved densities. Indeed,  $\rho$  is a density of (2.2) iff  $D_t \rho \in \text{Ker } E$  i.e.

$$E(D_t \rho) = 0 \tag{2.15}$$

Moreover,  $\rho$  is a non-trivial density if, in addition,  $E(\rho) \neq 0$ . If (2.15) holds, then  $D_t \rho \in \text{Im } \Delta$ . This implies the existence of a  $J$  for which (2.3) is satisfied.

For the example of the KvM lattice (2.6),  $\rho = 2u_0u_1 + u_1^2$  is a density since

$$\begin{aligned}
 E(D_t \rho) &= E(D_t (2u_0u_1 + u_1^2)) \\
 &= 2E(u_1u_0(u_1 - u_{-1}) + (u_0 + u_1)u_1(u_2 - u_0)) \\
 &= -2E(u_0u_{-1}u_1 - u_0u_1u_2 + u_0^2u_1 - u_1^2u_2) \\
 &= -2(D(u_0u_1) + D^0(u_{-1}u_1 - u_1u_2 + 2u_0u_1) \\
 &\quad + D^{-1}(u_{-1}u_0 - u_0u_2 + u_0^2 - 2u_1u_2) - D^{-2}(u_0u_1 + u_1^2)) \\
 &= -2(u_1u_2 + (u_{-1}u_1 - u_1u_2 + 2u_0u_1) \\
 &\quad + (u_{-2}u_{-1} - u_{-1}u_1 + u_{-1}^2 - 2u_0u_1) - (u_{-2}u_{-1} + u_{-1}^2)) \\
 &= 0.
 \end{aligned}$$

This density is non-trivial since  $E(\rho) = 2(u_{-1} + u_0 + u_1) \neq 0$ . One can quickly verify that  $J = -2(u_{-1}u_0u_1 + u_0^2u_1)$ .

Obviously,  $\rho = 2u_0u_1 + u_1^2 = 2u_0u_1 + u_0^2 + (u_1^2 - u_0^2) = 2u_0u_1 + u_0^2 + \Delta(u_0^2)$ . Hence,  $\rho = 2u_0u_1 + u_0^2$  and  $\tilde{\rho} = 2u_0u_1 + u_1^2$  are equivalent densities.

The computation of the action of the Euler operator quickly become tedious. However the Euler operator can be easily implemented in a computer algebra system like MAPLE, MATHEMATICA or MUPAD (Göktaş & Hereman 1997, Hereman & Göktaş 1999).

The computation of densities involves solving the integrability condition (2.15). This is a rather unusual ‘‘PDE’’ for  $\rho$  since it involves derivatives of both  $\rho$  and shifts of  $\rho$ . For example, if we assume that  $\rho$  depends (among others) on the variable  $u_k$  then  $D_t \rho$  will have a term

$$\left( \frac{\partial}{\partial u_k} \rho(\dots, u_k, \dots) \right) D^k f(u_{-l}, \dots, u_m).$$

This term will then give rise to terms of the form (the list is not exhaustive)

$$D^{-r} \left( \frac{\partial}{\partial u_r} \left( \left( \frac{\partial}{\partial u_k} \rho(\dots, u_k, \dots) \right) f(u_{k-l}, \dots, u_{k+m}) \right) \right),$$

where  $r$  ranges from  $k-l$  to  $k+m$  in the integrability condition (2.15). For example, with  $r = k$ , we get the term

$$\frac{\partial}{\partial u_0} \left( \left( \frac{\partial}{\partial u_0} \rho(\dots, u_0, \dots) \right) f(u_{-l}, \dots, u_m) \right),$$

whereas, with  $r = k+1$ , we get the term

$$\frac{\partial}{\partial u_0} \left( \left( \frac{\partial}{\partial u_{-1}} \rho(\dots, u_{-1}, \dots) \right) f(u_{1-l}, \dots, u_{m+1}) \right).$$

These two terms have derivatives of  $\rho$  evaluated with *different* arguments. It is then, perhaps, surprising that this equation is amenable to analysis. At this stage, it is best to proceed with an explicit example.

### 3. Computation of Conservation Laws for DDEs. An Example

Consider the modified Volterra (mV) lattice (Adler *et al.* 1999)

$$\frac{d v_n}{d t} = v_n^2(v_{n+1} - v_{n-1}), \quad (3.1)$$

or, equivalently,

$$\frac{d v_0}{d t} = v_0^2 (v_1 - v_{-1}). \quad (3.2)$$

To keep matters as clear as possible, we search for densities that depend only on the variables  $v_0, v_1, v_2$ ; viz.

$$\rho = \rho(v_0, v_1, v_2).$$

Attempting to find densities of greater generality would lead to messier, less illuminating calculations.

Upon differentiation

$$D_t \rho = v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho + v_1^2 (v_2 - v_0) \frac{\partial}{\partial v_1} \rho + v_2^2 (v_3 - v_1) \frac{\partial}{\partial v_2} \rho.$$

