Symbolic Computation of
Conserved Densities of Nonlinear Evolution
and Differential-Difference Equations

WILLY HEREMAN
Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887, USA

AMS Regional Meeting
Atlanta, Georgia
October 17-19, 1997
• **Purpose**

Design and implement an algorithm to compute polynomial conservation laws for nonlinear systems of evolution equations and differential-difference equations

• **Motivation**

- Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics

- For nonlinear PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability

- Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures

- Conservation laws can be used to test numerical integrators
PART I: Evolution Equations

• Conservation Laws for PDEs

Consider a single nonlinear evolution equation

\[ u_t = F(u, u_x, u_{2x}, \ldots, u_{nx}) \]

or a system of \( N \) nonlinear evolution equations

\[ u_t = F(u, u_x, \ldots, u_{nx}) \]

where \( u = [u_1, \ldots, u_N]^T \) and

\[ u_t \overset{\text{def}}{=} \frac{\partial u}{\partial t}, \quad u^{(n)} = u_{nx} \overset{\text{def}}{=} \frac{\partial^n u}{\partial x^n} \]

All components of \( u \) depend on \( x \) and \( t \)

*Conservation law:*

\[ D_t \rho + D_x J = 0 \]

\( \rho \) is the density, \( J \) is the flux

Both are polynomial in \( u, u_x, u_{2x}, u_{3x}, \ldots \)

Consequently

\[ P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant} \]

if \( J \) vanishes at infinity
• The Euler Operator (calculus of variations)

Useful tool to verify if an expression is a total derivative

**Theorem:**

If

\[ f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)}) \]

then

\[ \mathcal{L}_y(f) \equiv 0 \]

if and only if

\[ f = D_x g \]

where

\[ g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)}) \]

Notations:

\[ y = [y_1, \ldots, y_N]^T \]

\[ \mathcal{L}_y(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T \]

\[ 0 = [0, \ldots, 0]^T \]

(T for transpose)

and **Euler Operator:**

\[ \mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx} \left( \frac{\partial}{\partial y_i'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial}{\partial y_i^{''}} \right) + \cdots + (-1)^n \frac{d^n}{dx^n} \left( \frac{\partial}{\partial y_i^{(n)}} \right) \]
• **Example: Korteweg-de Vries (KdV) equation**

\[ u_t + uu_x + u_{3x} = 0 \]

Conserved densities:

\[
\begin{align*}
\rho_1 &= u, \quad (u)_t + \left( \frac{u^2}{2} + u_{2x} \right)_x = 0 \\
\rho_2 &= u^2, \quad (u^2)_t + \left( \frac{2u^3}{3} + 2uu_{2x} - u_x^2 \right)_x = 0 \\
\rho_3 &= u^3 - 3u_x^2, \\
\quad & \left( u^3 - 3u_x^2 \right)_t + \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x} \right)_x = 0 \\
\quad & \vdots \\
\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 + 720u^2u_{2x}^2 - 648uu_{3x}^2 + 216u_{4x}^2 \\
\quad & \vdots
\end{align*}
\]

**Note:** KdV equation and conservation laws are invariant under dilation (scaling) symmetry

\[ (x,t,u) \to (\lambda x, \lambda^3 t, \lambda^{-2} u) \]

\( u \) and \( t \) carry the weights of 2 and 3 derivatives with respect to \( x \)

\[ u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3} \]
• **Key Steps of the Algorithm**

  1. Determine weights (scaling properties) of variables & parameters
  2. Construct the form of the density (building blocks)
  3. Determine the unknown constant coefficients

• **Example: KdV equation**

\[ u_t + uu_x + u_{3x} = 0 \]

Compute the density of rank 6

(i) Compute the weights by solving a linear system

\[ w(u) + w\left( \frac{\partial}{\partial t} \right) = 2w(u) + w(x) = w(u) + 3w(x) \]

With \( w(x) = 1, \ w\left( \frac{\partial}{\partial t} \right) = 3, \ w(u) = 2. \)

Thus, \( (x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u) \)

(ii) Take all the variables, except \( \left( \frac{\partial}{\partial t} \right) \), with positive weight and list all possible powers of \( u \), up to rank 6: \( [u, u^2, u^3] \)

Introduce \( x \) derivatives to ‘complete’ the rank

\( u \) has weight 2, introduce \( \frac{\partial^4}{\partial x^4} \)

\( u^2 \) has weight 4, introduce \( \frac{\partial^2}{\partial x^2} \)

\( u^3 \) has weight 6, no derivatives needed
Apply the derivatives and remove terms that are total derivatives with respect to \( x \) or total derivative up to terms kept earlier in the list

\[
[u_{4x}] \rightarrow [\ ] \text{ empty list}
\]
\[
[u_x^2, uu_{2x}] \rightarrow [u_x^2] \text{ since } uu_{2x} = (uu_x)_x - u_x^2
\]
\[
[u^3] \rightarrow [u^3]
\]

Combine the building blocks: \( \rho = c_1 u^3 + c_2 u_x^2 \)

(iii) Determine the coefficients \( c_1 \) and \( c_2 \)

1. Compute \( D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt} \)

2. Replace \( u_t \) by \( -(uu_x + u_{3x}) \) and \( u_{xt} \) by \( -(uu_x + u_{3x})_x \)

3. Apply the Euler operator or integrate by parts

\[
D_t \rho = -\left[ \frac{3}{4}c_1 u^4 - (3c_1 - c_2)uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x} \right]_x \\
- (3c_1 + c_2)u_x^3
\]

4. The non-integrable term must vanish. Thus, \( c_1 = -\frac{1}{3}c_2. \)

Set \( c_2 = -3 \), hence, \( c_1 = 1 \)

Result:

\( \rho = u^3 - 3u_x^2 \)

Expression \([\ldots]\) yields

\[
J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_x^2 - 6u_x u_{3x}
\]
Example: Boussinesq equation

\[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0 \]

with nonzero parameter \( \alpha \). Can be written as

\[ u_t + v_x = 0 \]
\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0 \]

The terms \( u_x \) and \( \alpha u_{3x} \) are not uniform in rank.

