Continuous and Discrete Homotopy Operators
with Applications in Integrability Testing

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Talk dedicated to Ryan Sayers

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OUTLINE

Part I: Continuous Case
Integration by Parts on the Jet Space (by hand) + Mathematica Experiment
Exactness or Integrability Criterion: Continuous Euler Operator
Continuous Homotopy Operator
Application of Continuous Homotopy Operator
Demo of Mathematica software

Part II: Discrete Case
Inverting the Total Difference Operator (by hand)
Exactness or ‘Total Difference’ Criterion: Discrete Euler Operator
Discrete Homotopy Operator
Application of Discrete Homotopy Operator
Demo of Mathematica Software
Future Research
Problem Statement

For continuous case:
Given, for example,
\[ f = 3u'v^2 \sin(u) - u'^3 \sin(u) - 6v' \cos(u) + 2u'u'' \cos(u) + 8v'v'' \]
Find \( F \) so that \( f = D_x F \) or \( F = \int f \, dx \).
Result:
\[ F = 4v'^2 + u'^2 \cos(u) - 3v^2 \cos(u) \]
Can this be done without integration by parts?
Can the problem be reduced to a single integral in one variable?

For discrete case:
Given, for example,
\[ f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]
Find \( F_n \) so that \( f_n = \Delta F_n = F_{n+1} - F_n \) or \( F_n = \Delta^{-1} f_n \).
Result:
\[ F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1} \]
How can this be done algorithmically?
Can this be done in the same way as the continuous case?
Part I: Continuous Case

Integration by Parts on the Jet Space

• Given \( f \) involving \( u(x) \) and \( v(x) \) and their derivatives

\[
f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v''
\]

• Find \( F \) so that \( f = D_x F \) or \( F = \int f \, dx \).

Integrate by parts (compute \( F \) by hand)

<table>
<thead>
<tr>
<th>( f )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 8 v''v' )</td>
<td>( 4 v'^2 )</td>
</tr>
<tr>
<td>( 2 u' u'' \cos(u) )</td>
<td>( u'^2 \cos(u) )</td>
</tr>
<tr>
<td>( -u'^3 \sin(u) )</td>
<td></td>
</tr>
<tr>
<td>( -6 v v' \cos(u) )</td>
<td>( -3 v^2 \cos(u) )</td>
</tr>
<tr>
<td>( 3 u' v^2 \sin(u) )</td>
<td></td>
</tr>
</tbody>
</table>

• Integral:

\[
F = 4 v'^2 + u'^2 \cos(u) - 3 v^2 \cos(u)
\]

**Remark:** For simplicity we denote \( f(u, u', u'', \ldots, u^{(m)}) \) as \( f(u) \).
• **Exactness Criterion:**

**Continuous Euler Operator (variational derivative)**

**Definition** (exactness):

A function $f(u)$ is exact, i.e. can be integrated fully, if there exists a function $F(u)$, such that $f(u) = D_x F(u)$ or equivalently $F(u) = D_x^{-1} f(u) = \int_x f(u) \, dx$.

$D_x$ is the (total) derivative with respect to $x$.

**Theorem** (exactness or integrability test):

A necessary and sufficient condition for a function $f$ to be exact, i.e. the derivative of another function, is that $L^{(0)}(f) \equiv 0$ where $L^{(0)}$ is the continuous Euler operator (variational derivative) defined by

$$L^{(0)}_u = \sum_{k=0}^{m_0} (-D_x)^k \frac{\partial}{\partial u^{(k)}}$$

$$= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u'} + D_x^2 \frac{\partial}{\partial u''} + \cdots + (-1)^{m_0} D_x^{m_0} \frac{\partial}{\partial u^{(m_0)}}$$

where $m_0$ is the order (of $f$).

**Proof:**

See calculus of variations (derivation of Euler-Lagrange equations — the forgotten case!).
Example: Apply the continuous Euler operator to

\[ f(u) = 3u'v^2 \sin(u) - u'^3 \sin(u) - 6v' \cos(u) + 2u' u'' \cos(u) + 8v'v'' \]

Here \( u = (u, v) \).