The integrability condition (2.15) reads

$$\begin{aligned} E(D_t \rho) &= v_0^2 (v_1 - v_{-1}) \frac{\partial^2}{\partial v_0^2} \left( \rho(v_{-2}, v_{-1}, v_0) + \rho(v_{-1}, v_0, v_1) + \rho(v_0, v_1, v_2) \right) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial^2}{\partial v_{-1} \partial v_0} \left( \rho(v_{-2}, v_{-1}, v_0) + \rho(v_{-1}, v_0, v_1) \right) \\ &\quad + v_1^2 (v_2 - v_0) \frac{\partial^2}{\partial v_0 \partial v_1} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_0, v_1, v_2) \right) \\ &\quad + v_{-2}^2 (v_{-1} - v_{-3}) \frac{\partial^2}{\partial v_{-2} \partial v_0} \rho(v_{-2}, v_{-1}, v_0) \\ &\quad + v_2^2 (v_3 - v_1) \frac{\partial^2}{\partial v_0 \partial v_2} \rho(v_0, v_1, v_2) \\ &\quad + v_{-1}^2 \frac{\partial}{\partial v_{-1}} \left( \rho(v_{-3}, v_{-2}, v_{-1}) + \rho(v_{-2}, v_{-1}, v_0) + \rho(v_{-1}, v_0, v_1) \right) \\ &\quad - v_1^2 \frac{\partial}{\partial v_1} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_0, v_1, v_2) + \rho(v_1, v_2, v_3) \right) \\ &\quad + 2v_0 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \left( \rho(v_{-2}, v_{-1}, v_0) + \rho(v_{-1}, v_0, v_1) + \rho(v_0, v_1, v_2) \right) \\ &= 0. \end{aligned} \quad (3.3)$$

This illustrates the unusual nature of the integrability condition for  $\rho$ . It is a PDE for  $\rho$ . However the unknown function  $\rho$  occurs with different arguments. Indeed, there are terms that involve derivatives of  $\rho(v_0, v_1, v_2)$ ; other terms involve derivatives of  $D^{-1} \rho = \rho(v_{-1}, v_0, v_1)$ , for example. Nonetheless, we can decompose the equation by differentiating with respect to various  $v_k$  and separation of variables.

The highest shifted variable that occurs is  $v_3$ . Therefore, we differentiate with respect to  $v_3$ ,

$$v_2^2 \frac{\partial^2}{\partial v_0 \partial v_2} \rho(v_0, v_1, v_2) - v_1^2 \frac{\partial^2}{\partial v_1 \partial v_3} \rho(v_1, v_2, v_3) = 0. \quad (3.4)$$

A further differentiation with respect to  $v_0$  yields

$$v_2^2 \frac{\partial^3}{\partial v_0^2 \partial v_2} \rho(v_0, v_1, v_2) = 0.$$

So,

$$\begin{aligned}\rho(v_0, v_1, v_2) &= \rho^{(1)}(v_0, v_1) - \rho^{(2)}(v_0, v_1) + \rho^{(2)}(v_1, v_2) + \rho^{(3)}(v_1, v_2)v_0 \\ &= \rho^{(1)}(v_0, v_1) + \Delta \rho^{(2)}(v_0, v_1) + \rho^{(3)}(v_1, v_2)v_0\end{aligned}$$

for some unknown functions  $\rho^{(1)}, \rho^{(2)}$  and  $\rho^{(3)}$ . The term  $\Delta \rho^{(2)}(v_0, v_1)$  leads to a trivial density and can be ignored. Thus

$$\rho = \rho^{(1)}(v_0, v_1) + \rho^{(3)}(v_1, v_2)v_0.$$

The integrability condition (3.4) is now

$$v_2^2 \frac{\partial}{\partial v_2} \rho^{(3)}(v_1, v_2) - v_1^2 \frac{\partial}{\partial v_3} \rho^{(3)}(v_2, v_3) = 0.$$

A subsequent differentiation with respect to  $v_3$  gives

$$-v_1^2 \frac{\partial^2}{\partial v_3^2} \rho^{(3)}(v_2, v_3) = 0.$$

Consequently,

$$\rho^{(3)}(v_2, v_3) = \rho^{(4)}(v_2) + \rho^{(5)}(v_2)v_3.$$

Equation (3.4) has become

$$v_2^2 \rho^{(5)}(v_1) - v_1^2 \rho^{(5)}(v_2) = 0.$$

Thus,

$$\frac{1}{v_2^2} \rho^{(5)}(v_2) = \frac{1}{v_1^2} \rho^{(5)}(v_1),$$

from which we get

$$\rho^{(5)}(v_2) = c^{(1)} v_2^2$$

for some constant  $c^{(1)}$ .

The density now has the form

$$\rho = \rho^{(1)}(v_0, v_1) + (\rho^{(4)}(v_1) + c^{(1)}v_1^2v_2)v_0.$$

The term  $\rho^{(4)}(v_1)v_0$  can be absorbed into the term  $\rho^{(1)}(v_0, v_1)$ . Thus

$$\rho(v_0, v_1, v_2) = \rho^{(1)}(v_0, v_1) + c^{(1)}v_0v_1^2v_2. \tag{3.5}$$

At this stage we have reduced the problem from one unknown function of three variables to one unknown function of two variables and a constant.

Differentiating equation (3.3) with respect to  $v_2$  twice yields

$$v_1^2 \left( 2c^{(1)}v_1 - \frac{\partial^3}{\partial v_2^2 \partial v_1} \rho^{(1)}(v_1, v_2) \right) = 0, \tag{3.6}$$

which readily integrates into

$$\rho^{(1)}(v_1, v_2) = \frac{1}{2}c^{(1)}v_1^2v_2^2 + \rho^{(6)}(v_2) + \rho^{(7)}(v_1) + \rho^{(8)}(v_1)v_2.$$

Again we can write

$$\rho^{(6)}(v_2) = \Delta \rho^{(6)}(v_1) + \rho^{(6)}(v_1)$$

and absorb the term  $\rho^{(6)}(v_1)$  into  $\rho^{(7)}(v_1)$ . Thus,

$$\rho = \frac{1}{2}c^{(1)}v_0^2v_1^2 + \rho^{(7)}(v_0) + \rho^{(8)}(v_0)v_1 + c^{(1)}v_0v_1^2v_2.$$