Introduce auxiliary parameter \( \beta \) with weight. Replace the system by

\[ u_t + v_x = 0 \]
\[ v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0 \]

The system is invariant under the scaling symmetry

\( (x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta) \)

Hence

\[ w(u) = 2, \ w(\beta) = 2, \ w(v) = 3 \text{ and } w(\frac{\partial}{\partial t}) = 2 \]

or

\[ u \sim \beta \sim \frac{\partial^2}{\partial x^2}, \ v \sim \frac{\partial^3}{\partial x^3}, \ \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2} \]

Form \( \rho \) of rank 6

\[ \rho = c_1 \beta^2 u + c_2 \beta u^2 + c_3 u^3 + c_4 v^2 + c_5 u_x v + c_6 u_x^2 \]

Compute the \( c_i \). At the end set \( \beta = 1 \)

\[ \rho = u^2 - u^3 + v^2 + \alpha u_x^2 \]

which is no longer uniform in rank!
Application: A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters, and \( u \sim \frac{\partial^2}{\partial x^2} \)

Special cases:

\[
\begin{align*}
\alpha &= 30 \quad \beta = 20 \quad \gamma = 10 \quad \text{Lax} \\
\alpha &= 5 \quad \beta = 5 \quad \gamma = 5 \quad \text{Sawada – Kotera} \\
\alpha &= 20 \quad \beta = 25 \quad \gamma = 10 \quad \text{Kaup – Kupershmidt} \\
\alpha &= 2 \quad \beta = 6 \quad \gamma = 3 \quad \text{Ito}
\end{align*}
\]

Under what conditions for the parameters \( \alpha, \beta \) and \( \gamma \) does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)
  \[ \rho = u^2 \]
– **Rank 6:**

Condition:

\[ 10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2 \]

(Lax, SK, and KK cases)

\[ \rho = u^3 + \frac{15}{(-2\beta + \gamma)}u_x^2 \]

– **Rank 8:**

1. \( \beta = 2\gamma \) (Lax and Ito cases)

\[ \rho = u^4 - \frac{6\gamma}{\alpha}uu_x^2 + \frac{6}{\alpha}u_{2x}^2 \]

2. \( \alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45} \) (SK, KK and Ito cases)

\[ \rho = u^4 - \frac{135}{2\beta + \gamma}uu_x^2 + \frac{675}{(2\beta + \gamma)^2}u_{2x}^2 \]

– **Rank 10:**

Condition:

\[ \beta = 2\gamma \]

and

\[ 10\alpha = 3\gamma^2 \]

(Lax case)

\[ \rho = u^5 - \frac{50}{\gamma}u^2u_x^2 + \frac{100}{\gamma^2}uu_{2x}^2 - \frac{500}{7\gamma^3}u_{3x}^2 \]
What are the necessary conditions for the parameters $\alpha$, $\beta$ and $\gamma$ for this equation to admit infinitely many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If
  \[
  \alpha = -\frac{2\beta^2 - 7\beta \gamma + 4\gamma^2}{45}
  \]
  or
  \[
  \beta = 2\gamma
  \]
  then there is a conserved density of rank 8

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)
PART II: Differential-difference Equations

• Conservation Laws for DDEs

Consider a system of DDEs, continuous in time, discretized in space
\[ \dot{u}_n = F(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots) \]
\[ u_n \text{ and } F \text{ are vector dynamical variables} \]

Conservation law:
\[ \dot{\rho}_n = J_n - J_{n+1} \]
\[ \rho_n \text{ is the density, } J_n \text{ is the flux} \]

Both are polynomials in \( u_n \) and its shifts
\[ \frac{d}{dt} \left( \sum_n \rho_n \right) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

If \( J_n \) is bounded for all \( n \), with suitable boundary or periodicity conditions
\[ \sum_n \rho_n = \text{constant} \]

• Definitions

Define: \( D \) shift-down operator, \( U \) shift-up operator
\[ Dm = m|_{n\rightarrow n-1} \quad Um = m|_{n\rightarrow n+1} \]

For example,
\[ Du_{n+2}v_n = u_{n+1}v_{n-1} \quad Uu_{n-2}v_{n-1} = u_{n-1}v_n \]
Compositions of $D$ and $U$ define an \textit{equivalence relation}

All shifted monomials are \textit{equivalent}, e.g.

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}$$

Use \textit{equivalence criterion}:

If two monomials, $m_1$ and $m_2$, are equivalent, $m_1 \equiv m_2$, then

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}]$$

with $M_n = u_{n-2}u_n$

\textit{Main representative} of an equivalence class; the monomial with label $n$ on $u$ (or $v$)

For example, $u_nu_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts

For example, $u_nv_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.
• Algorithm: Toda Lattice

\[ m \ddot{y}_n = a [e^{(y_{n-1} - y_n)} - e^{(y_n - y_{n+1})}] \]

Take \( m = a = 1 \) (scale on \( t \)), and set \( u_n = \dot{y}_n, \quad v_n = e^{(y_n - y_{n+1})} \)

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}) \]

Simplest conservation law (by hand):

\[ \dot{u}_n = \dot{\rho}_n = v_{n-1} - v_n = J_n - J_{n+1} \quad \text{with} \quad J_n = v_{n-1} \]

First pair:

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1} \]

Second pair:

\[ \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1} \]

Key observation: The DDE and the two conservation laws, \( \dot{\rho}_n = J_n - J_{n+1} \), with

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1} \]

\[ \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1} \]

are invariant under the scaling symmetry

\[ (t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n) \]

Dimensional analysis:

\( u_n \) corresponds to one derivative with respect to \( t \)

For short, \( u_n \sim \frac{d}{dt} \), and similarly, \( v_n \sim \frac{d^2}{dt^2} \)
Our algorithm exploits this symmetry to find conserved densities:

1. Determining the weights
2. Constructing the form of density
3. Determining the unknown coefficients

**Step 1: Determine the weights**

The *weight*, $w$, of a variable is equal to the number of derivatives with respect to $t$ the variable carries.