For component \( u \) (order 2):

\[
\mathcal{L}_u^{(0)}(f) = \frac{\partial}{\partial u}(f) - D_x \frac{\partial}{\partial u'}(f) + D_x^2 \frac{\partial}{\partial u''}(f)
\]

\[
= 3u'v^2 \cos(u) - u'^3 \cos(u) + 6v' \sin(u) - 2u' u'' \sin(u) \\
- D_x[3v^2 \sin(u) - 3u'^2 \sin(u) + 2u'' \cos(u)] + D_x^2[2u' \cos(u)]
\]

\[
= 3u'v^2 \cos(u) - u'^3 \cos(u) + 6v' \sin(u) - 2u' u'' \sin(u) \\
- [3u'v^2 \cos(u) + 6v' \sin(u) - 3u'^3 \cos(u) - 6u u'' \sin(u) \\
- 2u' u'' \sin(u) + 2u''' \cos(u)]
\]

\[
+ [-2u''' \cos(u) - 6u' u'' \sin(u) + 2u''' \cos(u)]
\]

\[ \equiv 0 \]

For component \( v \) (order 2):

\[
\mathcal{L}_v^{(0)}(f) = \frac{\partial}{\partial v}(f) - D_x \frac{\partial}{\partial v'}(f) + D_x^2 \frac{\partial}{\partial v''}(f)
\]

\[
= 6u'v \sin(u) - 6v' \cos(u) - D_x[-6v \cos(u) + 8v''] + D_x^2[8v']
\]

\[
= 6u'v \sin(u) - 6v' \cos(u) - [6u'v \sin(u) - 6v' \cos(u) + 8v''' + 8v''']
\]

\[ \equiv 0 \]
• **Computation of the integral** $F$

**Definition** (higher Euler operators): The continuous higher Euler operators are defined by

$$\mathcal{L}_u^{(i)} = \sum_{k=i}^{m_i} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial u^{(k)}}$$

These Euler operators all terminate at some maximal order $m_i$.

**Examples** (for component $u$):

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u'} + D_x^2 \frac{\partial}{\partial u''} - D_x^3 \frac{\partial}{\partial u^{(4)}} + \cdots + (-1)^{m_0} D_x^{m_0} \frac{\partial}{\partial u^{(m_0)}}$$

$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u'} - 2D_x \frac{\partial}{\partial u''} + 3D_x^2 \frac{\partial}{\partial u^{(4)}} - 4D_x^3 \frac{\partial}{\partial u^{(5)}} + \cdots - (-1)^{m_1} m_1 D_x^{m_1-1} \frac{\partial}{\partial u^{(m_1)}}$$

$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u''} - 3D_x \frac{\partial}{\partial u^{(4)}} + 6D_x^2 \frac{\partial}{\partial u^{(5)}} - 10D_x^3 \frac{\partial}{\partial u^{(6)}} + \cdots + (-1)^{m_2} \binom{m_2}{2} D_x^{m_2-2} \frac{\partial}{\partial u^{(m_2)}}$$

$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u^{(4)}} - 4D_x \frac{\partial}{\partial u^{(5)}} + 10D_x^2 \frac{\partial}{\partial u^{(6)}} - 20D_x^3 \frac{\partial}{\partial u^{(7)}} + \cdots - (-1)^{m_3} \binom{m_3}{3} D_x^{m_3-3} \frac{\partial}{\partial u^{(m_3)}}$$

Similar formulae for component $\mathcal{L}_v^{(i)}$
**Definition** (homotopy operator):
The continuous homotopy operator is defined by
\[ \mathcal{H}_u = \int_0^1 \sum_{r=1}^N f_r(u)[\lambda u] \frac{d\lambda}{\lambda} \]
where
\[ f_r(u) = \sum_{i=0}^{p_r} D_x^i [u_r \mathcal{L}^{(i+1)}_{u_r}] \]
\( p_r \) is the maximum order of \( u_r \) in \( f \)
\( N \) is the number of dependent variables
\( f_r(u)[\lambda u] \) means that in \( f_r(u) \) one replaces \( u \to \lambda u, \ u' \to \lambda u' \), etc.