The  $v_2$  derivative (3.3) is now

$$v_1^2 \left( \frac{d}{dv_1} \rho^{(8)}(v_1) - \frac{d}{dv_0} \rho^{(8)}(v_0) \right) = 0,$$

which yields

$$\rho^{(8)}(v_0) = c^{(2)}v_0 + c^{(3)}$$

with constants  $c^{(2)}$  and  $c^{(3)}$ . As before, the term  $c^{(3)}v_1$  can be written as  $\Delta c^{(3)}v_0 + c^{(3)}v_0$ . Thus the density simplifies into

$$\rho = \frac{1}{2}c^{(1)}v_0^2v_1^2 + \rho^{(7)}(v_0) + c^{(2)}v_0v_1 + c^{(1)}v_0v_1^2v_2. \quad (3.7)$$

Eq. (3.3) is now reduced to

$$\begin{aligned} v_0^2(v_1 - v_{-1}) \frac{d^2}{dv_0^2} \rho^{(7)}(v_0) + 2v_0(v_1 - v_{-1}) \frac{d}{dv_0} \rho^{(7)}(v_0) \\ + v_{-1}^2 \frac{d}{dv_{-1}} \rho^{(7)}(v_{-1}) - v_1^2 \frac{d}{dv_1} \rho^{(7)}(v_1) = 0. \end{aligned}$$

Differentiating twice with respect to  $v_1$  yields the ordinary differential equation

$$\frac{d^2}{dv_1^2} \left( v_1^2 \frac{d}{dv_1} \rho^{(7)}(v_1) \right) = 0. \quad (3.8)$$

Integration gives

$$\rho^{(7)}(v_1) = c^{(4)} + c^{(5)} \frac{1}{v_1} + c^{(6)} \log v_1 \quad (3.9)$$

for some constants  $c^{(4)}$ ,  $c^{(5)}$  and  $c^{(6)}$ . The term  $c^{(4)}$  gives a trivial density. Hence,

$$\rho = \frac{1}{2}c^{(1)}(v_0^2v_1^2 + 2v_0v_1^2v_2) + c^{(2)}v_0v_1 + c^{(5)} \frac{1}{v_0} + c^{(6)} \log v_0. \quad (3.10)$$

Eq. (3.3) is now satisfied. Hence, for (3.2) we have four independent, non-trivial densities that depend on, at most, three shifts. Namely,

$$\begin{aligned} \rho_{(1)} &= \frac{1}{v_0} \\ \rho_{(2)} &= \log v_0 \\ \rho_{(3)} &= v_0v_1 \\ \rho_{(4)} &= v_0v_1^2 \left( \frac{1}{2}v_0 + v_2 \right), \end{aligned}$$

in agreement with the results in Adler *et al.* 1999.

The associated fluxes are computed via  $J_{(k)} = -\Delta^{-1}(D_t \rho_{(k)})$ . They are

$$\begin{aligned} J_{(1)} &= v_0 + v_{-1} \\ J_{(2)} &= -v_{-1}v_0 \\ J_{(3)} &= -v_{-1}v_0^2v_1 \\ J_{(4)} &= -v_{-1}v_0^2v_1^2(v_0 + v_2). \end{aligned}$$

This strategy of peeling off the terms that involve the highest (or lowest) shifted variables appears to work in many cases.

#### 4. Computing Fluxes and a Second Method to Compute Densities

Densities  $\rho$  and fluxes  $J$  are related by (2.3). Thus  $J = -\Delta^{-1}(D_t \rho)$  and so, in principle, one needs to invert the operator  $\Delta = D - I$ .

Working with the formal inverse,

$$\Delta^{-1} = D^{-1} + D^{-2} + D^{-3} + \dots, \quad (4.1)$$

is impractical, perhaps impossible.

We therefore present a simple algorithm which circumvents the above infinite formal series. Furthermore, starting from a candidate density the algorithm allows one to compute constraints for the density. For example, if the candidate has undetermined constant coefficients, one can derive the linear system for these unknown coefficients. If the candidate density is an unknown function, one can determine conditions for that function.

The idea is to split expressions into a total derivative term plus a term involving lower order shifts. Using

$$I = (D - I + I)D^{-1} = \Delta D^{-1} + D^{-1}, \quad (4.2)$$

any expression  $T$  can be split as follows,

$$T = \Delta D^{-1}T + D^{-1}T.$$

The first term will contribute to the flux whilst the second term has a strictly lower shift than the original expression. By selectively applying this decomposition to terms in a density  $\rho$ , we can compute the associated flux. More importantly, we can use this decomposition to find the obstruction for a candidate expression to be a density.

For any density  $\rho$  we compute  $D_t \rho$  and split it into

$$D_t \rho = A^{(0)} + A^{(1)},$$

where  $A^{(0)}$  is the part that does *not* depend on the lowest shifted variable  $u_p$  that appears in  $D_t \rho$ . Using (4.2), we write

$$D_t \rho = \Delta D^{-1}A^{(0)} + D^{-1}A^{(0)} + A^{(1)}.$$

Now, if  $D^{-1}A^{(0)} + A^{(1)} = 0$  we are finished and  $J = -D^{-1}A^{(0)}$ . If  $D^{-1}A^{(0)} + A^{(1)} \neq 0$  we again split this term,

$$D^{-1}A^{(0)} + A^{(1)} = A^{(2)} + A^{(3)},$$

where  $A^{(2)}$  does not depend on  $u_p$ . Applying (4.2) to  $A^{(2)}$  we get

$$D_t \rho = \Delta D^{-1}A^{(0)} + \Delta D^{-1}A^{(2)} + D^{-1}A^{(2)} + A^{(3)}.$$

If  $D^{-1}A^{(2)} + A^{(3)} = 0$  then the flux is  $J = -D^{-1}(A^{(0)} + A^{(2)})$ . If not, we repeat the process. This procedure must eventually terminate since  $\rho$  is a density.