Weights are positive, rational, and independent of $n$.

Requiring uniformity in rank for each equation

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}) \]

allows one to compute the weights of the dependent variables.

Solve the linear system

\[ w(u_n) + w\left(\frac{d}{dt}\right) = w(v_n) \]
\[ w(v_n) + w\left(\frac{d}{dt}\right) = w(v_n) + w(u_n) \]

Set $w\left(\frac{d}{dt}\right) = 1$, then $w(u_n) = 1$, and $w(v_n) = 2$

which is consistent with the scaling symmetry

\[ (t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1}u_n, \lambda^{-2}v_n) \]
• **Step 2: Construct the form of the density**

The *rank* of a monomial is the total weight of the monomial. For example, compute the form of the density of rank 3

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[
\mathcal{G} = \{ u_n^3, u_n^2, u_n v_n, u_n, v_n \}
\]

Next, for each monomial in \( \mathcal{G} \), introduce enough \( t \)-derivatives, so that each term exactly has weight 3. Use the DDE to remove \( \dot{u}_n \) and \( \dot{v}_n \)

\[
\frac{d^0}{dt^0}(u_n^3) = u_n^3, \quad \frac{d^0}{dt^0}(u_n v_n) = u_n v_n,
\]

\[
\frac{d}{dt}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n, \quad \frac{d}{dt}(v_n) = u_n v_n - u_{n+1} v_n,
\]

\[
\frac{d^2}{dt^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n
\]

Gather the resulting terms in a set

\[
\mathcal{H} = \{ u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n \}
\]

Identify members of the same equivalence classes and replace them by the main representatives.

For example, since \( u_n v_{n-1} \equiv u_{n+1} v_n \) both are replaced by \( u_n v_{n-1} \).

\( \mathcal{H} \) is replaced by

\[
\mathcal{I} = \{ u_n^3, u_n v_{n-1}, u_n v_n \}
\]

containing the building blocks of the density.

Form a linear combination of the monomials in \( \mathcal{I} \)

\[
\rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n
\]

with constant coefficients \( c_i \)
• **Step 3: Determine the unknown coefficients**

Require that the conservation law, $\dot{\rho}_n = J_n - J_{n+1}$, holds

Compute $\dot{\rho}_n$ and use the equations to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1}v_n$$
$$+ c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2$$

Use the equivalence criterion to modify $\dot{\rho}_n$

Replace $u_{n-1} u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n]$.

The goal is to introduce the main representatives. Therefore,

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n$$
$$+(c_3 - c_2)v_n v_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_n v_{n+1}]$$
$$+ c_2 u_n u_{n+1} v_n + [c_2 u_{n-1} u_n v_{n-1} - c_2 u_n u_{n+1} v_n]$$
$$+ c_2 v_n^2 + [c_2 v_{n-1}^2 - c_2 v_n^2] - c_3 u_n u_{n+1} v_n - c_3 v_n^2$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom. Rearrange the latter terms so that they match the pattern $[J_n - J_{n+1}]$. Hence,

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n$$
$$+(c_3 - c_2)v_n v_{n+1} + (c_2 - c_3)u_n u_{n+1} v_n + (c_2 - c_3)v_n^2$$
$$+ [(c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2]$$
$$- [(c_3 - c_2)v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 v_n^2]$$
The terms inside the square brackets determine:

\[ J_n = (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2 \]

The terms outside the square brackets must vanish, thus

\[ \mathcal{S} = \{ 3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0 \} \]

The solution is \( 3c_1 = c_2 = c_3 \). Choose \( c_1 = \frac{1}{3} \), thus \( c_2 = c_3 = 1 \)

\[ \rho_n = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2 \]

Analogously, conserved densities of rank \( \leq 5 \):

\[ \rho_n^{(1)} = u_n \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n \]

\[ \rho_n^{(3)} = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \]

\[ \rho_n^{(4)} = \frac{1}{4}u_n^4 + u_n^2(v_{n-1} + v_n) + u_nu_{n+1}v_n + \frac{1}{2}v_n^2 + v_nv_{n+1} \]

\[ \rho_n^{(5)} = \frac{1}{5}u_n^5 + u_n^3(v_{n-1} + v_n) + u_nu_{n+1}v_n(u_n + u_{n+1}) + u_nv_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_nv_n(v_{n-1} + v_n + v_{n+1}) \]
- **Application: A parameterized Toda lattice**

\[
\dot{u}_n = \alpha \, v_{n-1} - v_n, \quad \dot{v}_n = v_n \, (\beta \, u_n - u_{n+1})
\]

\(\alpha\) and \(\beta\) are non-zero parameters. The system is integrable if \(\alpha = \beta = 1\)

Compute the *compatibility conditions* for \(\alpha\) and \(\beta\), so that there is a conserved densities of, say, rank 3.

