Symbolic Computation of Travelling Wave Solutions
of Nonlinear Partial Differential Equations
and Differential-Difference Equations

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Thursday, December 23, 2004, 14:30

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Research supported in part by NSF
under Grants DMS-9912293 and CCR-9901929
OUTLINE

Purpose & Motivation

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Typical Examples

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Demo of Mathematica Software: DDESpecialSolutions.m

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Purpose & Motivation

- **Develop** and implement various **methods** to find closed form solutions of nonlinear PDEs and DDEs: direct methods, Lie symmetry methods, similarity methods, etc.

- Fully **automate** the hyperbolic and elliptic function methods to compute travelling solutions of nonlinear PDEs.

- Fully **automate** the hyperbolic tanh method to compute travelling wave solutions of nonlinear differential-difference equations (DDEs or lattices).

- **Class** of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations, and lattices.

  **Examples PDEs:** Korteweg-de Vries, Boussinesq, and Kuramoto-Sivashinsky equations.
  Fisher and FitzHugh-Nagumo equations.

  **Examples ODEs:** Duffing and nonlinear oscillator equations.

  **Examples DDEs:** Volterra, Toda, and Ablowitz-Ladik lattices.

- **PDEs:** Solutions of tanh (kink) or sech (pulse) type **model** solitary waves in fluids, plasmas, circuits, optical fibers, bio-genetics, etc.

  **DDEs:** discretizations of PDEs, lattice theory, queing and network problems, solid state and quantum physics.

- **Benchmark** solutions for numerical PDE and DDE solvers.
• **Research aspect:** Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations and lattices.

• **Educational aspect:** Software as course ware for courses in nonlinear PDEs and DDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.

  REU projects of NSF. Extreme Programming!

• **Users:** scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.
Typical Examples of ODEs and PDEs

- The Duffing equation:
  \[ u'' + u + \alpha u^3 = 0 \]
  Solutions in terms of elliptic functions:
  \[ u(x) = \pm \sqrt{\frac{c_1^2 - 1}{\alpha}} \text{cn}(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}), \]
  and
  \[ u(x) = \pm \sqrt{2(c_1^2 - 1)} \text{sn}(c_1 x + \Delta; 1 - \frac{c_1^2}{c_1^2}). \]

- The Korteweg-de Vries (KdV) equation:
  \[ u_t + 6\alpha uu_x + u_{3x} = 0. \]
  Solitary wave solution:
  \[ u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \Delta], \]
  or, equivalently,
  \[ u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \text{sech}^2 [c_1 x + c_2 t + \Delta]. \]
  Cnoidal wave solution:
  \[ u(x, t) = \frac{4c_1^3(1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \text{cn}^2 (c_1 x + c_2 t + \Delta; m), \]
  modulus \( m \).
• The modified Korteweg-de Vries (mKdV) equation:

\[ u_t + \alpha u^2 u_x + u_{3x} = 0. \]

Solitary wave solution:

\[ u(x, t) = \pm \sqrt{\frac{6}{\alpha}} c_1 \sech \left[ c_1 x - c_1^3 t + \Delta \right]. \]

• Three-dimensional modified Korteweg-de Vries equation:

\[ u_t + 6u^2 u_x + u_{xyz} = 0. \]

Solitary wave solution:

\[ u(x, y, z, t) = \pm \sqrt{c_2 c_3} \sech \left[ c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta \right]. \]

• The combined KdV-mKdV equation:

\[ u_t + 6\alpha uu_x + 6\beta u^2 u_x + \gamma u_{3x} = 0. \]

Real solitary wave solution:

\[ u(x, t) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\gamma}{\beta}} c_1 \sech \left( c_1 x + \frac{c_1}{2\beta}(3\alpha^2 - 2\beta \gamma c_1^2) t + \Delta \right). \]

Complex solutions:

\[ u(x, t) = -\frac{\alpha}{2\beta} \pm i \sqrt{\frac{\gamma}{\beta}} c_1 \tanh \left( c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + 4\beta \gamma c_1^2) t + \Delta \right), \]

\[ u(x, t) = -\frac{\alpha}{2\beta} + \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 \left( \sech \xi \pm i \tanh \xi \right), \]

and

\[ u(x, t) = -\frac{\alpha}{2\beta} - \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 \left( \sech \xi \mp i \tanh \xi \right) \]

with \( \xi = c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + \beta \gamma c_1^2) t + \Delta. \)
• The Fisher equation:

\[ u_t - u_{xx} - u(1-u) = 0. \]

