Symbolic Computation of Travelling Wave Solutions of Nonlinear Differential-Difference Equations

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OUTLINE

Purpose & Motivation

Typical Examples of ODEs, PDEs, and DDEs

Part I: Tanh Method for PDEs

Quick Review of Tanh Algorithm

A Typical Example

Other Types of Functions

Demo of Mathematica Software: PDESpecialSolutions.m

Part II: Tanh Method for DDEs (Lattices)

Algorithm for Tanh Solutions

Typical Examples

Table with Results

Demo of Mathematica Software: DDESpecialSolutions.m

Conclusions & Future Research

Research Papers & Software
Purpose & Motivation

• **Develop** and implement various **methods** to find closed form solutions of nonlinear PDEs and DDEs: direct methods, Lie symmetry methods, similarity methods, etc.

• Fully **automate** the hyperbolic and elliptic function methods to compute travelling solutions of nonlinear PDEs.

• Fully **automate** the hyperbolic tanh method to compute travelling wave solutions of nonlinear differential-difference equations (DDEs or lattices).

• **Class** of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations, and lattices.

  **Examples PDEs:** Korteweg-de Vries, Boussinesq, and Kuramoto-Sivashinsky equations.
  Fisher and FitzHugh-Nagumo equations.

  **Examples ODEs:** Duffing and nonlinear oscillator equations.

  **Examples DDEs:** Volterra, Toda, and Ablowitz-Ladik lattices.

• **PDEs:** Solutions of tanh (kink) or sech (pulse) type **model** solitary waves in fluids, plasmas, circuits, optical fibers, bio-genetics, etc.

  **DDEs:** discretizations of PDEs, lattice theory, queing and network problems, solid state and quantum physics.

• **Benchmark** solutions for numerical PDE and DDE solvers.
• **Research aspect:** Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations and lattices.

• **Educational aspect:** Software as course ware for courses in nonlinear PDEs and DDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.

REU projects of NSF. Extreme Programming!

• **Users:** scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.
Typical Examples of ODEs and PDEs

• The Duffing equation:

\[ u'' + u + \alpha u^3 = 0 \]

Solutions in terms of elliptic functions:

\[ u(x) = \pm \frac{\sqrt{c_1^2 - 1}}{\sqrt{\alpha}} \text{cn}(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}) , \]

and

\[ u(x) = \pm \frac{\sqrt{2(c_1^2 - 1)}}{\sqrt{\alpha}} \text{sn}(c_1 x + \Delta; \frac{1 - c_1^2}{c_1^2}) . \]

• The Korteweg-de Vries (KdV) equation:

\[ u_t + 6\alpha uu_x + u_{3x} = 0 . \]

Solitary wave solution:

\[ u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \Delta] \],

or, equivalently,

\[ u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \text{sech}^2 [c_1 x + c_2 t + \Delta] . \]

Cnoidal wave solution:

\[ u(x, t) = \frac{4c_1^3 (1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \text{cn}^2 (c_1 x + c_2 t + \Delta; m) , \]

modulus \( m \).
• The modified Korteweg-de Vries (mKdV) equation:
  \[ u_t + \alpha u^2 u_x + u_{3x} = 0. \]
  
  Solitary wave solution:
  \[ u(x, t) = \pm \sqrt{\frac{6}{\alpha}} c_1 \operatorname{sech} \left[ c_1 x - c_1^3 t + \Delta \right]. \]

• Three-dimensional modified Korteweg-de Vries equation:
  \[ u_t + 6u^2 u_x + u_{xyz} = 0. \]
  
  Solitary wave solution:
  \[ u(x, y, z, t) = \pm \sqrt{c_2 c_3 \operatorname{sech} \left[ c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta \right]} \].