In this case, we have \(\mathcal{S}\):

\[
\begin{align*}
3\alpha c_1 - c_2 &= 0, \\
\beta c_3 - 3c_1 &= 0, \\
\alpha c_3 - c_2 &= 0, \\
\beta c_2 - c_3 &= 0, \\
\alpha c_2 - c_3 &= 0
\end{align*}
\]

A non-trivial solution \(3c_1 = c_2 = c_3\) will exist iff \(\alpha = \beta = 1\)

Analogously, the parameterized Toda lattice has density

\[
\rho_n^{(1)} = u_n \text{ of rank 1 if } \alpha = 1
\]

and density

\[
\rho_n^{(2)} = \frac{\beta}{2} u_n^2 + v_n \text{ of rank 2 if } \alpha \beta = 1
\]

Only when \(\alpha = \beta = 1\) will the parameterized system have conserved densities of rank \(\geq 3\)
• **Example: Nonlinear Schrödinger (NLS) equation**

Ablowitz and Ladik discretization of the NLS equation:

\[
i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1})
\]

where \( u_n^* \) is the complex conjugate of \( u_n \).

Treat \( u_n \) and \( v_n = u_n^* \) as independent variables, add the complex conjugate equation, and absorb \( i \) in the scale on \( t \)

\[
\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1})
\]

\[
\dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1})
\]

Since \( v_n = u_n^* \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Circumvent this problem by introducing an auxiliary parameter \( \alpha \) with weight,

\[
\dot{u}_n = \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1})
\]

\[
\dot{v}_n = -\alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).
\]

Uniformity in rank requires that

\[
w(u_n) + 1 = w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n)
\]

\[
w(v_n) + 1 = w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n)
\]

which yields

\[
w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1
\]
Uniformity in rank is essential for the first two steps of the algorithm. After Step 2, you can already set $\alpha = 1$.

The computations now proceed as in the previous examples

Conserved densities:

$$
\rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}
$$

$$
\rho_n^{(2)} = c_1 \left( \frac{1}{2} u_n^2 v_{n-1}^2 + u_n u_{n+1} v_{n-1} v_n + u_n v_{n-2} \right) + c_2 \left( \frac{1}{2} u_n^2 v_{n+1}^2 + u_n u_{n+1} v_{n+1} v_{n+2} + u_n v_{n+2} \right)
$$

$$
\rho_n^{(3)} = c_1 \left[ \frac{1}{3} u_n^3 v_{n-1}^3 + u_n u_{n+1} v_{n-1} v_n (u_n v_{n-1} + u_{n+1} v_n + u_{n+2} v_{n+1}) + u_n v_{n-1} (u_n v_{n-2} + u_{n+1} v_{n-1}) + u_n v_n (u_{n+1} v_{n-2} + u_{n+2} v_{n-1}) + u_n v_{n-3} \right] + c_2 \left[ \frac{1}{3} u_n^3 v_{n+1}^3 + u_n u_{n+1} v_{n+1} v_{n+2} (u_n v_{n+1} + u_{n+1} v_{n+2} + u_{n+2} v_{n+3}) + u_n v_{n+2} (u_n v_{n+1} + u_{n+1} v_{n+2}) + u_n v_{n+3} (u_{n+1} v_{n+1} + u_{n+2} v_{n+2}) + u_n v_{n+3} \right]
$$
PART III: Symmetries of PDEs and DDEs

Symmetries of PDEs

Consider the system of PDEs
\[ u_t = F(x, t, u, u_x, u_{2x}, ..., u_{mx}) \]
space variable \( x \), time variable \( t \)
dynamical variables \( u = (u_1, u_2, ..., u_n) \) and \( F = (F_1, F_2, ..., F_n) \)

Definition of Symmetry

Vector function \( G(x, t, u, u_x, u_{2x}, ...) \) is a symmetry if and only if the PDE is invariant for the replacement
\[ u \rightarrow u + \epsilon G \]
within order \( \epsilon \). Hence
\[ \frac{\partial}{\partial t}(u + \epsilon G) = F(u + \epsilon G) \]
must hold up to order \( \epsilon \), or
\[ \frac{\partial G}{\partial t} = F'(u)[G] \]
where \( F' \) is the Gateaux derivative of \( F \)
\[ F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} \]

Equivalently, \( G \) is a symmetry if the compatibility condition
\[ \frac{\partial}{\partial \tau} F(x, t, u, u_x, u_{2x}, ..., u_{nx}) = \frac{\partial}{\partial t} G(x, t, u, u_x, u_{2x}, ...) \]
is satisfied, where \( \tau \) is the new time variable such that
\[ \frac{\partial u}{\partial \tau} = G(x, t, u, u_x, u_{2x}, ...) \]
– Example: The KdV Equation

\[ u_t = 6uu_x + u_{3x} \]

has infinitely many symmetries:

\[ G^{(1)} = u_x \quad G^{(2)} = 6uu_x + u_{3x} \]
\[ G^{(3)} = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x} \]
\[ G^{(4)} = 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} \]
\[ G^{(5)} = 630u^4u_x + 1260uu_x^3 + 2520u^2u_xu_{2x} + 1302u_xu_{2x}^2 + 420u^3u_{3x} + 966u_x^2u_{3x} + 1260uu_xu_{3x} + 756uu_xu_{4x} + 252u_{3x}u_{4x} + 126u^2u_{5x} + 168u_xu_{5x} + 72u_xu_{6x} + 18uu_{7x} + u_{9x} \]

The recursion operator connecting them is:

\[ R = D^2 + 4u + 2u_xD^{-1} \]

– Algorithm (KdV equation)

Use the dilation symmetry \((t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u)\)
\(\lambda\) is arbitrary parameter. Hence, \(u \sim \partial^2 / \partial x^2\) and \(\partial / \partial t \sim \partial^3 / \partial x^3\).

**Step 1: Determine the weights of variables**

We choose \(w(x) = -1\), then \(w(u) = 2\) and \(w(t) = -3\).