Solitary wave solution:

\[ u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \xi + \frac{1}{4} \tanh^2 \xi, \]

with

\[ \xi = \pm \frac{1}{2 \sqrt{6}} x \pm \frac{5}{12} t + \Delta. \]

• The generalized Kuramoto-Sivashinski equation:

\[ u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0. \]

Solitary wave solutions
(ignoring symmetry \( u \rightarrow -u, x \rightarrow -x, \sigma \rightarrow -\sigma \)):

For \( \sigma = 4 \):

\[ u(x, t) = 9 - 2c_2 - 15 \tanh \xi \left( 1 + \tanh \xi - \tanh^2 \xi \right) \]

with \( \xi = \frac{x}{2} + c_2 t + \Delta. \)

For \( \sigma = \frac{12}{\sqrt{47}} \):

\[ u(x, t) = \frac{45 \mp 4418c_2}{47 \sqrt{47}} \pm \frac{45}{47 \sqrt{47}} \tanh \xi - \frac{45}{47 \sqrt{47}} \tanh^2 \xi \pm \frac{15}{47 \sqrt{47}} \tanh^3 \xi \]

with \( \xi = \pm \frac{1}{2 \sqrt{47}} x + c_2 t + \Delta. \)
For $\sigma = 16/\sqrt{73}$:
\[
 u(x, t) = \frac{2 (30 \mp 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \tanh \xi - \frac{60}{73\sqrt{73}} \tanh^2 \xi \pm \frac{15}{73\sqrt{73}} \tanh^3 \xi
\]
with $\xi = \pm \frac{1}{2\sqrt{73}} x + c_2 t + \Delta$.

For $\sigma = 0$:
\[
 u(x, t) = -2 \left[ \frac{19}{11} c_2 - \frac{135}{19} \sqrt{\frac{11}{19}} \tanh \xi + \frac{165}{19} \sqrt{\frac{11}{19}} \tanh^3 \xi \right]
\]
with $\xi = \frac{1}{2\sqrt{11}} \left[ \frac{19}{11} x + c_2 t + \Delta \right]$.

- **The Boussinesq (wave) equation:**
  \[
  u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0,
  \]
  or written as a first-order system ($v$ auxiliary variable):
  \[
  u_t + v_x = 0,
  
  v_t + u_x - 3uu_x - \alpha u_{3x} = 0.
  \]

  Solitary wave solution:
  \[
  u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta],
  
  v(x, t) = b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta].
  \]

- **The Broer-Kaup system:**
  \[
  u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xyy} = 0,
  
  v_t + 2(uv)_x + v_{xx} = 0.
  \]

  Solitary wave solution:
  \[
  u(x, t) = -\frac{c_3}{2c_1} + c_1 \tanh [c_1 x + c_2 y + c_3 t + \Delta],
  
  v(x, t) = c_1 c_2 - c_1 c_2 \tanh^2 [c_1 x + c_2 y + c_3 t + \Delta].
  \]
• System of three nonlinear coupled equations (Gao & Tian, 2001):

\[
\begin{align*}
    u_t - u_x - 2v &= 0, \\
    v_t + 2uw &= 0, \\
    w_t + 2uv &= 0.
\end{align*}
\]

Solutions:

\[
\begin{align*}
    u(x, t) &= \pm c_2 \tanh \xi, \\
    v(x, t) &= \mp \frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi, \\
    w(x, t) &= -\frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi,
\end{align*}
\]

and

\[
\begin{align*}
    u(x, t) &= \pm ic_2 \sech \xi, \\
    v(x, t) &= \pm \frac{1}{2} ic_2 (c_1 - c_2) \tanh \xi \sech \xi, \\
    w(x, t) &= \frac{1}{4} c_2 (c_1 - c_2) \left( 1 - 2 \sech^2 \xi \right),
\end{align*}
\]

and also

\[
\begin{align*}
    u(x, t) &= \pm \frac{1}{2} ic_2 \left( \sech \xi + i \tanh \xi \right), \\
    v(x, t) &= \pm \frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right), \\
    w(x, t) &= -\frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right)
\end{align*}
\]

with \( \xi = c_1 x + c_2 t + \Delta \).
• Nonlinear sine-Gordon equation (light cone coordinates):
\[ \Phi_{xt} = \sin \Phi. \]

Set \( u = \Phi_x, \ v = \cos(\Phi) - 1, \)
\[ u_{xt} - u - uv = 0, \]
\[ u_t^2 + 2v + v^2 = 0. \]