We illustrate this algorithm for the mV lattice (3.2) with density  $\rho = \log v_0$ . Now,

$$D_t \log v_0 = \frac{v_0^2(v_1 - v_{-1})}{v_0} = v_0(v_1 - v_{-1}).$$

So, we have  $A^{(0)} = v_0v_1$  and  $A^{(1)} = -v_{-1}v_0$ . Thus

$$D_t \log v_0 = \Delta D^{-1}v_0v_1 + D^{-1}v_0v_1 - v_{-1}v_0 = \Delta v_{-1}v_0,$$

giving  $J = -v_{-1}v_0$ . In this simple case the algorithm terminated after one step.

A similar approach can be used in the more interesting case of computing densities. Without loss of generality, we can assume that  $\rho = \rho(u_0, \dots, u_q)$ . For (2.2), we have

$$\begin{aligned} D_t \rho &= \sum_{k=0}^q \frac{\partial \rho}{\partial u_k} D^k f(u_{-l}, \dots, u_m) \\ &= \frac{\partial \rho}{\partial u_0} f(u_{-l}, \dots, u_m) + \sum_{k=1}^q \frac{\partial \rho}{\partial u_k} D^k f(u_{-l}, \dots, u_m). \end{aligned}$$

Applying (4.2) to the second term we obtain

$$\begin{aligned} D_t \rho &= \frac{\partial \rho}{\partial u_0} f + (\Delta D^{-1} + D^{-1}) \left( \sum_{k=1}^q \frac{\partial \rho}{\partial u_k} D^k f \right) \\ &= \frac{\partial \rho}{\partial u_0} f + (\Delta + \mathbb{I}) \left( \sum_{k=1}^q \frac{\partial D^{-1} \rho}{\partial u_{k-1}} D^{k-1} f \right) \\ &= \frac{\partial \rho}{\partial u_0} f + \Delta \sum_{k=0}^{q-1} \frac{\partial D^{-1} \rho}{\partial u_k} D^k f + \sum_{k=0}^{q-1} \frac{\partial D^{-1} \rho}{\partial u_k} D^k f \\ &= \left( \frac{\partial}{\partial u_0} (\rho + D^{-1} \rho) \right) f + \Delta \sum_{k=0}^{q-1} \frac{\partial D^{-1} \rho}{\partial u_k} D^k f + \sum_{k=1}^{q-1} \frac{\partial D^{-1} \rho}{\partial u_k} D^k f \end{aligned}$$

with  $f = f(u_{-l}, \dots, u_m)$ . Next, we repeat this procedure by applying (4.2) to the last term. After a further  $q - 2$  applications, we get

$$\begin{aligned}
 D_t \rho &= \left( \frac{\partial}{\partial u_0} \sum_{j=0}^2 D^{-j} \rho \right) f + \Delta \left( \sum_{k=0}^{q-1} \frac{\partial D^{-1} \rho}{\partial u_k} D^k f + \sum_{k=0}^{q-2} \frac{\partial D^{-2} \rho}{\partial u_k} D^k f \right) \\
 &\quad + \sum_{k=1}^{q-2} \frac{\partial D^{-2} \rho}{\partial u_k} D^k f \\
 &= \dots \\
 &= \left( \frac{\partial}{\partial u_0} \sum_{j=0}^q D^{-j} \rho \right) f + \Delta \left( \sum_{j=1}^q \sum_{k=0}^{q-j} \frac{\partial D^{-j} \rho}{\partial u_k} D^k f \right) \\
 &= E(\rho) f + \Delta \left( \sum_{j=1}^q \sum_{k=0}^{q-j} \frac{\partial D^{-j} \rho}{\partial u_k} D^k f \right)
 \end{aligned}$$

by (2.11).

If  $E(\rho) f = 0$  then  $\rho$  is a trivial density. For  $\rho$  to be a non-trivial density, we require

$$E(\rho) f = \Delta h \quad (4.3)$$

for some  $h$  with  $h \neq 0$ . In this case the associated flux is

$$J = -h - \sum_{j=1}^q \sum_{k=0}^{q-j} \frac{\partial D^{-j} \rho}{\partial u_k} D^k f.$$

One could apply the discrete Euler operator to  $E(\rho) f$  to determine conditions such that (4.3) holds. Alternatively, one could repeat the above strategy by splitting this expression into a part,  $A^{(0)}$ , that does not depend on the lowest shifted variable and the remaining terms. One then applies (4.2) repeatedly to  $A^{(0)}$  (removing the total difference terms generated by (4.2) from  $A^{(0)}$ ) until one obtains a term that involves both the lowest and highest shifted variables. By noting that, from (2.13), a *necessary* condition that an  $g = g(u_p, \dots, u_q)$  is a total difference is

$$\frac{\partial^2}{\partial u_p \partial u_q} g = 0, \quad (4.4)$$

one then obtains conditions for (4.3) to hold. The splitting followed by the application of (4.4) is repeated until the only terms that remain depend on the single variable  $u_p$ . This condition that this expression vanishes is the last obstruction for  $\rho$  to be a density.

We now illustrate this approach by recomputing low order densities for the mV lattice.

### *The Modified Volterra Lattice Revisited*

To keep the exposition simple, we search for densities of the form

$$\rho = \rho(v_0, v_1, v_2)$$

for the mV lattice (3.2), where  $f = v_0^2(v_1 - v_{-1})$ . The construction of densities involving a greater spread of dynamic variables could be attempted with a computer algebra package.