**Step 2: Construct the form of the symmetry**

Compute the form of the symmetry with rank 7
List all monomials in \(u\) of rank 7 or less

\[ \mathcal{L} = \{1, u, u^2, u^3\} \]

Introduce \(x\)-derivatives so that each term has weight 7

\[ \frac{\partial}{\partial x}(u^3) = 3u^2u_x, \quad \frac{\partial^3}{\partial x^3}(u^2) = 6u_xu_{2x} + 2uu_{3x}, \quad \frac{\partial^5}{\partial x^5}(u) = u_{5x}, \quad \frac{\partial^7}{\partial x^7}(1) = 0 \]

Gather the non-zero resulting terms in a set

\[ \mathcal{R} = \{u^2u_x, u_xu_{2x}, uu_{3x}, u_{5x}\} \]

which contains the building blocks of the symmetry
Linear combination of the monomials in \( R \) determines the symmetry

\[
G = c_1 u^2 u_x + c_2 u_x u_{2x} + c_3 u u_{3x} + c_4 u_{5x}
\]

**Step 3: Determine the unknown coefficients in the symmetry**

Requiring that

\[
\frac{\partial}{\partial \tau} F(x, t, u, u_x, u_{2x}, \ldots, u_{nx}) = \frac{\partial}{\partial t} G(x, t, u, u_x, u_{2x}, \ldots)
\]

holds. Compute \( G_t \) and \( F_\tau \)

Use the PDE,

\[
u_t = F
\]

to replace \( u_t, u_{tx}, u_{txx}, \ldots \)

Use

\[
u_\tau = G(x, t, u, u_x, u_{2x}, \ldots)
\]

to replace \( u_\tau, u_{\tau x}, u_{\tau xx}, \ldots \)

After grouping the terms

\[
F_\tau - G_t = (12c_1 - 18c_2)u^2 u_x u_{2x} + (6c_1 - 18c_3)uu^2 u_{2x} + (6c_1 - 18c_3)uu_x u_{3x} + (3c_2 - 60c_4)u^2 u_{3x} + (3c_2 + 3c_3 - 90c_4)u_x u_{4x} + (3c_3 - 30c_4)u_x u_{5x}
\]

\[
\equiv 0
\]

This yields

\[
S = \{12c_1 - 18c_2 = 0, \ 6c_1 - 18c_3 = 0, \ 3c_2 - 60c_4 = 0, \ 3c_2 + 3c_3 - 90c_4 = 0, \ 3c_3 - 30c_4 = 0\}
\]

Choosing \( c_4 = 1 \), the solution is \( c_1 = 30, \ c_2 = 20, \ c_3 = 10 \)

Hence

\[
G = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}
\]

which leads to Lax equation (in the KdV hierarchy)

\[
u_t + 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}
\]
- x-t Dependent Symmetries

Algorithm can be used provided the **degree** in \( x \) or \( t \) is given

Compute the symmetry of the KdV equation with rank 2 (**linear** in \( x \) or \( t \))

Build list of monomials in \( u, x \) and \( t \) of rank 2 or less

\[ \mathcal{L} = \{1, u, x, xu, t, tu, tu^2\} \]

Introduce the correct number of \( x \)-derivatives to make each term weight 2

\[
\begin{align*}
\frac{\partial}{\partial x}(xu) &= u + xu_x, \\
\frac{\partial}{\partial x}(tu^2) &= 2tuu_x, \\
\frac{\partial^3}{\partial x^3}(tu) &= tu_{3x}, \\
\frac{\partial^2}{\partial x^2}(1) &= \frac{\partial^3}{\partial x^3}(x) = \frac{\partial^5}{\partial x^5}(t) = 0
\end{align*}
\]

Gather the non-zero resulting terms

\[ \mathcal{R} = \{u, xu_x, tuu_x, tu_{3x}\} \]

Linearly combine the monomials to obtain

\[ G = c_1 u + c_2 xu_x + c_3 tuu_x + c_4 tu_{3x} \]

Determine the coefficients \( c_1 \) through \( c_4 \)

Compute \( G_t \) and \( F_\tau \) and remove all \( t \) and \( \tau \) derivatives (as before)

Group the terms

\[
F_\tau - G_t = (6c_1 + 6c_2 - c_3)uu_x + (3c_3 - 18c_4)tu_{2x}^2 + (3c_2 - c_4)u_{3x} + (3c_3 - 18c_4)tu_xu_{3x} \equiv 0
\]

This yields

\[ \mathcal{S} = \{6c_1 + 6c_2 - c_3 = 0, 3c_3 - 18c_4 = 0, 3c_2 - c_4 = 0\} \]

The solution is \( c_1 = \frac{2}{3}, c_2 = \frac{1}{3}, c_3 = 6, c_4 = 1 \)

Hence

\[ G = \frac{2}{3}u + \frac{1}{3}xu_x + 6tuu_x + tu_{3x} \]

These are two \( x-t \) dependent symmetries (of rank 0 and 2)

\[ G = 1 + 6tu_x \quad \text{and} \quad G = 2u + xu_x + t(6uu_x + u_{3x}) \]
Symmetries of DDEs

Consider a system of DDEs (continuous in time, discretized in space)

\[ \dot{u}_n = F(..., u_{n-1}, u_n, u_{n+1}, ...) \]

\( u_n \) and \( F \) have any number of components

**Definition of Symmetry**

A vector function \( G(..., u_{n-1}, u_n, u_{n+1}, ...) \) is called a symmetry of the DDE if the infinitesimal transformation

\[ u \rightarrow u + \epsilon G(..., u_{n-1}, u_n, u_{n+1}, ...) \]

leaves the DDE invariant within order \( \epsilon \)