Solitary wave solution (kink):
\[ u = \pm \frac{1}{\sqrt{-c}} \text{sech}\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right], \]
\[ v = 1 - 2 \text{sech}^2\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right]. \]

Solution:
\[ \Phi(x, t) = \int u(x, t)dx = \pm 4 \arctan \left[ \exp \left( \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right) \right]. \]

• ODEs from quantum field theory:
\[ u_{xx} = -u + u^3 + avu^2, \]
\[ v_{xx} = bv + cv^3 + av(u^2 - 1). \]

Solitary wave solutions:
\[ u = \pm \tanh\left[ \frac{a^2 - c}{2(a - c)}x + \Delta \right], \]
\[ v = \pm \left[ \frac{1 - a}{a - c} \right] \text{sech}\left[ \frac{a^2 - c}{2(a - c)}x + \Delta \right], \]
provided \( b = \sqrt{\frac{a^2 - c}{2(a - c)}}. \)
Typical Examples of DDEs (lattices)

• The Volterra lattice:

\[
\begin{align*}
\dot{u}_n &= u_n(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n).
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= -c_1 \coth(d_1) + c_1 \tanh[d_1 n + c_1 t + \delta], \\
v_n(t) &= -c_1 \coth(d_1) - c_1 \tanh[d_1 n + c_1 t + \delta].
\end{align*}
\]

• The Toda lattice:

\[
\begin{align*}
\ddot{u}_n &= (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}).
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= a_{10} \pm \sinh(d_1) \tanh[d_1 n \pm \sinh(d_1) t + \delta].
\end{align*}
\]

• The Relativistic Toda lattice:

\[
\begin{align*}
\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= -\frac{1}{\alpha} - c_1 \coth(d_1) + c_1 \tanh[d_1 n + c_1 t + \delta], \\
v_n(t) &= \frac{c_1 \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh[d_1 n + c_1 t + \delta].
\end{align*}
\]
• The Ablowitz-Ladik lattice:

\[
\dot{u}_n(t) = (\alpha + u_nv_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\
\dot{v}_n(t) = -(\alpha + u_nv_n(v_{n+1} + v_{n-1}) + 2\alpha v_n.
\]

Travelling wave solution:

\[
u_n(t) = \frac{\alpha \sinh^2(d_1)}{a_{21}} (\pm 1 - \tanh[d_1n + 2\alpha \sinh^2(d_1) + \delta]), \\
v_n(t) = a_{21}(\pm 1 + \tanh[d_1n + 2\alpha \sinh^2(d_1)t + \delta]).
\]

• 2D Toda lattice:

\[
\frac{\partial^2 u_n}{\partial x \partial t}(x, t) = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}).
\]

Travelling wave solution:

\[
u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh\left[d_1n + \frac{\sinh^2(d_1)}{c_2}x + c_2t + \delta\right].
\]

• Hybrid lattice:

\[
\dot{u}_n(t) = (1 + \alpha u_n + \beta u_n^2)(u_{n-1} - u_{n+1}),
\]

Travelling wave solution:

\[
u_n(t) = -\alpha \pm \frac{\sqrt{\alpha^2 - 4\beta}}{2\beta} \tanh(d_1) \tanh\left[d_1n + \frac{\alpha^2 - 4\beta}{2\beta} \tanh(d_1)t + \delta\right].
\]
Algorithm for Tanh Solutions for system of PDEs

System of nonlinear PDEs of order $m$

$$\Delta(u(x), u'(x), u''(x), \ldots, u^{(m)}(x)) = 0.$$ 

Dependent variable $u$ has $M$ components $u_i$ (or $u, v, w, \ldots$). 
Independent variable $x$ has $N$ components $x_j$ (or $x, y, z, \ldots, t$).

