A note on the Zakharov equation
and Lie symmetry vector fields

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Abstract. The construction of Lie symmetry vector fields of a partial differential equation is well described in literature. In this note we evaluate the Lie symmetry vector fields for the Zakharov equation. We also consider an extended Zakharov equation.

In the study of Langmuir solitons and their stability Zakharov [1] derived the following system of partial differential equations

\[ i \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} = nE \]  \hspace{1cm} (1a)

\[ \frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2} |E|^2. \]  \hspace{1cm} (1b)

The wave fields $E$ and $n$ are complex and real, respectively. Several authors [2-6] studied this system. System (1) has an important particular solution: the Langmuir soliton

\[ E(x, t) = E_0(x - ct) \exp \left[ \frac{i c}{2} + i \left( \lambda^2 - \frac{c^2}{4} \right) t \right] \]  \hspace{1cm} (2a)
where
\[
E_0(x - ct) = \frac{\lambda \sqrt{2(1 - c)^2}}{\cosh \lambda (x - x_0 - ct)}
\]  
(2b)

We determine the Lie symmetry vector fields for (1). To find the Lie symmetries of system (1) we set \(E = u + iv\), where \(u\) and \(v\) are real fields. Then we obtain from (1)

\[
\frac{\partial v}{\partial t} - \frac{\partial^2 u}{\partial x^2} + nu = 0
\]  
(3a)
\[
\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} - nv = 0
\]  
(3b)
\[
\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} - 2 \left( u \frac{\partial^2 u}{\partial x^2} + \left( \frac{\partial u}{\partial x} \right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left( \frac{\partial v}{\partial x} \right)^2 \right) = 0.
\]  
(3c)

The evaluation of Lie symmetry vector fields of differential equations is well described in literature [7 – 14]. We adopt the jet bundle formalism [12, 13]. Within the jet bundle formalism we consider instead of (1) the submanifold

\[
F_1(x, t, u, v, n, u_x, \ldots) \equiv v_t - u_{xx} + nu = 0
\]  
(4a)
\[
F_2(x, t, u, v, n, u_x, \ldots) \equiv u_t + v_{xx} - nv = 0
\]  
(4b)
\[
F_3(x, t, u, v, n, u_x, \ldots) \equiv n_t - n_{xx} - 2(uu_{xx} + (u_x)^2 + vv_{xx} + (v_x)^2) = 0
\]  
(4c)

and its differential consequences. A Lie symmetry vector field is given by

\[
V = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v} + \eta_3 \frac{\partial}{\partial n}
\]  
(5)

where \(\xi_1, \xi_2, \eta_1, \eta_2\) and \(\eta_3\) are functions of \(x, t, u, v, n\). The corresponding vertical vector field is given by

\[
V_v = (-\xi_1 u_x - \xi_2 u_t + \eta_1) \frac{\partial}{\partial u} + (-\xi_1 v_x - \xi_2 v_t + \eta_2) \frac{\partial}{\partial v} + (-\xi_1 n_x - \xi_2 n_t + \eta_3) \frac{\partial}{\partial n}.
\]  
(6)

The determining equations for the smooth functions \(\xi_1, \xi_2, \eta_1, \eta_2\) and \(\eta_3\) are given by \(L_{V_v} F = 0\), where \(L_{V_v} (.)\) denotes the Lie derivative. The determining equations have been evaluated with different computer algebra packages.
(see [7, 14, 15] for a discussion). From the determining equations we find that the Lie algebra of the Lie symmetry vector fields is spanned by the following generators

\[
Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial t}, \quad Z_3 = -u \frac{\partial}{\partial v} + v \frac{\partial}{\partial u} \quad (7)
\]

\[
Z_4 = -ut \frac{\partial}{\partial v} + vt \frac{\partial}{\partial u} + \frac{\partial}{\partial n} \equiv t Z_3 + \frac{\partial}{\partial n} \quad (8)
\]

\[
Z_5 = -\frac{u}{2} t^2 \frac{\partial}{\partial v} + \frac{v}{2} t^2 \frac{\partial}{\partial u} + t \frac{\partial}{\partial n} \equiv \frac{t^2}{2} Z_3 + t \frac{\partial}{\partial n}. \quad (9)
\]

Obviously \( Z_1 \) and \( Z_2 \) are related to the space and time translation, respectively. The symmetry vector field \( Z_3 \) is related to a rotation in the \( u-v \) space. The non-vanishing commutators are given by \([Z_2, Z_4] = Z_3, \ [Z_2, Z_5] = Z_4\).

Recently, Hadouaj et al [5] have studied the generalized Zakharov equation

\[
i \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + 2\lambda |E|^2 E + 2nE = 0 \quad (10a)
\]

\[
\frac{\partial^2 n}{\partial t^2} - c^2 \frac{\partial^2 n}{\partial x^2} + \mu \frac{\partial^2}{\partial t \partial x^2} |E|^2 - \gamma \frac{\partial^3 n}{\partial t \partial x^2} = 0. \quad (10b)
\]

where \( c, \lambda, \mu \) and \( \gamma \) are real parameters. If \( \lambda \neq 0 \), then we assume that direct self-interaction of the dispersive wave is presumed. For this generalized Zakharov system we find with the approach described above the Lie symmetry vector fields are given by

\[
Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial t}, \quad Z_3 = -u \frac{\partial}{\partial v} + v \frac{\partial}{\partial u} \quad (11)
\]

\[
Z_4 = 2ut \frac{\partial}{\partial v} - 2vt \frac{\partial}{\partial u} + \frac{\partial}{\partial n} \quad (12)
\]

\[
Z_5 = ut^2 \frac{\partial}{\partial v} - vt^2 \frac{\partial}{\partial u} + t \frac{\partial}{\partial n}. \quad (13)
\]
REFERENCES

13) Rogers C. and Shadwick W. F. Bäcklund Transformations and Their Applications (New York, Academic, 1982)