We write

$$D_t \rho = \sigma + \Delta K$$

with

$$\begin{aligned} \sigma &= E(\rho) f \\ &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \left( \rho(v_0, v_1, v_2) + \rho(v_{-1}, v_0, v_1) + \rho(v_{-2}, v_{-1}, v_0) \right), \end{aligned}$$

and

$$\begin{aligned} K &= \sum_{j=1}^q \sum_{k=0}^{q-j} \frac{\partial D^{-j} \rho}{\partial u_k} D^k f \\ &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho(v_{-1}, v_0, v_1) + v_1^2 (v_2 - v_0) \frac{\partial}{\partial v_1} \rho(v_{-1}, v_0, v_1) \\ &\quad + v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho(v_{-2}, v_{-1}, v_0). \end{aligned}$$

The application of (4.4) directly to  $\sigma$  yields nothing since no term depends on both the lowest shifted variable ( $v_{-2}$ ) and the highest shifted variable ( $v_2$ ). We therefore split  $\sigma$  into a term which depends on  $v_{-2}$  and a term that does not. We then apply (4.2) to the latter term.

Thus,

$$\begin{aligned} \sigma &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho(v_{-2}, v_{-1}, v_0) \\ &\quad + (\Delta + I) D^{-1} \left( v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \left( \rho(v_0, v_1, v_2) + \rho(v_{-1}, v_0, v_1) \right) \right) \\ &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho(v_{-2}, v_{-1}, v_0) \\ &\quad + (\Delta + I) \left( v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_{-2}, v_{-1}, v_0) \right) \right). \end{aligned}$$

Next, we update  $\sigma$  and  $K$  by removing the total difference term from  $\sigma$  and adding it to  $K$ . Thus,

$$\begin{aligned} \sigma &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \rho(v_{-2}, v_{-1}, v_0) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_{-2}, v_{-1}, v_0) \right), \end{aligned}$$

and

$$\begin{aligned} K &= v_0^2 (v_1 - v_{-1}) \frac{\partial}{\partial v_0} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_{-2}, v_{-1}, v_0) \right) \\ &\quad + v_1^2 (v_2 - v_0) \frac{\partial}{\partial v_1} \rho(v_{-1}, v_0, v_1) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \left( \rho(v_{-1}, v_0, v_1) + \rho(v_{-2}, v_{-1}, v_0) \right). \end{aligned}$$

Now we are ready to apply (4.4) to  $\sigma$ . Thus,

$$\frac{\partial^2}{\partial v_{-2} \partial v_1} \sigma = v_0^2 \frac{\partial^2}{\partial v_{-2} \partial v_0} \rho(v_{-2}, v_{-1}, v_0) - v_{-1}^2 \frac{\partial^2}{\partial v_{-1} \partial v_1} \rho(v_{-1}, v_0, v_1) = 0.$$

Note that we get (3.4) of our first derivation by applying to up-shifts to the above equation. Therefore, the solution is given by (3.5):

$$\rho = \rho^{(1)}(v_0, v_1) + c^{(1)} v_0 v_1^2 v_2.$$

This procedure is repeated until  $\sigma$  is reduced to 0. When this happens  $-K$  will be the associated flux. Hence,

$$\begin{aligned} \sigma &= c^{(1)} v_{-2} v_{-1}^3 v_0 (v_0 - 2v_{-2}) - v_{-2} v_{-1}^2 \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-1}, v_0) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-2}, v_{-1}) + c^{(1)} v_{-1}^2 v_0^3 v_1 \\ &\quad + v_{-1}^2 v_0 \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-1}, v_0) \\ &= c^{(1)} v_{-2} v_{-1}^3 v_0 (v_0 - 2v_{-2}) - v_{-2} v_{-1}^2 \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-1}, v_0) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-2}, v_{-1}) \\ &\quad + (\Delta + \mathbf{I}) \left( c^{(1)} v_{-2}^2 v_{-1}^3 v_0 + v_{-2}^2 v_{-1} \frac{\partial}{\partial v_{-2}} \rho^{(1)}(v_{-2}, v_{-1}) \right) \end{aligned}$$

by (4.2). As before, we update  $\sigma$  by moving the total derivative term into  $K$ . Therefore,

$$\begin{aligned} \sigma &= c^{(1)} v_{-2} v_{-1}^3 v_0 (v_0 - v_{-2}) - v_{-2} v_{-1}^2 \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-1}, v_0) \\ &\quad + v_{-1}^2 (v_0 - v_{-2}) \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-2}, v_{-1}) + v_{-2}^2 v_{-1} \frac{\partial}{\partial v_{-2}} \rho^{(1)}(v_{-2}, v_{-1}), \end{aligned}$$

and

$$\begin{aligned} K &= c^{(1)} v_{-1} v_0 (v_{-1} v_0^2 v_1 - v_0 v_1^2 v_2 - v_{-2} v_{-1}^2 v_0 - v_0^2 v_1^2 + 3v_{-2}^2 v_{-1}^2) \\ &\quad + v_0^2 (v_{-1} - v_1) \frac{\partial}{\partial v_0} \rho^{(1)}(v_{-1}, v_0) + v_{-2}^2 v_{-1} \frac{\partial}{\partial v_{-2}} \rho^{(1)}(v_{-2}, v_{-1}) \\ &\quad + v_{-1}^2 (v_{-2} - v_0) \left( \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-1}, v_0) + \frac{\partial}{\partial v_{-1}} \rho^{(1)}(v_{-2}, v_{-1}) \right). \end{aligned}$$

Applying (4.4) to  $\sigma$  yields

$$\begin{aligned} \frac{\partial^2 \sigma}{\partial v_{-2} \partial v_0} &= v_{-1}^2 \left( 2c^{(1)} v_{-1} (v_0 - v_{-2}) - \frac{\partial^2}{\partial v_{-1} \partial v_0} \rho^{(1)}(v_{-1}, v_0) \right. \\ &\quad \left. + \frac{\partial^2}{\partial v_{-2} \partial v_{-1}} \rho^{(1)}(v_{-2}, v_{-1}) \right) \\ &= 0. \end{aligned}$$