Equivalently

\[ \frac{d}{d\tau} F(..., u_{n-1}, u_n, u_{n+1}, ...) = \frac{d}{dt} G(..., u_{n-1}, u_n, u_{n+1}, ...) \]

where \( \tau \) is the new time variable such that

\[ \frac{d}{d\tau} u = G(..., u_{n-1}, u_n, u_{n+1}, ...) \]

**Algorithm**

Consider the one-dimensional Toda lattice

\[ \ddot{y}_n = \exp (y_{n-1} - y_n) - \exp (y_n - y_{n+1}) \]

Change the variables

\[ u_n = \dot{y}_n, \quad v_n = \exp (y_n - y_{n+1}) \]

to write the lattice in algebraic form

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}) \]
This system is invariant under the scaling symmetry

\( (t, u_n, v_n) \to (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n) \)

\( \lambda \) is an arbitrary parameter. Hence, \( u_n \sim \frac{d}{dt} \) and \( v_n \sim \frac{d^2}{dt^2} \)

**Step 1: Determine the weights of variables**

Set \( w(t) = -1 \). Then \( w(u_n) = 1 \), and \( w(v_n) = 2 \)

**Step 2: Construct the form of the symmetry**

Compute the form of the symmetry of ranks \( \{3, 4\} \)

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less

\[ \mathcal{L}_1 = \{ u_n^3, u_n^2, u_n v_n, u_n, v_n \} \]

and of rank 4 or less

\[ \mathcal{L}_2 = \{ u_n^4, u_n^3, u_n^2 v_n, u_n^2, u_n v_n, u_n, v_n^2, v_n \} \]

For each monomial in both lists, introduce the adjusting number of \( t \)-derivatives so that each term exactly has weight 3 and 4, resp.

For the monomials in \( \mathcal{L}_1 \)

\[
\begin{align*}
\frac{d^0}{dt^0}(u_n^3) &= u_n^3, \\
\frac{d^0}{dt^0}(u_n v_n) &= u_n v_n, \\
\frac{d}{dt}(u_n^2) &= 2u_n u_n = 2u_n v_n - 2u_n v_n, \\
\frac{d}{dt}(v_n) &= \dot{v}_n = u_n v_n - u_{n+1} v_n, \\
\frac{d^2}{dt^2}(u_n) &= \frac{d}{dt}(u_n) = \frac{d}{dt}(v_n - v_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n
\end{align*}
\]

Gather the resulting terms in a set

\[ \mathcal{R}_1 = \{ u_n^3, u_{n-1} v_{n-1}, u_n v_{n-1}, u_n v_n, u_{n+1} v_n \} \]

Similarly

\[
\begin{align*}
\mathcal{R}_2 &= \{ u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1} u_n v_{n-1}, u_n^2 v_{n-1}, v_n - 2v_{n-1}, v_{n-1}^2, u_n^2 v_n, \\
&\quad u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_{n-1} v_n, v_n^2, v_n v_{n+1} \}
\end{align*}
\]
Linear combination of the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$ determines

$$G_1 = c_1 u_n^3 + c_2 u_{n-1} v_{n-1} + c_3 u_n v_{n-1} + c_4 u_n v_n + c_5 u_{n+1} v_n$$

$$G_2 = c_6 u_n^4 + c_7 u_{n-1}^2 v_{n-1} + c_8 u_{n-1} u_n v_{n-1} + c_9 u_n^2 v_{n-1} + c_{10} u_{n-2} v_{n-1} + c_{11} v_{n-1}^2 + c_{12} u_n^2 v_n + c_{13} u_n u_{n+1} v_n + c_{14} u_{n+1}^2 v_n + c_{15} u_{n-1} v_n + c_{16} v_n^2 + c_{17} v_n v_{n+1}$$

**Step 3: Determine the unknown coefficients in the symmetry**

Requiring that $F_\tau = G_t$ holds

Compute $\frac{d}{dt} G_1, \frac{d}{dt} G_2, \frac{d}{d\tau} F_1$ and $\frac{d}{d\tau} F_2$ and remove all $u_n, v_n, \frac{d}{d\tau} u_n, \frac{d}{d\tau} v_n$

Require that

$$\frac{d}{d\tau} F_1 - \frac{d}{dt} G_1 \equiv 0, \quad \frac{d}{d\tau} F_2 - \frac{d}{dt} G_2 \equiv 0$$

which gives

$$c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0,$$

$$-c_2 = -c_{12} = c_4 = c_5 = -c_{14} = -c_{15} = c_{17}$$

With $c_{17} = 1$ the symmetry is

$$G_1 = u_n v_n - u_{n-1} v_{n-1} + u_{n+1} v_n - u_n v_{n-1}$$

$$G_2 = u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1} v_n$$
• **Scope and Limitations of Algorithm & Software**

- Systems of PDEs or DDEs must be polynomial in dependent variables
- Only one space variable (continuous $x$ for PDEs, discrete $n$ for DDEs) is allowed
- No terms should *explicitly* depend on $x$ and $t$ for PDEs, or $n$ for DDEs
- Program only computes polynomial conserved densities; only polynomials in the dependent variables and their derivatives; no explicit dependencies on $x$ and $t$ for PDEs (or $n$ for DDEs)
- No limit on the number of PDEs or DDEs.
  In practice: time and memory constraints
- Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that densities (of a given rank) exist
- Systems can also have parameters with (unknown) weight. Allows one to test PDEs or DDEs of non-uniform rank
- For systems where one or more of the weights are free, the program prompts the user to enter values for the free weights
- Negative weights are not allowed
- Fractional weights and ranks are permitted
- Form of $\rho$ can be given in the data file (testing purposes)
• Conserved Densities Software