**Example:**
For a two-component system \((N = 2)\) where \( u = (u, v) \):
\[ \mathcal{H}_u = \int_0^1 \{f_1(u)[\lambda u] + f_2(u)[\lambda u]\} \frac{d\lambda}{\lambda} \]
with
\[ f_1(u) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}^{(i+1)}_u] \]
and
\[ f_2(u) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}^{(i+1)}_v] \]

**Theorem** (integration via homotopy operator):
Given an integrable function \( f \)
\[ F = D_x^{-1} f = \int f \, dx = \mathcal{H}_u(f) \]

**Proof:** Olver’s book ‘Applications of Lie Groups to Differential Equations’, p. 372. Proof is given in terms of differential forms.
Work of Henri Poincaré (1854-1912), George de Rham (1950), and Ian Anderson & Tom Duchamp (1980).
Proofs based on calculus: Deconinck and Hereman.
Example: Apply the continuous homotopy operator to integrate

\[ f(u) = 3u'v^2 \sin(u) - u'^3 \sin(u) - 6vv' \cos(u) + 2u'u'' \cos(u) + 8v'v'' \]

For component \( u \) (order 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( {\mathcal{L}}^{(i+1)}_u(f) )</th>
<th>( D^i_x \left( u {\mathcal{L}}^{(i+1)}_u(f) \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\partial}{\partial u'} f - 2D_x \left( \frac{\partial}{\partial u''} f \right) )</td>
<td>( 3v^2 \sin u - 3u'^2 \sin u + 2u'' \cos u )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -2D_x (2u' \cos u) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = 3v^2 \sin u + uu'' \sin u - 2uu'' \cos u )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{\partial}{\partial u''} f = 2u' \cos u )</td>
<td>( D_x [2uu' \cos u] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = 2u'^2 \cos u + 2uu'' \cos u - 2uu'^2 \sin u )</td>
</tr>
</tbody>
</table>

Hence, \( f_1(u)(f) = 3uv^2 \sin(u) - uu'^2 \sin(u) + 2u'^2 \cos(u) \)

For component \( v \) (order 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( {\mathcal{L}}^{(i+1)}_v(f) )</th>
<th>( D^i_x [v {\mathcal{L}}^{(i+1)}_v(f)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( -6v \cos(u) + 8v'' - 2D_x [8v'] )</td>
<td>( -6v^2 \cos(u) - 8vv'' )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = -6v \cos(u) - 8v'' )</td>
</tr>
<tr>
<td>1</td>
<td>( 8v' )</td>
<td>( D_x [8vv'] = 8v'^2 + 8vv'' )</td>
</tr>
</tbody>
</table>

Hence, \( f_2(u)(f) = -6v'^2 \cos(u) + 8v'^2 \)

The homotopy operator leads to an integral for (auxiliary) variable \( \lambda \).

(Use standard integration by parts to work the integral).

\[
F(u) = \mathcal{H}_u(f) = \int_0^1 \left\{ f_1(u)(f) [\lambda u] + f_2(u)(f) [\lambda u] \right\} \frac{d\lambda}{\lambda}
= \int_0^1 [3\lambda^2 uv^2 \sin(\lambda u) - \lambda^2 uu'^2 \sin(\lambda u) + 2\lambda uu'^2 \cos(\lambda u) - 6\lambda v^2 \cos(\lambda u) + 8\lambda v'^2] \, d\lambda
= 4v'^2 + uu'^2 \cos(u) - 3v^2 \cos(u)
\]
Application: Conserved densities and fluxes for PDEs with transcendental nonlinearities

**Definition** (conservation law):

\[ D_t \rho + D_x J = 0 \quad \text{(on PDE)} \]

conserved density \( \rho \) and flux \( J \).