• The combined KdV-mKdV equation:
  \[ u_t + 6\alpha uu_x + 6\beta u^2 u_x + \gamma u_{3x} = 0. \]
  
  Real solitary wave solution:
  \[ u(x, t) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\gamma}{\beta}} c_1 \operatorname{sech} \left( c_1 x + \frac{c_1}{2\beta} (3\alpha^2 - 2\beta \gamma c_1^2) t + \Delta \right). \]
  
  Complex solutions:
  \[ u(x, t) = -\frac{\alpha}{2\beta} \pm i \sqrt{\frac{\gamma}{\beta}} c_1 \operatorname{tanh} \left( c_1 x + \frac{c_1}{2\beta} (3\alpha^2 + 4\beta \gamma c_1^2) t + \Delta \right), \]
  
  \[ u(x, t) = -\frac{\alpha}{2\beta} + \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 \left( \operatorname{sech} \xi \pm i \operatorname{tanh} \xi \right), \]
  
  and
  \[ u(x, t) = -\frac{\alpha}{2\beta} - \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 \left( \operatorname{sech} \xi \mp i \operatorname{tanh} \xi \right) \]
  
  with \( \xi = c_1 x + \frac{c_1}{2\beta} (3\alpha^2 + \beta \gamma c_1^2) t + \Delta \).
• The Fisher equation:

\[ u_t - u_{xx} - u(1 - u) = 0. \]

Solitary wave solution:

\[ u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \xi + \frac{1}{4} \tanh^2 \xi, \]

with

\[ \xi = \pm \frac{1}{2\sqrt{6}} x \pm \frac{5}{12} t + \Delta. \]

• The generalized Kuramoto-Sivashinski equation:

\[ u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0. \]

Solitary wave solutions (ignoring symmetry \( u \to -u, x \to -x, \sigma \to -\sigma \)):

For \( \sigma = 4 \):

\[ u(x, t) = 9 - 2c^2 - 15 \tanh \xi (1 + \tanh \xi - \tanh^2 \xi) \]

with \( \xi = \frac{x}{2} + c^2 t + \Delta. \)

For \( \sigma = \frac{12}{\sqrt{47}} \):

\[ u(x, t) = \frac{45 \mp 4418c^2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh \xi - \frac{45}{47\sqrt{47}} \tanh^2 \xi \pm \frac{15}{47\sqrt{47}} \tanh^3 \xi \]

with \( \xi = \pm \frac{1}{2\sqrt{47}} x + c^2 t + \Delta. \)
For $\sigma = 16/\sqrt{73}$:

$$u(x, t) = \frac{2}{73^{\sqrt{73}}} \left( 30 \pm 5329c_2 \right) \pm \frac{75}{73^{\sqrt{73}}} \tanh \xi - \frac{60}{73^{\sqrt{73}}} \tanh^2 \xi \pm \frac{15}{73^{\sqrt{73}}} \tanh^3 \xi$$

with $\xi = \pm \frac{1}{2\sqrt{73}} x + c_2 t + \Delta$.

For $\sigma = 0$:

$$u(x, t) = -2 \left\{ \sqrt{\frac{19}{11}} c_2 - \frac{135}{19} \sqrt{\frac{11}{19}} \tanh \xi + \frac{165}{19} \sqrt{\frac{11}{19}} \tanh^3 \xi \right\}$$

with $\xi = \frac{1}{2\sqrt{19}} x + c_2 t + \Delta$.

- The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3u u_{2x} + 3u_x^2 + \alpha u_{4x} = 0,$$

or written as a first-order system ($v$ auxiliary variable):

$$u_t + v_x = 0,$$

$$v_t + u_x - 3uu_x - \alpha u_{3x} = 0.$$

Solitary wave solution:

$$u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_4^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta],$$

$$v(x, t) = b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta].$$

- The Broer-Kaup system:

$$u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xyy} = 0,$$

$$v_t + 2(uv)_x + v_{xx} = 0.$$

Solitary wave solution:

$$u(x, t) = -\frac{c_3}{2c_1} + c_1 \tanh [c_1 x + c_2 y + c_3 t + \Delta],$$

$$v(x, t) = c_1 c_2 - c_1 c_2 \tanh^2 [c_1 x + c_2 y + c_3 t + \Delta].$$
• System of three nonlinear coupled equations (Gao & Tian, 2001):

\[ u_t - u_x - 2v = 0, \]
\[ v_t + 2uw = 0, \]
\[ w_t + 2uv = 0. \]