  – Conserved densities in DELiA by Bocharov (Pascal, 1990)
  – Conserved densities and formal symmetries FS by Gerdt and Zharkov (Reduce, 1993)
  – Formal symmetry approach by Mikhailov and Yamilov (MuMath, 1990)
  – Recursion operators and symmetries by Roelofs, Sanders and Wang (Reduce 1994, Maple 1995, Form 1995-present)
  – Conserved densities condens.m by Hereman and Göktaş (Mathematica, 1996)
  – Conservation laws, based on CRACK by Wolf (Reduce, 1995)
  – Conservation laws by Hickman (Maple, 1996)
  – Conserved densities by Ahner et al. (Mathematica, 1995). Project halted.
  – Conserved densities diffdens.m by Göktaş and Hereman (Mathematica, 1997)
• Conclusions and Further Research

– Two Mathematica programs are available:
  * condens.m for evolution equations (PDEs)
  * diffdens.m for differential-difference equations (DDEs)

– Usefulness

  * Testing models for integrability
  * Study of classes of nonlinear PDEs or DDEs

– Comparison with other programs

  * Parameter analysis is possible
  * Not restricted to uniform rank equations
  * Not restricted to evolution equations provided that one can write the equation(s) as a system of evolution equations

– Future work

  * Generalization towards broader classes of equations (e.g. $u_{xt}$)
  * Generalization towards more space variables (e.g. KP equation)
  * Conservation laws with time and space dependent coefficients
  * Conservation laws with $n$ dependent coefficients
* Exploit other symmetries in the hope to find conserved densities of non-polynomial form

* Constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)

- Research supported in part by NSF under Grant CCR-9625421
- In collaboration with Ünal Göktaş and Grant Erdmann
- Software: available via FTP, ftp site mines.edu in subdirectories
  
  pub/papers/math_cs_dept/software/condens
  pub/papers/math_cs_dept/software/diffdens

  or via the Internet

  URL: http://www.mines.edu/fs_home/whereman/
• More Examples

• Nonlinear Schrödinger Equation

\[ iq_t - q_{2x} + 2|q|^2q = 0 \]

Program can not handle complex equations

Replace by

\[ u_t - v_{2x} + 2v(u^2 + v^2) = 0 \]
\[ v_t + u_{2x} - 2u(u^2 + v^2) = 0 \]

where \( q = u + iv \)

Scaling properties

\[ u \sim v \sim \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2} \]

First seven conserved densities:

\[ \rho_1 = u^2 + v^2 \]
\[ \rho_2 = vu_x \]
\[ \rho_3 = u^4 + 2u^2v^2 + v^4 + u_x^2 + v_x^2 \]
\[ \rho_4 = u^2vu_x + \frac{1}{3}v^3u_x - \frac{1}{6}vu_{3x} \]
\[ \rho_5 = -\frac{1}{2} u^6 - \frac{3}{2} u^4 v^2 - \frac{3}{2} u^2 v^4 - \frac{1}{2} v^6 - \frac{5}{2} u^2 u_x^2 - \frac{1}{2} v^2 u_x^2 - \frac{3}{2} u^2 v_x^2 - \frac{5}{2} v^2 v_x^2 + u v^2 u_{2x} - \frac{1}{4} u_{2x}^2 - \frac{1}{4} v_{2x}^2 \]

\[ \rho_6 = -\frac{3}{4} u^4 v u_x - \frac{1}{2} u^2 v^3 u_x - \frac{3}{20} v^5 u_x + \frac{1}{4} v u_x^3 - \frac{1}{4} v u_x v_x^2 + u v u_x u_{2x} + \frac{1}{4} u^2 v u_{3x} + \frac{1}{12} v^3 u_{3x} - \frac{1}{40} v u_{5x} \]

\[ \rho_7 = \frac{5}{4} u^8 + 5 u^6 v^2 + \frac{15}{2} u^4 v^4 + 5 u^2 v^6 + \frac{5}{4} v^8 + \frac{35}{2} u^4 u_x^2 - 5 u^2 v u_x^2 + \frac{5}{2} v^4 u_x^2 - \frac{7}{4} u_x^4 + \frac{15}{2} u v^4 u_x^2 + 25 u^2 v^2 v_x^2 + \frac{35}{2} v^4 v_x^2 - \frac{5}{2} u x v_x^2 - \frac{7}{4} v_x^4 - 10 u^3 v^2 u_{2x} - 5 uv^4 u_{2x} - 5 u v u_x u_{2x} + \frac{7}{2} u^2 u_{2x}^2 + \frac{1}{2} v^2 u_{2x}^2 + \frac{5}{2} u^2 v_{2x}^2 + \frac{7}{2} v^2 v_{2x}^2 - v^2 u_x u_{3x} + \frac{1}{4} u_{3x}^2 + \frac{1}{4} v_{3x}^2 + u v^2 u_{4x} \]
The Ito system

\[
\begin{align*}
    u_t - u_{3x} - 6uu_x - 2vv_x &= 0 \\
    v_t - 2u_xv - 2uv_x &= 0
\end{align*}
\]

\[
\begin{align*}
    u &\sim \frac{\partial^2}{\partial x^2}, & v &\sim \frac{\partial^2}{\partial x^2}
\end{align*}
\]

\[
\begin{align*}
    \rho_1 &= c_1u + c_2v \\
    \rho_2 &= u^2 + v^2 \\
    \rho_3 &= 2u^3 + 2uv^2 - u_x^2 \\
    \rho_4 &= 5u^4 + 6u^2v^2 + v^4 - 10uu_x^2 + 2v^2u_{2x} + u_{2x}^2 \\
    \rho_5 &= 14u^5 + 20u^3v^2 + 6uv^4 - 70u^2u_x^2 + 10v^2u_{x}^2 \\
            &- 4v^2v_{x}^2 + 20uv^2u_{2x} + 14uu_{2x}^2 - u_{3x}^2 + 2v^2u_{4x}
\end{align*}
\]