**Step T1:** 

- Seek solution $u(x) = U(T)$, with 
  $$T = \tanh \xi = \tanh \left[ \sum_j c_j x_j + \delta \right].$$ 

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, all derivative of $T$ are polynomial in $T$. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule 
  $$\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d\bullet}{dT} = c_j (1 - T^2) \frac{d\bullet}{dT}$$

  Produces a nonlinear system of ODEs 
  $$\Delta(T, U(T), U'(T), U''(T), \ldots, U^{(m)}(T)) = 0.$$ 

**NOTE:** Compare with the ultra-spherical (linear) ODE: 

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0$$

with integer $n \geq 0$ and $\alpha$ real. Includes:

* Legendre equation ($\alpha = \frac{1}{2}$),
* ODE for Chebyshev polynomials of type I ($\alpha = 0$),
* ODE for Chebyshev polynomials of type II ($\alpha = 1$).
• Example: For the Boussinesq system

\[
\begin{align*}
  u_t + v_x &= 0, \\
  v_t + u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
\]

after cancelling common factors \(1 - T^2\),

\[
\begin{align*}
  c_2 U' + c_1 V' &= 0, \\
  c_2 V' + c_1 U' - 3c_1 UU' &+ \alpha c_1^3 \left[ 2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2 U''' \right] = 0.
\end{align*}
\]

**Step T2:**

• Seek polynomial solutions

\[
U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.
\]

Determine the highest exponents \(M_i \geq 1\).

Substitute \(U_i(T) = T^{M_i}\) into the LHS of ODE.

Gives polynomial \(P(T)\).

For every \(P_i\) consider all possible balances of the highest exponents in \(T\).

Solve the resulting linear system(s) for the unknowns \(M_i\).

• Example: Balance highest exponents for the Boussinesq system

\[
M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.
\]

So, \(M_1 = M_2 = 2\).

Hence,

\[
\begin{align*}
  U(T) &= a_{10} + a_{11} T + a_{12} T^2, \\
  V(T) &= a_{20} + a_{21} T + a_{22} T^2.
\end{align*}
\]
Step T3:

- Derive algebraic system for the unknown coefficients $a_{ij}$ by setting to zero the coefficients of the power terms in $T$.

- Example: Algebraic system for Boussinesq case

$$a_{11} c_1 (3a_{12} + 2\alpha c_1^2) = 0,$$
$$a_{12} c_1 (a_{12} + 4\alpha c_1^2) = 0,$$
$$a_{21} c_1 + a_{11} c_2 = 0,$$
$$a_{22} c_1 + a_{12} c_2 = 0,$$
$$a_{11} c_1 - 3a_{10} a_{11} c_1 + 2\alpha a_{11} c_1^3 + a_{21} c_2 = 0,$$
$$-3a_{11}^2 c_1 + 2 a_{12} c_1 - 6a_{10} a_{12} c_1 + 16\alpha a_{12} c_1^3 + 2a_{22} c_2 = 0.$$

Step T4:

- Solve the nonlinear algebraic system with parameters.

- Example: Solution for Boussinesq system

$$a_{10} = \frac{c_2^2 - c_1^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0,$$
$$a_{12} = -4\alpha c_1^2, \quad a_{20} = \text{free},$$
$$a_{21} = 0, \quad a_{22} = 4\alpha c_1 c_2.$$

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.

- Example: Solitary wave solution for Boussinesq system:

$$u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \delta],$$
$$v(x, t) = a_{20} + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \delta].$$
Other Types of Solutions for PDEs

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<th>Chain Rule</th>
</tr>
</thead>
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<tr>
<td>( \tanh(\xi) )</td>
<td>( y' = 1 - y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j \left(1 - \frac{\bullet}{T}\right) \frac{d\bullet}{dT} )</td>
</tr>
<tr>
<td>( \text{sech}(\xi) )</td>
<td>( y' = -y \sqrt{1 - y^2} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = -c_j \frac{\bullet}{s} \sqrt{1 - s^2} \frac{d\bullet}{ds} )</td>
</tr>
<tr>
<td>( \tan(\xi) )</td>
<td>( y' = 1 + y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j \left(1 + \frac{\bullet}{\tan^2}\right) \frac{d\bullet}{d\tan} )</td>
</tr>
<tr>
<td>( \exp(\xi) )</td>
<td>( y' = y )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j \frac{\bullet}{E} \frac{d\bullet}{dE} )</td>
</tr>
<tr>
<td>( \text{cn}(\xi; m) )</td>
<td>( y' = \sqrt{(1 - y^2)(1 - m + m y^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{1 - \text{cn}^2(1 - m + m \text{cn}^2)} \frac{d\bullet}{d\text{cn}} )</td>
</tr>
<tr>
<td>( \text{sn}(\xi; m) )</td>
<td>( y' = \sqrt{(1 - y^2)(1 - m y^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j \sqrt{1 - \text{sn}^2(1 - m \text{sn}^2)} \frac{d\bullet}{d\text{sn}} )</td>
</tr>
</tbody>
</table>
Algorithm for Tanh Solutions for System of DDEs