This equation is a shifted version of the  $v_2$  derivative of (3.3). In turn, its  $v_0$  derivative is a shifted version of (3.6). The solution is thus given by (3.7) after an appropriate shift:

$$\rho = \frac{1}{2}c^{(1)}v_0^2v_1^2 + \rho^{(\tau)}(v_0) + c^{(2)}v_0v_1 + c^{(1)}v_0v_1^2v_2.$$

At this stage

$$\sigma = v_{-2}^2v_{-1}\frac{d}{dv_{-2}}\rho^{(\tau)}(v_{-2}) - v_2v_{-1}^2\frac{d}{dv_{-1}}\rho^{(\tau)}(v_{-1}),$$

and

$$\begin{aligned} K &= v_{-2}^2v_{-1}\frac{d}{dv_{-2}}\rho^{(\tau)}(v_{-2}) - v_{-1}^2v_{-2}\frac{d}{dv_{-1}}\rho^{(\tau)}(v_{-1}) + v_0^3c^{(1)}v_{-1}v_1^2 \\ &\quad + v_{-1}^2v_0\frac{d}{dv_{-1}}\rho^{(\tau)}(v_{-1}) + c^{(1)}v_{-1}v_0^2v_1^2v_2 + v_0^2c^{(2)}v_{-1}v_1. \end{aligned}$$

One more application of (4.4) to  $\sigma$  yields

$$\frac{\partial^2\sigma}{\partial v_{-2}\partial v_{-1}} = \frac{d}{dv_{-2}}\left(v_{-2}^2\frac{d}{dv_{-2}}\rho^{(\tau)}(v_{-2})\right) - \frac{d}{dv_{-1}}\left(v_{-1}^2\frac{d}{dv_{-1}}\rho^{(\tau)}(v_{-1})\right) = 0.$$

Differentiating this equation with respect to  $v_{-2}$  yields a shifted version of (3.8). Thus the solution is given by (3.10):

$$\rho = \frac{1}{2}c^{(1)}(v_0^2v_1^2 + 2v_0v_1^2v_2) + c^{(2)}v_0v_1 + c^{(5)}\frac{1}{v_0} + c^{(6)}\log v_0.$$

Consequently,

$$\sigma = c^{(5)}v_{-2} - c^{(5)}v_{-1} = -c^{(5)}\Delta v_{-2},$$

which can be absorbed into  $K$  to yield  $\sigma = 0$ . Formally we could apply (4.2) to  $\sigma$  but in this case the result is obvious. Finally, the associated flux is

$$J = -K = -c^{(1)}v_{-1}v_0^2v_1^2(v_0 + v_2) - c^{(2)}v_{-1}v_0^2v_1 + c^{(5)}(v_0 + v_{-1}) - c^{(6)}v_{-1}v_0.$$

Splitting the density  $\rho$  and flux  $J$  according to the independent constants  $c^{(i)}$  we obtain the densities and fluxes listed at the end of Section 3.

## 5. Miura Map

In this section we investigate the connection between the KvM and mV lattices.

In contrast to the approach in Yamilov 1994, we construct an explicit homotopic deformation of the KvM lattice which preserves a conserved density. This deformation can then be restricted to retrieve the Muira map between the KvM and mV lattices. Following Muira *et al.* 1968, the recursive use of this map allows one to find the conserved densities for the KvM lattice order by order.

Starting from the KvM lattice (2.6),

$$\frac{du_0}{dt} = u_0(u_1 - u_{-1}),$$

consider the non-local map

$$u_0 = v_0 + \epsilon F(v_{-1}, v_0), \quad (5.1)$$

where we assume that  $F$  is analytic at  $\epsilon = 0$  (for notational simplicity, we suppress the dependency on  $\epsilon$ ) so that  $u_0 = v_0$  at  $\epsilon = 0$ . Under this transformation, the KvM lattice is deformed into

$$\frac{dv_0}{dt} + \epsilon \left( \left( \frac{\partial}{\partial v_{-1}} F(v_{-1}, v_0) \right) \frac{dv_{-1}}{dt} + \left( \frac{\partial}{\partial v_0} F(v_{-1}, v_0) \right) \frac{dv_0}{dt} \right) = (v_0 + \epsilon F(v_{-1}, v_0))(v_1 + \epsilon F(v_0, v_1)) - v_{-1} - \epsilon F(v_{-2}, v_{-1}). \quad (5.2)$$

This equation involves the shifted variables  $v_{-2}, v_{-1}, v_0$  and  $v_1$ . So, we obtain an implicit non-local lattice for  $v_0$  given by

$$\frac{dv_0}{dt} = g(v_{-1}, v_0, v_1). \quad (5.3)$$

Note that there is no guarantee that an explicit function  $g$  will exist. In other words, the KvM lattice may be mapped onto a lattice which is not of type (2.2). Again, for clarity, the  $\epsilon$  dependency of  $g$  is suppressed. Eq. (5.2) is regarded as an integrability condition on  $F$  so that  $v_0$  does indeed satisfy (5.3). Map (5.1) is viewed as deforming (2.6) which is recovered by setting the parameter  $\epsilon$  to 0.