Solutions:

\[ u(x, t) = \pm c_2 \tanh \xi, \]
\[ v(x, t) = \pm \frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi, \]
\[ w(x, t) = -\frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi, \]

and

\[ u(x, t) = \pm ic_2 \sech \xi, \]
\[ v(x, t) = \pm \frac{1}{2} ic_2 (c_1 - c_2) \tanh \xi \sech \xi, \]
\[ w(x, t) = \frac{1}{4} c_2 (c_1 - c_2) \left( 1 - 2 \sech^2 \xi \right), \]

and also

\[ u(x, t) = \pm \frac{1}{2} ic_2 \left( \sech \xi + i \tanh \xi \right), \]
\[ v(x, t) = \pm \frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right), \]
\[ w(x, t) = -\frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right) \]

with \( \xi = c_1 x + c_2 t + \Delta. \)
• Nonlinear sine-Gordon equation (light cone coordinates):

\[ \Phi_{xt} = \sin \Phi. \]

Set \( u = \Phi_x, \ v = \cos(\Phi) - 1, \)

\[ u_{xt} - u - u v = 0, \]
\[ u_t^2 + 2v + v^2 = 0. \]

Solitary wave solution (kink):

\[ u = \pm \frac{1}{\sqrt{-c}} \sech\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right], \]
\[ v = 1 - 2 \sech^2\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right]. \]

Solution:

\[ \Phi(x, t) = \int u(x, t) \, dx = \pm 4 \arctan \left[ \exp \left( \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right) \right]. \]

• ODEs from quantum field theory:

\[ u_{xx} = -u + u^3 + auv^2, \]
\[ v_{xx} = bv + cv^3 + av(u^2 - 1). \]

Solitary wave solutions:

\[ u = \pm \tanh\left[ \sqrt{\frac{a^2 - c}{2(a - c)}}x + \Delta \right], \]
\[ v = \pm \sqrt{\frac{1 - a}{a - c}} \sech\left[ \sqrt{\frac{a^2 - c}{2(a - c)}}x + \Delta \right], \]

provided \( b = \sqrt{\frac{a^2 - c}{2(a - c)}}. \)
Typical Examples of DDEs (lattices)

- The Volterra lattice:
  \[
  \begin{aligned}
  \dot{u}_n &= u_n(v_n - v_{n-1}), \\
  \dot{v}_n &= v_n(u_{n+1} - u_n).
  \end{aligned}
  \]

  Travelling wave solution:
  \[
  \begin{aligned}
  u_n(t) &= -c_1 \coth(d_1) + c_1 \tanh [d_1 n + c_1 t + \delta], \\
  v_n(t) &= -c_1 \coth(d_1) - c_1 \tanh [d_1 n + c_1 t + \delta].
  \end{aligned}
  \]

- The Toda lattice:
  \[
  \begin{aligned}
  \ddot{u}_n &= (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}).
  \end{aligned}
  \]

  Travelling wave solution:
  \[
  u_n(t) = a_{10} \pm \sinh(d_1) \tanh [d_1 n \pm \sinh(d_1) t + \delta].
  \]

- The Relativistic Toda lattice:
  \[
  \begin{aligned}
  \dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\
  \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
  \end{aligned}
  \]

  Travelling wave solution:
  \[
  \begin{aligned}
  u_n(t) &= -\frac{1}{\alpha} - c_1 \coth(d_1) + c_1 \tanh [d_1 n + c_1 t + \delta], \\
  v_n(t) &= \frac{c_1 \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh [d_1 n + c_1 t + \delta].
  \end{aligned}
  \]
• The Ablowitz-Ladik lattice:

\[ \dot{u}_n(t) = (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \]

\[ \dot{v}_n(t) = -(\alpha + u_n v_n)(v_{n+1} + v_{n-1}) + 2\alpha v_n. \]

Travelling wave solution:

\[ u_n(t) = \frac{\alpha \sinh^2(d_1)}{a_{21}} (\pm 1 - \tanh [d_1 n + 2\alpha t \sinh^2(d_1) + \delta]), \]

\[ v_n(t) = a_{21} (\pm 1 + \tanh [d_1 n + 2\alpha \sinh^2(d_1)t + \delta]). \]

• 2D Toda lattice:

\[ \frac{\partial^2 u_n}{\partial x \partial t}(x, t) = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}). \]

Travelling wave solution:

\[ u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[ d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right]. \]

• Hybrid lattice:

\[ \dot{u}_n(t) = (1 + \alpha u_n + \beta u_n^2)(u_{n-1} - u_{n+1}), \]

Travelling wave solution:

\[ u_n(t) = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta} \tanh(d_1)}{2\beta} \tanh \left[ d_1 n + \frac{\alpha^2 - 4\beta}{2\beta} \tanh(d_1)t + \delta \right]. \]
Algorithm for Tanh Solutions for system of PDEs

System of nonlinear PDEs of order $m$

$$\Delta(u(x), u'(x), u''(x), \cdots u^{(m)}(x)) = 0.$$ 

Dependent variable $u$ has $M$ components $u_i$ (or $u, v, w, \ldots$).