Nonlinear differential-difference equations (DDEs) of order \( m \)
\[
\Delta(u_{n+p_1}(x), u_{n+p_2}(x), \ldots, u_{n+p_k}(x), u'_{n+p_1}(x), u'_{n+p_2}(x), \ldots, u'_{n+p_k}(x), \ldots, u^{(r)}_{n+p_1}(x), u^{(r)}_{n+p_2}(x), \ldots, u^{(r)}_{n+p_k}(x)) = 0.
\]

Dependent variable \( u_n \) has \( M \) components \( u_{i,n} \) (or \( u_n, v_n, w_n, \ldots \))

Independent variable \( x \) has \( N \) components \( x_i \) (or \( t, x, y, \ldots \)).

Shift vectors \( p_i \in \mathbb{Z}^Q \).

\( u^{(r)}(x) \) is collection of mixed derivatives of order \( r \).

Simplest case for independent variable \( (t) \), and one lattice point \( (n) \):
\[
\Delta(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots, u'_{n-1}, u'_{n}, u'_{n+1}, \ldots, u^{(r)}_{n-1}, u^{(r)}_n, u^{(r)}_{n+1}, \ldots) = 0.
\]

**Step D1:**

- Seek solution \( u_n(x) = U_n(T_n) \), with \( T_n = \tanh(\xi_n) \),
  \[
  \xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = d \cdot n + c \cdot x + \delta.
  \]

- Repeatedly apply the operator rule
  \[
  \frac{d\bullet}{dx_j} = \frac{\partial \xi_n}{\partial x_j} \frac{dT_n}{dT_n} \frac{d\bullet}{dT_n} = c_j(1 - T_n^2) \frac{d\bullet}{dT_n},
  \]
  transforms DDE into
  \[
  \Delta(U_{n+p_1}(T_n), \ldots, U_{n+p_k}(T_n), U'_{n+p_1}(T_n), \ldots, U'_{n+p_k}(T_n), \ldots, U^{(r)}_{n+p_1}(T_n), \ldots, U^{(r)}_{n+p_k}(T_n)) = 0.
  \]

**Note:** \( U_{n+p_s} \) is function of \( T_n \) not of \( T_{n+p_s} \).
Example: Toda lattice
\[ \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}) \]
transforms into
\[ c_1^2(1 - T_n^2)
   \left[ 2T_n U' - (1 - T_n^2)U'' \right] + \left[ 1 + c_1(1 - T_n^2)U' \right] \left[ U_{n-1} - 2U_n + U_{n+1} \right] = 0. \]

Step D2:

- Seek polynomial solutions
  \[ U_{i,n}(T_n) = \sum_{j=0}^{M_i} a_{ij} T_n^j. \]

Use \( \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \) to deal with the shift:
\[ T_{n+p} = \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s}, \]
where
\[ \phi_s = p_s \cdot d = p_s d_1 + p_s d_2 + \cdots + p_s Q d_Q, \]
Substitute \( U_{i,n}(T_n) = T_n^{M_i} \), and
\[ U_{i,n+p}(T_n) = T_n^{M_i} = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i}, \]
and balance the highest exponents in \( T_n \) to determine \( M_i \).

Note: \( U_{i,n+0}(T_n) = T_n^{M_i} \) is of degree \( M_i \) in \( T_n \).
\[ U_{i,n+p}(T_n) = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i} \] is of degree zero in \( T_n \).
• Example: Balance of exponents for Toda lattice

\[ 2M_1 + 1 = M_1 + 2. \]

So, \( M_1 = 1 \). Hence,

\[ U_n(T_n) = a_{10} + a_{11}T_n, \]

\[ U_{n\pm1}(T(n \pm 1)) = a_{10} + a_{11}T(n \pm 1) = a_{10} + a_{11} \frac{T_n \pm \tanh(d_1)}{1 \pm T_n \tanh(d_1)}. \]

**Step D3:**

• Determine the algebraic system for the unknown coefficients \( a_{ij} \) by setting to zero the coefficients of the powers in \( T_n \).

• Example: Algebraic system for Toda lattice

\[ c_1^2 - \tanh^2(d_1) - a_{11}c_1 \tanh^2(d_1) = 0, \]

\[ c_1 - a_{11} = 0. \]

**Step D4:**

• Solve the nonlinear algebraic system with parameters.

• Example: Solution of algebraic system for Toda lattice

\[ a_{10} = \text{free}, \quad a_{11} = \pm \sinh(d_1), \quad c_1 = \pm \sinh(d_1). \]

**Step D5:**

• Return to the original variables. Test solution(s) of DDE. Reject trivial ones.