The KvM lattice (2.6) has density  $\rho = \log u_0$ . We impose the additional condition on map (5.1) that it preserves this density. That is, for all  $\epsilon$  in a neighborhood of 0,  $\log v_0$  should be a density for (5.3). The resulting integrability condition is

$$\begin{aligned} E(D_t \log v_0) &= E \left( \frac{g(v_{-1}, v_0, v_1)}{v_0} \right) \\ &= -\frac{1}{v_0^2} g(v_{-1}, v_0, v_1) + \frac{1}{v_0} \frac{\partial}{\partial v_0} g(v_{-1}, v_0, v_1) \\ &\quad + \frac{1}{v_1} \frac{\partial}{\partial v_0} g(v_0, v_1, v_2) + \frac{1}{v_{-1}} \frac{\partial}{\partial v_0} g(v_{-2}, v_{-1}, v_0) \\ &= 0. \end{aligned} \quad (5.4)$$

Differentiating the above with respect to  $v_{-2}$  yields

$$\frac{\partial^2}{\partial v_0 \partial v_{-2}} g(v_{-2}, v_{-1}, v_0) = 0,$$

and so

$$g(v_{-1}, v_0, v_1) = g^{(1)}(v_{-1}, v_0) + g^{(2)}(v_0, v_1).$$

The  $v_{-1}$  derivative of (5.4) can be written in the form

$$\frac{\partial^2}{\partial v_{-1} \partial v_0} \left( \frac{1}{v_0} g^{(1)}(v_{-1}, v_0) + \frac{1}{v_{-1}} g^{(2)}(v_{-1}, v_0) \right) = 0.$$

Thus we obtain

$$g^{(2)}(v_{-1}, v_0) = -\frac{v_{-1}}{v_0} g^{(1)}(v_{-1}, v_0) + g^{(3)}(v_{-1}) + v_{-1} g^{(4)}(v_0).$$

Eq. (5.4) now reads

$$\frac{\partial}{\partial v_0} \left( \frac{1}{v_0} g^{(3)}(v_0) + g^{(4)}(v_0) \right) = 0,$$

and so

$$g^{(4)}(v_0) = -\frac{1}{v_0} g^{(3)}(v_0) + c,$$

where  $c$  is a constant. The integrability condition is now satisfied, and

$$g(v_{-1}, v_0, v_1) = g^{(1)}(v_{-1}, v_0) - \frac{v_0}{v_1} g^{(1)}(v_0, v_1) + g^{(3)}(v_0) + v_0 \left( -\frac{1}{v_1} g^{(3)}(v_1) + c \right).$$

Absorbing  $g^{(3)}(v_0)$  into  $g^{(1)}(v_{-1}, v_0)$ , the most general lattice of form (5.3) that has a density  $\log v_0$  is

$$\frac{d v_0}{d t} = G(v_{-1}, v_0) - \frac{v_0}{v_1} G(v_0, v_1) + c v_0, \quad (5.5)$$

where  $G$  is an arbitrary function of its variables. Note that for the choice  $c = 0$  and

$$G(v_{-1}, v_0) = -v_{-1}v_0 - v_0^2$$

(5.5) is the KvM lattice (2.6). For the special case  $c = 0$  and

$$G(v_{-1}, v_0) = -v_{-1}v_0^2$$

(5.5) reduces to the mV lattice (3.2).

Continuing with arbitrary  $G(v_{-1}, v_0)$ , we now ask if there exists a deformation (5.1) of the KvM lattice (2.6) such that the deformed lattice has the form (5.5) for all  $\epsilon$  in a neighborhood of 0. Obviously, this will further restrict the admissible functions  $F$  and  $G$ .

The necessary condition is obtained by substituting (5.5) into (5.2). Differentiating the resulting expression with respect to  $v_{-2}$  yields

$$\epsilon \left( \frac{\partial}{\partial v_{-1}} F(v_{-1}, v_0) \right) \frac{\partial}{\partial v_{-2}} G(v_{-2}, v_{-1}) = -(v_0 + \epsilon F(v_{-1}, v_0)) \epsilon \frac{\partial}{\partial v_{-2}} F(v_{-2}, v_{-1}).$$

Assuming that the deformation (5.1) is genuinely non-local, consequently, that  $F(v_{-1}, v_0)$  depends non-trivially on  $v_{-1}$ , we may rewrite this equation as

$$\frac{\frac{\partial}{\partial v_{-2}} G(v_{-2}, v_{-1})}{\frac{\partial}{\partial v_{-2}} F(v_{-2}, v_{-1})} = -\frac{v_0 + \epsilon F(v_{-1}, v_0)}{\frac{\partial}{\partial v_{-1}} F(v_{-1}, v_0)}.$$

Since the left hand side is independent of  $v_0$  and  $F$  is analytic at  $\epsilon = 0$ ,

$$F(v_{-1}, v_0) = v_0 F^{(1)}(v_{-1})$$

with

$$\frac{\frac{\partial}{\partial v_{-2}} G(v_{-2}, v_{-1})}{\frac{d}{d v_{-2}} F^{(1)}(v_{-2})} = -\frac{v_{-1} (1 + \epsilon F^{(1)}(v_{-1}))}{\frac{d}{d v_{-1}} F^{(1)}(v_{-1})}.$$

The right hand side is independent of  $v_{-2}$ . Thus,

$$G(v_{-2}, v_{-1}) = -\frac{v_{-1} (1 + \epsilon F^{(1)}(v_{-1}))}{\frac{d}{dv_{-1}} F^{(1)}(v_{-1})} F^{(1)}(v_{-2}) + G^{(1)}(v_{-1}).$$

Differentiating (5.2) with respect to both  $v_{-1}$  and  $v_1$  yields

$$\begin{aligned} v_1^2 F^{(1)}(v_0) \frac{d^2 F^{(1)}(v_1)}{dv_1^2} (1 + \epsilon F^{(1)}(v_1)) \\ + \left( \frac{d F^{(1)}(v_1)}{dv_1} \right)^2 \left( v_1^2 + v_1 \frac{d G^{(1)}(v_1)}{dv_1} - G^{(1)}(v_1) \right) = 0. \end{aligned}$$

