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THE RAMAN-MATH EQUATIONS REVISITED. II. OBLIQUE INCIDENCE OF THE LIGHT -
BRAGG REFLECTION

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The N-th order approximation (NOA) method is applied to the diffraction of light by ultrasonic. Of, in cases of oblique incidence of the light. A truncated system of Raman-Math equations is integrated numerically by means of an eigenvalue method. The results obtained theoretical curves for the light intensities, with varying Raman-Math parameter \( q \) and different Klein-Cook parameters \( C \), are compared with previous approximations and experimental data. The theoretical predicted symmetries of the diffraction spectra with respect to the various Bragg angles are verified.

INTRODUCTION

Recently, the intensities of the diffracted lightwaves in acoustooptical problems were successfully determined by analytical-numerical solution of the Raman-Math (RM) equations in the simplified case of normal incidence of light.\(^1\)\(^2\)\(^3\)\(^4\). For that, the authors used a N-th order approximation (NOA) method, introduced by Nagendra Math\(^5\) for N=1 and extended by Hartnas\(^6\). In the present paper we treat the approximate solutions of a similar RM system, however comprising all cases of oblique light incidence. A solution in the NOA can be obtained from a truncated Raman system of 2N+1 equations, relating the amplitudes \( A_0, A_1, \ldots, A_N, \ldots \) thus ignoring the amplitudes \( A_{2N+1}, \ldots \). The solution of this finite system is then reduced in a classical way to an eigenvalue problem, distinctly suited for numerical treatment. The case N=1 is already long due to Nagahousiana\(^7\)\'s actual computer facilities however, we have calculated for large values of N, yielding nearly exact solutions not only in the RM and Bragg regimes but also in the intermediate region. In the latter two cases the diffraction spectrum is symmetric with respect to the zero order line. This symmetry with regard to the intensities clearly results in an unequal number of positive and negative lines, determined by the computer routine itself. A comparison is made with the experiments of Mayer\(^8\) and Klein et al.\(^9\). The light intensities of zero and first order are shown versus the angle of incidence for different values of the Klein-Cook parameter \( h = 1 \) and with the RM parameter \( h = 2 \). The fitting of the theoretical curves with the experimental points is excellent.

THE NOA METHOD FOR OBLIQUE INCIDENCE

Restricting ourselves to the problem of a light beam diffracted by a progressive ultrasonic wave in an isotropic medium, where the direction of propagation of the light beam makes an angle \( \theta \) with the ultrasonic wave front, the amplitudes \( A_n(\theta) \) of the diffracted light waves satisfy the infinite set of RM equations \(^{12}\)\(^{13}\):

\[
\frac{dA_n}{d\theta} = (\eta - \text{im}(\pi n)k)A_n,
\]

with boundary conditions

\[
A_n(\theta) = 0, \quad n = 0, 1, 2, \ldots
\]

In (1), \( \xi = \sqrt{\eta^2 - \pi^2 n^2} \), \( \xi = \sqrt{\eta^2 - \pi^2 n^2} \cos \theta \), \( \xi = \sqrt{\eta^2 - \pi^2 n^2} \sin \theta \), where \( \xi \) is the peak variation of the relative permittivity \( \varepsilon \) of the medium, \( \eta \) the wave length of light in vacuum, \( \lambda \) the wave length of ultrasonic. If \( \phi = \eta \), with \( \phi = \eta \), then

\[
\phi(x) = \frac{\phi(x)}{\sqrt{\eta^2 - \pi^2 n^2}} \sin \theta
\]

where \( \phi(x) \) is called the Bragg angle of order \( p \). The \( x \)-axis is being parallel with the sound wave front; \( s = 0 \) and \( s = \lambda \) correspond with the boundaries of the ultrasonic field. In what follows we shall make use of the RM parameter \( v = C\lambda / s \) and the Klein-Cook parameter \( q = \eta \), the latter being independent of \( \xi \).

Since in practice \( v \) is very small we set \( v \approx 1 \). In the NOA method one neglects the energy in the diffraction orders higher than \( N \), i.e. \( A_{N+1}(\theta) = A_{N+2}(\theta) = \cdots = 0 \). Hence (1) is replaced by the truncated system of 2N+1 equations

\[
2\eta \frac{dA_n}{d\theta} + (\eta - \text{im}(\pi n)k)A_n = \text{im}(\pi n)A_n,
\]

with boundary conditions

\[
A_n(\theta) = 0, \quad n = 0, 1, 2, \ldots
\]

Projecting this solution

\[
A_n = \bar{A}_n \exp(i\pi n \cos \theta), \quad n = 0, 1, 2, \ldots
\]

providing the integration of system (3) is then reduced to an eigenvalue problem with matrix equation

\[
(M+\pi X)A = \bar{C} \bar{X}.
\]