**Example:** Sine-Gordon system (type \( u_t = F \))

\[
\begin{align*}
  u_t &= v \\
  v_t &= u_{xx} + \alpha \sin(u)
\end{align*}
\]

has scaling symmetry

\[(t, x, u, v, \alpha) \to (\lambda^{-1}t, \lambda^{-1}x, \lambda^0u, \lambda v, \lambda^2\alpha)\]

In terms of weights:

\[ w(D_x) = 1, \ w(D_t) = 1, \ w(u) = 0, \ w(v) = 1, \ w(\alpha) = 2 \]

Conserved densities and fluxes

\[
\begin{align*}
  \rho_{(1)} &= 2\alpha \cos(u) + v^2 + u_x^2 & J_{(1)} &= -2u_xv \\
  \rho_{(2)} &= u_xv & J_{(2)} &= -\left[\frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha \cos(u)\right] \\
  \rho_{(3)} &= 12 \cos(u)v u_x + 2v^3u_x + 2vu_x^3 - 16v_xu_{2x} \\
  \rho_{(4)} &= 2\cos^2(u) - 2\sin^2(u) + v^4 + 6v^2u_x^2 + u_x^4 + 4\cos(u)v^2 \\
  &\quad + 20\cos(u)u_x^2 - 16v_x^2 - 16u_{2x}^2.
\end{align*}
\]

are all scaling invariant!

**Remark:** \( J_{(3)} \) and \( J_{(4)} \) are not shown (too long).
Algorithm for Conserved Densities and Fluxes

**Example:** Density and flux of rank 2 for sine-Gordon system

**Step 1:** Construct the form of the density

\[ \rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)u_xv \]

where \( h_i(u) \) are unknown functions.

**Step 2:** Determine the functions \( h_i \)

Compute

\[
E = D_t\rho = \frac{\partial \rho}{\partial t} + \rho'(u)[F] \quad \text{(on PDE)}
\]

\[
= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m_1} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t + \sum_{k=0}^{m_2} \frac{\partial \rho}{\partial v_{kx}} D_x^k v_t
\]

Since \( E = D_t\rho = -D_xJ \), the expression \( E \) must be integrable.

Require that \( L_u^0(E) \equiv 0 \) and \( L_v^0(E) \equiv 0 \).

Solve the system of linear mixed system (algebraic eqs. and ODEs):

\[
\begin{align*}
    h_2(u) - h_3(u) &= 0 \\
    h_2'(u) &= 0 \\
    h_3'(u) &= 0 \\
    h_4'(u) &= 0 \\
    h_2''(u) &= 0 \\
    h_4''(u) &= 0 \\
    2h_2'(u) - h_3'(u) &= 0 \\
    2h_2''(u) - h_3''(u) &= 0 \\
    h_1'(u) + 2\sin(u)h_2(u) &= 0 \\
    h_1''(u) + 2\sin(u)h_2'(u) + 2\cos(u)h_2(u) &= 0
\end{align*}
\]
Solution:

\[ h_1(u) = 2c_1 \cos(u) + c_3 \]
\[ h_2(u) = h_3(u) = c_1 \]
\[ h_4(u) = c_2 \]

(with arbitrary constants \( c_i \)).

Substitute in \( \rho = c_1(2\alpha \cos(u) + v^2 + u_x^2) + c_2(u_xv) + c_3\alpha \)

**Step 3: Compute the flux \( J \)**

First, compute

\[ E = D_t \rho = c_1(-2\alpha u_t \sin u + 2vv_t + 2u_xu_{xt}) + c_2(u_{xt}v + u_xv_t) \]
\[ = c_1(-2\alpha v \sin u + 2v(u_{2x} + \alpha \sin u) + 2u_xv_x) \]
\[ + c_2(v_xv + u_x(u_{2x} + \alpha \sin u)) \]
\[ = c_1(2u_{2x}v + 2u_xv_x) + c_2(vv_x + u_xu_{2x} + \alpha u_x \sin u) \]

Since \( E = D_t \rho = -D_x J \), one must integrate \( f = -E \).

Apply the homotopy operator for each component of \( \mathbf{u} = (u, v) \).