Independent variable $x$ has $N$ components $x_j$ (or $x, y, z, \ldots, t$).

Step T1:

- Seek solution $u(x) = U(T)$, with

  $$T = \tanh \xi = \tanh \left[ \sum_j^N c_j x_j + \delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, all derivative of $T$ are polynomial in $T$. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

  $$\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d \bullet}{dT} = c_j (1 - T^2) \frac{d \bullet}{dT}$$

  Produces a nonlinear system of ODEs

  $$\Delta(T, U(T), U'(T), U''(T), \ldots, U^{(m)}(T)) = 0.$$ 

NOTE: Compare with the ultra-spherical (linear) ODE:

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0$$

with integer $n \geq 0$ and $\alpha$ real. Includes:

- Legendre equation ($\alpha = \frac{1}{2}$),
- ODE for Chebyshev polynomials of type I ($\alpha = 0$),
- ODE for Chebyshev polynomials of type II ($\alpha = 1$).
• Example: For the Boussinesq system

\[ u_t + v_x = 0, \]
\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \]

after cancelling common factors \( 1 - T^2 \),

\[ c_2 U' + c_1 V' = 0, \]
\[ c_2 V' + c_1 U' - 3c_1 U U' \]
\[ + \alpha c_1^3 \left[ 2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2 U''' \right] = 0. \]

Step T2:

• Seek polynomial solutions

\[ U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j. \]

Determine the highest exponents \( M_i \geq 1 \).

Substitute \( U_i(T) = T^{M_i} \) into the LHS of ODE.

Gives polynomial \( P(T) \).

For every \( P_i \) consider all possible balances of the highest exponents in \( T \).

Solve the resulting linear system(s) for the unknowns \( M_i \).

• Example: Balance highest exponents for the Boussinesq system

\[ M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1. \]

So, \( M_1 = M_2 = 2 \).

Hence,

\[ U(T) = a_{10} + a_{11} T + a_{12} T^2, \]
\[ V(T) = a_{20} + a_{21} T + a_{22} T^2. \]
Step T3:

- Derive algebraic system for the unknown coefficients \( a_{ij} \) by setting to zero the coefficients of the power terms in \( T \).

- Example: Algebraic system for Boussinesq case
  
  \[
  a_{11} c_1 (3a_{12} + 2\alpha c_1^2) = 0, \\
  a_{12} c_1 (a_{12} + 4\alpha c_1^2) = 0, \\
  a_{21} c_1 + a_{11} c_2 = 0, \\
  a_{22} c_1 + a_{12} c_2 = 0, \\
  a_{11} c_1 - 3a_{10} a_{11} c_1 + 2\alpha a_{11} c_1^3 + a_{21} c_2 = 0, \\
  -3a_{11}^2 c_1 + 2a_{12} c_1 - 6a_{10} a_{12} c_1 + 16\alpha a_{12} c_1^3 + 2a_{22} c_2 = 0.
  \]

Step T4:

- Solve the nonlinear algebraic system with parameters.

- Example: Solution for Boussinesq system
  
  \[
  a_{10} = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0, \\
  a_{12} = -4\alpha c_1^2, \quad a_{20} = \text{free}, \\
  a_{21} = 0, \quad a_{22} = 4\alpha c_1 c_2.
  \]

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.