A further differentiation with respect to  $v_0$  gives

$$v_1^2 \left( \frac{d F^{(1)}(v_0)}{dv_0} \right) \left( \frac{d^2 F^{(1)}(v_1)}{dv_1^2} \right) (1 + \epsilon F^{(1)}(v_1)) = 0,$$

and so

$$F^{(1)}(v_1) = v_1 + a$$

for some constant  $a$ . The coefficient of  $v_1$  can be taken 1 by rescaling  $\epsilon$  if necessary. Differentiating (5.2) twice with respect to  $v_{-1}$  yields

$$\epsilon v_0 \left( 2 + \frac{d^2}{dv_{-1}^2} G^{(1)}(v_{-1}) \right) = 0,$$

and so

$$G^{(1)}(v_{-1}) = -v_{-1}^2 + b^{(1)}v_{-1} + b^{(2)}$$

for some constants  $b^{(1)}$  and  $b^{(2)}$ . Equation (5.2) now reads

$$-\left( c + \epsilon a c + \epsilon b^{(2)} \right) v_0 v_1 - 2 c \epsilon v_{-1} v_0 v_1 + b^{(2)} (1 + a \epsilon) (v_0 - v_1) + b^{(2)} \epsilon v_{-1} v_0 = 0.$$

Thus  $c = 0$  and  $b^{(2)} = 0$ . Therefore the required deformation (5.1) is

$$u_0 = v_0 + \epsilon(v_{-1} + a)v_0 = (1 + \epsilon a)v_0 + \epsilon v_{-1} v_0, \quad (5.6)$$

and the deformed lattice (5.5) is

$$\frac{dv_0}{dt} = v_0(1 + \epsilon a + \epsilon v_0)(v_1 - v_{-1}). \quad (5.7)$$

For  $a \neq 0$ , (5.7) is the mV lattice (3.2) when  $\epsilon = -a^{-1}$ . Thus the deformation (5.6) is a homotopy between the KvM and the mV lattices. For  $a = 0$  the mV lattice is obtained in the limit  $\epsilon \rightarrow \infty$ . For  $a = -1$  and  $\epsilon = 1$ , the upshifted version of (5.6) is a well-known nonlinear transformation (Adler *et al.* 1999, Levi & Yamilov 1997),

$$u_0 = v_0 v_1, \quad (5.8)$$

which maps solutions of (3.2) into solutions of (2.6).

Map (5.6) is the analogue of the famous Miura transformation connecting the KdV and modified KdV equations (Miura 1968). For other non-zero values of  $\epsilon$ , (5.7) is a combined KvM–mV lattice.

Finally, we use the homotopy (5.6) to generate conserved densities of the KvM lattice. The strategy is similar to the one for the continuous case (Miura *et al.* 1968) where the Miura transformation was used to find low-order conserved densities for the KdV equation.

Since  $\log v_0$  is a density for all the deformed equations (that is, independent of  $\epsilon$ ), we can compute the densities of the KvM lattice (2.6) by recursively substituting (5.6) into  $\log v_0$  and expanding in powers of  $\epsilon$ . Each coefficient must therefore be a density of the KvM lattice. Conveniently, we set  $a = 0$ . The deformation (5.6) can be recursively solved for  $v_0$

$$v_0 = \frac{u_0}{1 + \epsilon v_{-1}},$$

and so

$$\rho = \log v_0 = \log \left( \frac{u_0}{1 + \epsilon v_{-1}} \right) = \log u_0 + O(\epsilon).$$

Not surprisingly,  $\log u_0$  is a density of the KvM lattice. Repeating this procedure, we obtain

$$\begin{aligned} \rho &= \log \left( \frac{u_0}{1} + \frac{\epsilon u_{-1}}{1 + \epsilon v_{-2}} \right) \\ &= \log u_0 - u_{-1} \epsilon + O(\epsilon^2). \end{aligned}$$

Therefore  $u_{-1}$  (equivalently  $u_0$ ) is also a density of the KvM lattice. Repeating this procedure once more,

$$\begin{aligned} \rho &= \log \left( u_0 \left( 1 + \epsilon u_{-1} \left( 1 + \frac{\epsilon u_{-2}}{1 + \epsilon v_{-3}} \right)^{-1} \right)^{-1} \right) \\ &= \log u_0 - u_{-1} \epsilon + \frac{1}{2} u_{-1} (2 u_{-2} + u_{-1}) \epsilon^2 + O(\epsilon^3). \end{aligned}$$

After a shift, the next density of the KvM lattice is  $u_0(2u_{-1} + u_0)$ .

Although cumbersome, higher-order densities of the KvM lattice could be obtained this way.

Conversely, apart from density  $\frac{1}{v_0}$ , the densities (3.11) can be obtained from the densities of the KvM lattice using (5.8).

## 6. Conclusion

The two methods discussed in this paper allow one to find conserved densities and fluxes of differential-difference equations.

Starting with a conserved density of a particular form, the first method relied on finding a closed-form solution of the integrability condition which follows from the application of the discrete Euler operator. For the example of the mV lattice, the solution was obtained by repeated differentiation and separation of variables. The more general the assumed form of the density, the more cumbersome the computations are, and the more suitable they would be for computer algebra.

The second, more algorithmic method was used in two different ways. If the density  $\rho$  is known, we found the associated flux by splitting  $D_t \rho$  into a piece that lies in the image of the operator  $\Delta$  and a piece that does not. Sophisticated use of splitting technique allowed us to simultaneously find densities and fluxes at increased computational effort. The method was illustrated for the mV lattice.

A homotopic deformation of the KvM lattice was constructed. Further restriction of the deformation lead to the Miura map which links the KvM and mV lattices. Although cumbersome, recursive use of the Miura map allowed us to generate densities of the KvM lattice order by order.

MH wishes to thank the Department of Mathematical and Computer Sciences of the Colorado School of Mines for its hospitality during his sabbatical visit.

WH gratefully acknowledges the support of the National Science Foundation. This effort was sponsored in part under Grant CCR-9901929.

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