• Example: Solitary wave solution for Toda lattice:

\[ u_n(t) = a_{10} \pm \sinh(d_1) \tanh[d_1n \pm \sinh(d_1)t + \delta]. \]
Example of System of DDEs: Relativistic Toda Lattice

\[ \begin{align*}
\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
\end{align*} \]

Change of variables

\[ u_n(t) = U_n(T_n), \quad v_n(t) = V_n(T_n), \]

with

\[ T_n(t) = \tanh [d_1 n + c_1 t + \delta]. \]

gives

\[ \begin{align*}
c_1(1 - T^2)U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) &= 0, \\
c_1(1 - T^2)V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) &= 0.
\end{align*} \]

Seek polynomial solutions

\[ U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \quad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j. \]

Balance the highest exponents in \( T_n \) to determine \( M_1 \), and \( M_2 \):

\[ M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2. \]

So, \( M_1 = M_2 = 1 \). Hence,

\[ U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n. \]
Algebraic system for $a_{ij}$:

\[-a_{11} c_1 + a_{21} \tanh(d_1) + \alpha a_{10} a_{21} \tanh(d_1) = 0,\]
\[a_{11} \tanh(d_1) (\alpha a_{21} + c_1) = 0,\]
\[-a_{21} c_1 + a_{11} a_{20} \tanh(d_1) + 2\alpha a_{20} a_{21} \tanh(d_1) = 0,\]
\[\tanh(d_1) (a_{11} a_{21} + 2\alpha a_{21}^2 - a_{11} a_{20} \tanh(d_1)) = 0,\]
\[a_{21} \tanh^2(d_1) (c_1 - a_{11}) = 0.\]

Solution of the algebraic system

\[a_{10} = -\frac{1}{\alpha} - c_1 \coth(d_1),\]
\[a_{11} = c_1,\]
\[a_{20} = \frac{c_1 \coth(d_1)}{\alpha},\]
\[a_{21} = -\frac{c_1}{\alpha}.\]

Solitary wave solution in original variables:

\[u_n(t) = -\frac{1}{\alpha} - c_1 \coth(d_1) + c_1 \tanh [d_1 n + c_1 t + \Delta],\]
\[v_n(t) = \frac{c_1 \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh [d_1 n + c_1 t + \Delta].\]
Multi-dimensional Example: 2D Toda Lattice

2D Toda lattice:

\[
\frac{\partial^2 y_n}{\partial x \partial t} = \exp (y_{n-1} - y_n) - \exp (y_n - y_{n+1}),
\]

\(y_n(x, t)\) is displacement from equilibrium of the \(n\)-th unit mass under an exponential decaying interaction force between nearest neighbors.

Set

\[
\frac{\partial u_n}{\partial t} = \exp (y_{n-1} - y_n) - 1. \quad (*)
\]

Then,

\[\exp (y_{n-1} - y_n) = \frac{\partial u_n}{\partial t} + 1,\]

and the 2D-Toda lattice becomes

\[
\frac{\partial^2 y_n}{\partial x \partial t} = \frac{\partial u_n}{\partial t} + 1 - \left( \frac{\partial u_{n+1}}{\partial t} + 1 \right) = \frac{\partial u_n}{\partial t} - \frac{\partial u_{n+1}}{\partial t}.
\]

Integrate with respect to \(t\) to get

\[\frac{\partial y_n}{\partial x} = u_n - u_{n+1}.
\]

Differentiate (*) with respect to \(x\) and

\[
\frac{\partial^2 u_n}{\partial x \partial t} = \frac{\partial}{\partial x} \left( \exp (y_{n-1} - y_n) - 1 \right)
\]

\[= \exp (y_{n-1} - y_n) \left( \frac{\partial y_{n-1}}{\partial x} - \frac{\partial y_n}{\partial x} \right),
\]

\[= \left( \frac{\partial u_n}{\partial t} + 1 \right) [(u_{n-1} - u_n) - (u_n - u_{n+1})],
\]

\[= \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}).
\]
So, the 2D Toda lattice is written in polynomial form:

\[
\frac{\partial^2 u_n(x, t)}{\partial x \partial t} = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}).
\]

Travelling wave solution:

\[
u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[ d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right].
\]