- Example: Solitary wave solution for Boussinesq system:
  
  \[
  u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \delta], \\
  v(x, t) = a_{20} + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \delta].
  \]
Other Types of Solutions for PDEs

<table>
<thead>
<tr>
<th>Function</th>
<th>ODE ( y' = \frac{dy}{d\xi} )</th>
<th>Chain Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tanh(\xi) )</td>
<td>( y' = 1 - y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{dT} )</td>
</tr>
<tr>
<td>( \sech(\xi) )</td>
<td>( y' = -y \sqrt{1 - y^2} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = -c_j s \sqrt{1 - s^2} \frac{d \bullet}{ds} )</td>
</tr>
<tr>
<td>( \tan(\xi) )</td>
<td>( y' = 1 + y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j (1 + \tan^2) \frac{d \bullet}{dT \tan} )</td>
</tr>
<tr>
<td>( \exp(\xi) )</td>
<td>( y' = y )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j E \frac{d \bullet}{dE} )</td>
</tr>
<tr>
<td>( \text{cn}(\xi; m) )</td>
<td>( y' = -\sqrt{(1 - y^2)(1 - m + m y^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{(1 - \text{cn}^2)(1 - m + m \text{cn}^2)} \frac{d \bullet}{d \text{cn}} )</td>
</tr>
<tr>
<td>( \text{sn}(\xi; m) )</td>
<td>( y' = \sqrt{(1 - y^2)(1 - m y^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j \sqrt{(1 - \text{sn}^2)(1 - m \text{sn}^2)} \frac{d \bullet}{d \text{sn}} )</td>
</tr>
</tbody>
</table>
Algorithm for Tanh Solutions for System of DDEs

Nonlinear differential-difference equations (DDEs) of order $m$

$$\Delta(u_{n+p_1}(x), u_{n+p_2}(x), \ldots, u_{n+p_k}(x), u'_{n+p_1}(x), u'_{n+p_2}(x), \ldots, u'_{n+p_k}(x), \ldots, u^{(r)}_{n+p_1}(x), u^{(r)}_{n+p_2}(x), \ldots, u^{(r)}_{n+p_k}(x)) = 0.$$ 

Dependent variable $u_n$ has $M$ components $u_{i,n}$ (or $u_n, v_n, w_n, \ldots$) 
Independent variable $x$ has $N$ components $x_i$ (or $t, x, y, \ldots$). 
Shift vectors $p_i \in \mathbb{Z}^Q$. 
$u^{(r)}(x)$ is collection of mixed derivatives of order $r$.

Simplest case for independent variable ($t$), and one lattice point ($n$): 

$$\Delta(..., u_{n-1}, u_n, u_{n+1}, ..., \dot{u}_{n-1}, \dot{u}_n, \dot{u}_{n+1}, ..., u^{(r)}_{n-1}, u^{(r)}_n, u^{(r)}_{n+1}, ...) = 0.$$ 

Step D1:

- Seek solution $u_n(x) = U_n(T_n)$, with $T_n = \tanh(\xi_n), \quad \xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = d \cdot n + c \cdot x + \delta.$

- Repeatedly apply the operator rule

  $$\frac{d \bullet}{dx_j} = \frac{\partial \xi_n}{\partial x_j} \frac{dT_n}{dT_n} \frac{d \bullet}{dT_n} = c_j (1 - T_n^2) \frac{d \bullet}{dT_n},$$

  transforms DDE into

  $$\Delta(U_{n+p_1}(T_n), \ldots, U_{n+p_k}(T_n), U'_{n+p_1}(T_n), \ldots, U'_{n+p_k}(T_n), \ldots, U^{(r)}_{n+p_1}(T_n), \ldots, U^{(r)}_{n+p_k}(T_n)) = 0.$$ 

Note: $U_{n+p_s}$ is function of $T_n$ not of $T_{n+p_s}$.
• Example: Toda lattice

\[ \ddot{u}_n = (1 + \dot{u}_n) (u_{n-1} - 2u_n + u_{n+1}) \]

transforms into

\[ c_1^2 (1 - T_n^2) \left[ 2T_n U'_n - (1 - T_n^2) U''_n \right] + \left[ 1 + c_1 (1 - T_n^2) U'_n \right] [U_{n-1} - 2U_n + U_{n+1}] = 0. \]

**Step D2:**

• Seek polynomial solutions

\[ U_{i,n}(T_n) = \sum_{j=0}^{M_i} a_{ij} T_n^j. \]

Use \( \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \) to deal with the shift:

\[ T_{n+p_s} = \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s}, \]

where

\[ \phi_s = p_s \cdot d = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q, \]

Substitute \( U_{i,n}(T_n) = T_n^{M_i} \), and

\[ U_{i,n+p_s}(T_n) = T_n^{M_i} = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i}, \]

and balance the highest exponents in \( T_n \) to determine \( M_i \).