The Hermitian (2N+1) by (2N+1) matrix \( M \) has diagonal elements \( M_{ij} = p(\pi+1)(\pi+2) \), \( p = 1, 2, \ldots, 2N+1 \), and off-diagonal elements \( M_{ij} = 0 \), \( p = 1, 2, \ldots, 2N+1 \), and off-diagonal elements \( M_{ij} = 0 \), \( p = 2, 3, \ldots, 2N+1 \), the remaining elements being zero; \( X \) is the 2N+1 unit matrix and \( A = A_{N+1} \). For a nonzero vector solution \( A \), the eigenvalues \( \lambda \) must be the 2N+1 obviously real roots of the characteristic equation

\[
\det(M+\pi X) = 0.
\]

Next, the eigenvector \( A(\lambda) = A(\lambda)A_{N+1} \) associated with the eigenvalue \( \lambda \) can be determined from the linear homogeneous system (5). The general solution of the linear system (5) may then be written as

\[
A_{(\lambda)} = \sum_{k=1}^{2N+1} C_{(\lambda)} \exp(2\pi i n \cos \theta) s_n, \quad n = 0, 1, 2, \ldots
\]

Regarding the boundary conditions (2), the 2N+1 real constants \( C_{(\lambda)} \) follow from

\[
C_{(\lambda)} = \delta_{n0}, \quad n = 0, 1, 2, \ldots
\]

Finally, one can calculate the intensities at \( x = L \), yielding
The characteristic equation (7) of degree 2N+1 in \( a \) can be solved analytically only for \( N=1 \); explicit expressions were obtained by Nagabhushana Rao \(^8\); otherwise the problem has to be treated numerically in the following steps: (i) determine the eigenvalues and eigenvectors of matrix \( M \); (ii) solve the linear system (6) for \( C_k \); (iii) substitute these results into (10).

**NUMERICAL RESULTS AND DISCUSSION**

(i) In Fig. 1 we compare \( I_0 \) and \( I_1 \) versus \( \psi \) obtained from the NOA method (for \( N=7 \)) with the experimental results of Mayer \(^9\) for \( \Psi=0.28 \). The fitting of both curves is very good. A similar result is obtained for \( \Psi=0.36 \). The purpose of Mayer's experiments was to show the deviation from BN's elementary theory \(^9\), confirmed here theoretically. Similar results were also obtained by Leroy and Blomme \(^15\) by a so-called NOA-GA method.

(ii) Further we compare \( I_0 \) and \( I_1 \) versus \( s \), calculated with the NOA method (\( N=7 \), although \( N=5 \) suffices for all values considered) with the experimental points obtained by Klein et al. \(^5\) for different combinations of \( q \)'s (0.5, 1.2, 2.5, 3.75, 6.28 and 9.32) and \( \psi = (2 \text{ and } 3) \). In Figs. 2, 3, 4, 5 we represent our numerical and Klein's experimental results respectively for \( Q=0.57 \) and \( \psi = (2.25 \text{ and } 3), Q=0.28 \) and \( \psi = 2, Q=9.3 \) and \( \psi = 3 \). There is excellent agreement with the measured values. Our curves fit the data even better than those calculated by Klein et al. using a direct numerical integration of the BE equations. Furthermore the numerical integration used here results in accurate plots for even larger values of \( \psi \). From the figures the following theoretical properties are confirmed:

- Intensities \( I_0 \) are odd when \( n \) even and \( I_1 \) are even when \( n \) odd.
- Intensity distributions of \( I_0 \) are symmetric about \( s=0 \).
- Intensity distribution of \( I_1 \) is not symmetric about \( s=0 \).

(iii) In addition to this, for a comparison with previous experiments we have calculated the intensities \( I_0, I_1, I_2 \) versus \( \psi \), for \( \psi = 2.25 \) with \( \psi = 1 \) and \( \psi = 2 \), and \( \psi = 2.25 \), and \( \psi = 3 \) (Figs. 6 and 7), showing clearly the symmetry of the spectrum. Up to \( \psi = 10 \), \( N=5 \) largely suffices and even \( N=4 \) would have given excellent results.

(iv) Finally, we have computed \( I_0, I_1 \) versus \( \psi \) for \( \Psi=1 \), namely \( Q=0.1 \), with \( \psi = 1 \), \( \psi = 2 \), and \( \psi = 3 \) using the NOA method (\( N=8 \)). Compared with the formulae from the elementary BN theory, the agreement here is good and \( I_1 = I_0 = J_2(\psi) (n=0,1) \), \( I_1 = \psi \cos \theta(\Psi=1/2) \) (Fig. 8).

These results are displayed in the following diagrams (Fig. 8). This case the symmetry of the diffraction pattern is also displayed.

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