**Complicated Example: Ablowitz-Ladik Lattice**

The Ablowitz-Ladik lattice:

\[
\begin{align*}
\dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\
\dot{v}_n(t) &= -(\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n.
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= \frac{\alpha \sinh^2(d_1)}{a_{21}} \left( \pm 1 - \tanh \left[ d_1 n + 2\alpha t \sinh^2(d_1) + \delta \right] \right), \\
v_n(t) &= a_{21} \left( \pm 1 + \tanh \left[ d_1 n + 2\alpha \sinh^2(d_1)t + \delta \right] \right).
\end{align*}
\]
Analyzing and Solving Nonlinear Parameterized Systems

Assumptions:

- All \( c_i \neq 0 \) and \( d_i \neq 0 \) (and modulus \( m \neq 0 \)).
- Parameters \((\alpha, \beta, \gamma, \ldots)\). Otherwise the maximal exponents \( M_i \) may change.
- All \( M_i \geq 1 \).
- All \( a_i M_i \neq 0 \). Highest power terms in \( U_i \) must be present, except in mixed sech-tanh-method.
- Solve for \( a_{ij} \), then \( c_i, \tanh(d_i) \), and \( m \). Then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in \( a_{ij} \) first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in \( c_i, \tanh(d_i), m \) if they are free of \( a_{ij} \).
- Solve linear equations in parameters if they free of \( a_{ij}, c_i, \tanh(d_i), m \).
- Solve quasi-linear equations for \( a_{ij}, c_i, \tanh(d_i), m \).
- Solve quadratic equations for \( a_{ij}, c_i, \tanh(d_i), m \).
- Eliminate cubic terms for \( a_{ij}, c_i, \tanh(d_i), m \), without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner bases techniques.
- Use Ritt-Wu characteristic sets method.
- Use combinatorics on coefficients \( a_{ij} = 0 \) or \( a_{ij} \neq 0 \).
Implementation Issues – Software Demo – Future Work

• Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and tanh function for DDEs.

• Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.

• Implement various methods: Lie symmetry methods, etc.

• Look at other types of explicit solutions involving
  – other hyperbolic and elliptic functions sinh, cosh, dn,....
  – complex exponentials combined with sech or tanh.

• Other applications (of the nonlinear algebraic solver):
  Computation of conservation laws, symmetries, first integrals, etc. leading to \textbf{linear} parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).
• Papers:
  

• Software:
  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/PDESpecialSolutions; or via Internet URL: http://www.mines.edu/fs_home/whereman/
  
  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/DDESpecialSolutions; or via Internet URL: http://www.mines.edu/fs_home/whereman/
Appendix: A Complicated Case

Class of fifth-order evolution equations with parameters:

\[ u_t + \alpha \gamma^2 u^2 u_x + \beta \gamma u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0. \]

Well-Known Special cases

Lax case: \( \alpha = \frac{3}{10}, \beta = 2, \gamma = 10 \). Two solutions:

\[ u(x, t) = 4c_1^2 - 6c_1^2 \tanh^2 \left[ c_1 x - 56c_1^5 t + \Delta \right], \]

and

\[ u(x, t) = a_0 - 2c_1^2 \tanh^2 \left[ c_1 x - 2(15a_0^2 c_1 - 40a_0 c_1^3 + 28c_1^5) t + \Delta \right], \]

where \( a_0 \) is arbitrary.

Sawada-Kotera case: \( \alpha = \frac{1}{5}, \beta = 1, \gamma = 5 \). Two solutions:

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 16c_1^5 t + \Delta \right], \]

and

\[ u(x, t) = a_0 - 6c_1^2 \tanh^2 \left[ c_1 x - (5a_0^2 c_1 - 40a_0 c_1^3 + 76c_1^5) t + \Delta \right], \]

where \( a_0 \) is arbitrary.

Kaup-Kupershmidt case: \( \alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10 \). Two solutions:

\[ u(x, t) = c_1^2 - \frac{3}{2} c_1^2 \tanh^2 \left[ c_1 x - c_1^5 t + \Delta \right] \]

and

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 176c_1^5 t + \Delta \right]. \]

No free constants!

Ito case: \( \alpha = \frac{2}{5}, \beta = 2, \gamma = 3 \). One solution:

\[ u(x, t) = 20c_1^2 - 30c_1^2 \tanh^2 \left[ c_1 x - 96c_1^5 t + \Delta \right]. \]
What about the General case?

Q1: Can we retrieve the special solutions?

Q2: What are the condition(s) on the parameters $\alpha, \beta, \gamma$ for solutions of tanh-type to exist?