**Note:** \( U_{i,n+0}(T_n) = T_n^{M_i} \) is of degree \( M_i \) in \( T_n \).

\[ U_{i,n+p_s}(T_n) = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i} \] is of degree zero in \( T_n \).
• Example: Balance of exponents for Toda lattice

\[ 2M_1 + 1 = M_1 + 2. \]

So, \( M_1 = 1 \). Hence,

\[
U_n(T_n) = a_{10} + a_{11}T_n, \\
U_{n\pm1}(T(n \pm 1)) = a_{10} + a_{11}T(n \pm 1) = a_{10} + a_{11} \frac{T_n \pm \tanh(d_1)}{1 \pm T_n \tanh(d_1)}.
\]

**Step D3:**

• Determine the algebraic system for the unknown coefficients \( a_{ij} \) by setting to zero the coefficients of the powers in \( T_n \).

• Example: Algebraic system for Toda lattice

\[
c_1^2 - \tanh^2(d_1) - a_{11}c_1 \tanh^2(d_1) = 0, \\
c_1 - a_{11} = 0.
\]

**Step D4:**

• Solve the nonlinear algebraic system with parameters.

• Example: Solution of algebraic system for Toda lattice

\[
a_{10} = \text{free}, \quad a_{11} = \pm \sinh(d_1), \quad c_1 = \pm \sinh(d_1).
\]

**Step D5:**

• Return to the original variables. Test solution(s) of DDE. Reject trivial ones.

• Example: Solitary wave solution for Toda lattice:

\[
u_n(t) = a_{10} \pm \sinh(d_1) \tanh\left[ d_1n \pm \sinh(d_1) t + \delta \right].
\]
Example of System of DDEs: Relativistic Toda Lattice

\[ \dot{u}_n = (1 + \alpha u_n)(v_n - v_{n-1}), \]
\[ \dot{v}_n = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \]

Change of variables

\[ u_n(t) = U_n(T_n), \quad v_n(t) = V_n(T_n), \]

with

\[ T_n(t) = \tanh [d_1 n + c_1 t + \delta]. \]

gives

\[ c_1 (1 - T^2) U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) = 0, \]
\[ c_1 (1 - T^2) V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) = 0. \]

Seek polynomial solutions

\[ U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \quad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j. \]

Balance the highest exponents in \( T_n \) to determine \( M_1 \), and \( M_2 \):

\[ M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2. \]

So, \( M_1 = M_2 = 1 \). Hence,

\[ U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n. \]
Algebraic system for $a_{ij}$:

\[-a_{11}c_1 + a_{21}\tanh(d_1) + \alpha a_{10}a_{21}\tanh(d_1) = 0,\]
\[a_{11}\tanh(d_1)(\alpha a_{21} + c_1) = 0,\]
\[-a_{21}c_1 + a_{11}a_{20}\tanh(d_1) + 2\alpha a_{20}a_{21}\tanh(d_1) = 0,\]
\[\tanh(d_1)(a_{11}a_{21} + 2\alpha a_{21}^2 - a_{11}a_{20}\tanh(d_1)) = 0,\]
\[a_{21}\tanh^2(d_1)(c_1 - a_{11}) = 0.\]

Solution of the algebraic system

\[a_{10} = -\frac{1}{\alpha} - c_1\coth(d_1),\]
\[a_{11} = c_1,\]
\[a_{20} = \frac{c_1\coth(d_1)}{\alpha},\]
\[a_{21} = -\frac{c_1}{\alpha}.\]

Solitary wave solution in original variables:

\[u_n(t) = -\frac{1}{\alpha} - c_1\coth(d_1) + c_1\tanh[d_1n + c_1t + \Delta],\]
\[v_n(t) = \frac{c_1\coth(d_1)}{\alpha} - \frac{c_1}{\alpha}\tanh[d_1n + c_1t + \Delta].\]
Multi-dimensional Example: 2D Toda Lattice

2D Toda lattice:

\[ \frac{\partial^2 y_n}{\partial x \partial t} = \exp (y_{n-1} - y_n) - \exp (y_n - y_{n+1}), \]

\( y_n(x,t) \) is displacement from equilibrium of the \( n \)-th unit mass under an exponential decaying interaction force between nearest neighbors.