and more conservation laws
The dispersiveless long-wave system

\[ u_t + vu_x + uv_x = 0 \]
\[ v_t + u_x + vv_x = 0 \]

\[ u \sim 2v \quad w(v) \text{ is free} \]

choose \[ u \sim \frac{\partial}{\partial x} \] and \[ 2v \sim \frac{\partial}{\partial x} \]

\[ \rho_1 = v \]
\[ \rho_2 = u \]
\[ \rho_3 = uv \]
\[ \rho_4 = u^2 + uv^2 \]
\[ \rho_5 = 3u^2v + uv^3 \]

\[ \rho_6 = \frac{1}{3}u^3 + u^2v^2 + \frac{1}{6}uv^4 \]

\[ \rho_7 = u^3v + u^2v^3 + \frac{1}{10}uv^5 \]

\[ \rho_8 = \frac{1}{3}u^4 + 2u^3v^2 + u^2v^4 + \frac{1}{15}uv^6 \]

and more

Always the same set irrespective the choice of weights
• A generalized Schamel equation

\[ n^2 u_t + (n + 1)(n + 2)u^{\frac{n}{2}} u_x + u_{3x} = 0 \]

where \( n \) is a positive integer

\[ \rho_1 = u, \quad \rho_2 = u^2 \]

\[ \rho_3 = \frac{1}{2} u_x^2 - \frac{n^2}{2} u^{2+\frac{2}{n}} \]

For \( n \neq 1, 2 \) no further conservation laws
• Three-Component Korteweg-de Vries Equation

\[ u_t - 6uu_x + 2vv_x + 2ww_x - u_{3x} = 0 \]
\[ v_t - 2vu_x - 2uv_x = 0 \]
\[ w_t - 2wu_x - 2uw_x = 0 \]

Scaling properties
\[ u \sim v \sim w \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3} \]

First five densities:
\[ \rho_1 = c_1u + c_2v + c_3w \]
\[ \rho_2 = u^2 - v^2 - w^2 \]
\[ \rho_3 = -2u^3 + 2uv^2 + 2uw^2 + u_x^2 \]
\[ \rho_4 = -\frac{5}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 + 3u^2w^2 - v^2w^2 - \frac{1}{2}w^4 \]
\[ + 5uu_x^2 + v^2u_{2x} + w^2u_{2x} - \frac{1}{2}w_{2x}^2 \]
\[ \rho_5 = -\frac{7}{10}u^5 + u^3v^2 - \frac{3}{10}uv^4 + u^3w^2 - \frac{3}{5}uv^2w^2 - \frac{3}{10}uw^4 \]
\[ + \frac{7}{2}u^2u_x^2 + \frac{1}{2}v^2u_x^2 + \frac{1}{2}w^2u_x^2 + \frac{1}{5}v^2v_x^2 \]
\[ - \frac{1}{5}w^2v_x^2 + \frac{1}{5}w^2w_x^2 + uv^2u_{2x} + uw^2u_{2x} - \frac{7}{10}uu_{2x}^2 \]
\[ - \frac{1}{5}vw^2v_{2x} + \frac{1}{20}u_{3x}^2 + \frac{1}{10}v^2u_{4x} + \frac{1}{10}w^2u_{4x} \]
- The Deconinck-Meuris-Verheest equation

Consider the modified vector derivative NLS equation:
\[
\frac{\partial B_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 B_\perp) + \alpha B_\perp B_\perp_0 \cdot \frac{\partial B_\perp}{\partial x} + e_x \times \frac{\partial^2 B_\perp}{\partial x^2} = 0
\]

Replace the vector equation by
\[
\begin{align*}
  u_t + (u(u^2 + v^2) + \beta u - v_x)_x &= 0 \\
v_t + (v(u^2 + v^2) + u_x)_x &= 0
\end{align*}
\]

\(u\) and \(v\) denote the components of \(B_\perp\) parallel and perpendicular to \(B_\perp_0\) and \(\beta = \alpha B_{\perp 0}^2\)

\[
\begin{align*}
u^2 &\sim \frac{\partial}{\partial x}, & v^2 &\sim \frac{\partial}{\partial x}, & \beta &\sim \frac{\partial}{\partial x}
\end{align*}
\]

First 6 conserved densities

\[
\begin{align*}
  \rho_1 &= c_1 u + c_2 v \\
  \rho_2 &= u^2 + v^2 \\
  \rho_3 &= \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2 \\
  \rho_4 &= \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)
\end{align*}
\]
\[ \rho_5 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{2x} - u_{2x} v_x) + \frac{4}{5}(u u_x + v v_x)^2 \]

\[ + \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(u v_x - u_x v) \]

\[ + \frac{\beta}{5}(2 u_x^2 - 4 u^3 v_x + 2 u^6 + 3 u^4 v^2 - v^6) + \frac{\beta^2}{5} u^4 \]

\[ \rho_6 = \frac{7}{16}(u^2 + v^2)^5 + \frac{1}{2}(u_{2x}^2 + v_{2x}^2) \]

\[ - \frac{5}{2}(u^2 + v^2)(u_x v_{2x} - u_{2x} v_x) + 5(u^2 + v^2)(u u_x + v v_x)^2 \]

\[ + \frac{15}{4}(u^2 + v^2)^2(u_x^2 + v_x^2) - \frac{35}{16}(u^2 + v^2)^3(u v_x - u_x v) \]

\[ + \frac{\beta}{8}(5 u^8 + 10 u^6 v^2 - 10 u^2 v^6 - 5 v^8 + 20 u^2 u_x^2 \]

\[ - 12 u^5 v_x + 60 u v^4 v_x - 20 v^2 v_x^2) \]

\[ + \frac{\beta^2}{4}(u^6 + v^6) \]