Tanh solutions:

$$u(x, t) = a_0 + a_1 \tanh [c_1x + c_2t + \Delta] + a_2 \tanh^2 [c_1x + c_2t + \Delta].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha \gamma^2 a_2^2 + 6\gamma a_2 c_1^2 + 2\beta \gamma a_2 c_1^2 + 24c_1^4) = 0,$$

$$a_1(\alpha \gamma^2 a_1^2 + 6\alpha \gamma^2 a_0 a_2 + 6\gamma a_0 c_1^2 - 18\gamma a_2 c_1^2 - 12\beta \gamma a_2 c_1^2 - 120c_1^4) = 0,$$

$$\alpha \gamma^2 a_2^2 + 12\gamma a_2 c_1^2 + 6\beta \gamma a_2 c_1^2 + 360c_1^4 = 0,$$

$$2\alpha \gamma^2 a_1^2 + 2\alpha \gamma^2 a_0 a_2 + 3\gamma a_1^2 c_1^2 + \beta \gamma a_1^2 c_1^2 + 12\gamma a_0 a_2 c_1^2 - 8\gamma a_2^2 c_1^2 - 8\beta \gamma a_2^2 c_1^2 - 480a_2 c_1^4 = 0,$$

$$a_1(\alpha \gamma^2 a_0^2 c_1 - 2\gamma a_0 c_1^3 + 2\beta \gamma a_2 c_1^3 + 16c_1^5 + c_2) = 0,$$

$$\alpha \gamma^2 a_0 a_1^2 c_1 + \alpha \gamma^2 a_0 a_2 c_1 - \gamma a_1^2 c_1^3 - \beta \gamma a_1^2 c_1^3 - 8\gamma a_0 a_2 c_1^3 + 2\beta \gamma a_2^2 c_1^3 + 136a_2 c_1^5 + a_2 c_2 = 0.$$

Unknowns: $a_0, a_1, a_2$.

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

**Solve** and **Reduce** cannot be used on the whole system!
**Actual Solution:** Two major cases:

CASE 1: \( a_1 = 0 \), two subcases

Subcase 1-a:

\[
a_2 = -\frac{3}{2}a_0, \]

\[
c_2 = c_1^3(24c_1^2 - \beta \gamma a_0),
\]

where \( a_0 \) is one of the two roots of the quadratic equation:

\[
\alpha \gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta \gamma a_0 c_1^2 + 160 c_1^4 = 0.
\]

Subcase 1-b: If \( \beta = 10\alpha - 1 \), then

\[
a_2 = -\frac{6}{\alpha \gamma} c_1^2,
\]

\[
c_2 = -\frac{1}{\alpha}(\alpha^2 \gamma^2 a_0^2 c_1 - 8\alpha \gamma a_0 c_1^3 + 12 c_1^5 + 16 \alpha c_1^5),
\]

where \( a_0 \) is arbitrary.

CASE 2: \( a_1 \neq 0 \), then

\[
\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)
\]

and

\[
a_2 = -\frac{168}{\gamma(3 + 2\beta)} c_1^2,
\]

provided \( \beta \) is root of

\[
(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0.
\]
Subcase 2-a: If \( \beta^2 = -\frac{1}{104}(886\beta + 1487) \), then

\[
\alpha = -\frac{2\beta + 5}{26},
\]

\[
a_0 = -\frac{49c_1^2(9983 + 4378\beta)}{26\gamma(8 + 3\beta)(3 + 2\beta)^2},
\]

\[
a_1 = \pm\frac{336c_1^2}{\gamma(3 + 2\beta)},
\]

\[
a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},
\]

\[
c_2 = -\frac{364c_1^5(3851 + 1634\beta)}{6715 + 2946\beta}.
\]

Subcase 2-b: If \( \beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2) \), then

\[
\alpha = \frac{39 + 38\beta + 8\beta^2}{392},
\]

\[
a_0 = \frac{28c_1^2(6483 + 5529\beta + 1066\beta^2)}{(3 + 2\beta)(23 + 6\beta)(81 + 26\beta)\gamma},
\]

\[
a_1^2 = \frac{28224c_1^4(4\beta - 1)(26\beta - 17)}{(3 + 2\beta)^2(23 + 6\beta)(81 + 26\beta)\gamma^2},
\]

\[
a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},
\]

\[
c_2 = -\frac{8c_1^5(1792261977 + 1161063881\beta + 188900114\beta^2)}{959833473 + 632954969\beta + 105176786\beta^2}.
\]