Set

\[ \frac{\partial u_n}{\partial t} = \exp (y_{n-1} - y_n) - 1. \quad (*) \]

Then,

\[ \exp (y_{n-1} - y_n) = \frac{\partial u_n}{\partial t} + 1, \]

and the 2D-Toda lattice becomes

\[ \frac{\partial^2 y_n}{\partial x \partial t} = \frac{\partial u_n}{\partial t} + 1 - \left( \frac{\partial u_{n+1}}{\partial t} + 1 \right) = \frac{\partial u_n}{\partial t} - \frac{\partial u_{n+1}}{\partial t}. \]

Integrate with respect to \( t \) to get

\[ \frac{\partial y_n}{\partial x} = u_n - u_{n+1}. \]

Differentiate (*) with respect to \( x \) and

\[ \frac{\partial^2 u_n}{\partial x \partial t} = \frac{\partial}{\partial x} (\exp (y_{n-1} - y_n) - 1) \]

\[ = \exp (y_{n-1} - y_n) \left( \frac{\partial y_{n-1}}{\partial x} - \frac{\partial y_{n}}{\partial x} \right), \]

\[ = \left( \frac{\partial u_n}{\partial t} + 1 \right) \left( u_{n-1} - u_n - (u_n - u_{n+1}) \right), \]

\[ = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}). \]
So, the 2D Toda lattice is written in polynomial form:

\[
\frac{\partial^2 u_n}{\partial x \partial t} = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}).
\]

Travelling wave solution:

\[
u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[ d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right].
\]

**Complicated Example: Ablowitz-Ladik Lattice**

The Ablowitz-Ladik lattice:

\[
\dot{u}_n(t) = (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n,
\]

\[
\dot{v}_n(t) = - (\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n.
\]

Travelling wave solution:

\[
u_n(t) = \frac{\alpha \sinh^2(d_1)}{a_{21}} \left( \pm 1 - \tanh \left[ d_1 n + 2\alpha \sinh^2(d_1) + \delta \right] \right),
\]

\[
v_n(t) = a_{21} (\pm 1 + \tanh \left[ d_1 n + 2\alpha \sinh^2(d_1) t + \delta \right]).
\]
Analyzing and Solving Nonlinear Parameterized Systems

Assumptions:

• All $c_i \neq 0$ and $d_i \neq 0$ (and modulus $m \neq 0$).
• Parameters ($\alpha, \beta, \gamma, \ldots$). Otherwise the maximal exponents $M_i$ may change.
• All $M_i \geq 1$.
• All $a_i M_i \neq 0$. Highest power terms in $U_i$ must be present, except in mixed sech-tanh-method.
• Solve for $a_{ij}$, then $c_i$, tanh$(d_i)$, and $m$. Then find conditions on parameters.

Strategy followed by hand:

• Solve all linear equations in $a_{ij}$ first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
• Solve linear equations in $c_i$, tanh$(d_i)$, $m$ if they are free of $a_{ij}$.
• Solve linear equations in parameters if they free of $a_{ij}, c_i$, tanh$(d_i)$, $m$.
• Solve quasi-linear equations for $a_{ij}, c_i$, tanh$(d_i)$, $m$.
• Solve quadratic equations for $a_{ij}, c_i$, tanh$(d_i)$, $m$.
• Eliminate cubic terms for $a_{ij}, c_i$, tanh$(d_i)$, $m$, without solving.
• Show remaining equations, if any.

Alternatives:

• Use (adapted) Gröbner bases techniques.
• Use Ritt-Wu characteristic sets method.
• Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$. 
Implementation Issues – Software Demo – Future Work

- Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and tanh function for DDEs.

- Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.

- Implement various methods: Lie symmetry methods, etc.

- Look at other types of explicit solutions involving
  - other hyperbolic and elliptic functions sinh, cosh, dn, ....
  - complex exponentials combined with sech or tanh.

- Other applications (of the nonlinear algebraic solver):
  Computation of conservation laws, symmetries, first integrals, etc. leading to linear parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).
• Preprints:


  Available from http://www.mines.edu/fs_home/whereman/

• Software:

  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/pde-sols;
  or via Internet URL: http://www.mines.edu/fs_home/whereman/


  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/dde-sols;
  or via Internet URL: http://www.mines.edu/fs_home/whereman/