For component \( u \) (order 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}^{(i+1)}_u(f) )</th>
<th>( D_x^i(u\mathcal{L}^{(i+1)}_u(f)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 2c_1v_x + c_2(u_{2x} - \alpha \sin u) )</td>
<td>( 2c_1uv_x + c_2(uu_{2x} - \alpha u \sin u) )</td>
</tr>
<tr>
<td>1</td>
<td>( -2c_1v - c_2u_x )</td>
<td>( -2c_1(u_xv + uv_x) - c_2(u_x^2 + uu_{2x}) )</td>
</tr>
</tbody>
</table>

Hence, \( f_1(\mathbf{u})(f) = -2c_1u_xv - c_2(u_x^2 + \alpha u \sin u) \)
For component $v$ (order 1):

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\mathcal{L}_v^{(i+1)}(f)$</th>
<th>$D_x^i (v\mathcal{L}_v^{(i+1)}(f))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-2c_1u_x - c_2v$</td>
<td>$-2c_1u_xv - c_2v^2$</td>
</tr>
</tbody>
</table>

Hence, $f_2(u)(f) = -2c_1u_xv - c_2v^2$

The homotopy operator leads to an integral for (one) variable $\lambda$:

$$J(u) = \mathcal{H}_u(f) = \int_0^1 (f_1(u)(f)[\lambda u] + f_2(u)(f)[\lambda u]) \frac{d\lambda}{\lambda}$$

$$= -\int_0^1 \left( 4c_1\lambda u_xv + c_2(\lambda u_x^2 + \alpha u \sin(\lambda u) + \lambda v^2) \right) d\lambda$$

$$= -2c_1u_xv - c_2\left( \frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha \cos u \right)$$

Split the density and flux in independent pieces (for $c_1$ and $c_2$):

$$\rho(1) = 2\alpha \cos u + v^2 + u_x^2 \quad J(1) = -2u_xv$$

$$\rho(2) = u_xv \quad J(2) = -\frac{1}{2}v^2 - \frac{1}{2}u_x^2 + \alpha \cos u$$

**Remark:** Computation of $J(3)$ and $J(4)$ requires integration with the homotopy operator!
Computer Demos

(1) Use continuous homotopy operator to integrate
\[ f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v'' \]

(2) Compute densities of rank 8 and fluxes for 5th-order Korteweg-de Vries equation with three parameters:
\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]
(\(\alpha, \beta, \gamma\) are nonzero constant parameters).

(3) Compute density of rank 4 and flux for sine-Gordon system:
\[
\begin{align*}
    u_t &= v \\
    v_t &= u_{xx} + \alpha \sin(u)
\end{align*}
\]
### Analogy PDEs and DDEs

<table>
<thead>
<tr>
<th>System</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_t = F(u, u_x, u_{2x}, ...)$</td>
<td>$\dot{u}<em>n = F(..., u</em>{n-1}, u_n, u_{n+1}, ...)$</td>
<td></td>
</tr>
</tbody>
</table>

| Conservation Law | $D_t \rho + D_x J = 0$ | $\dot{\rho}_n + J_{n+1} - J_n = 0$ |

| Symmetry | $D_t G = F'(u)[G]$ | $D_t G = F'(u_n)[G]$ |

| Recursion Operator | $D_t R + [\mathcal{R}, F'(u)] = 0$ | $D_t \mathcal{R} + [\mathcal{R}, F'(u_n)] = 0$ |

### Table 1: Conservation Laws and Symmetries

<table>
<thead>
<tr>
<th>Equation</th>
<th>KdV Equation</th>
<th>Volterra Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Densities</td>
<td>$\rho = u, \quad \rho = u^2$</td>
<td>$\rho_n = u_n, \quad \rho_n = u_n(\frac{1}{2}u_n + u_{n+1})$</td>
</tr>
<tr>
<td>$\rho = u^3 - \frac{1}{2}u_x^2$</td>
<td>$\rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$</td>
<td></td>
</tr>
</tbody>
</table>

| Symmetries | $G = u_x, \quad G = 6uu_x + u_{3x}$ | $G = u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$ |
| $G = 30u^2u_x + 20uxu_{2x}$ | $-u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n)$ |
| $+10uu_{3x} + u_{5x}$ | |

| Recursion Operator | $\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1}$ | $\mathcal{R} = u_n(I + D)(u_nD - D^{-1}u_n)$ |
| $(D - I)^{-1} \frac{1}{u_n}$ | |

### Table 2: Prototypical Examples


Part II: Discrete Case

Definitions (shift and total difference operators):
D is the **up-shift** (forward or right-shift) operator if for $F_n$

$$DF_n = F_{n+1} = F_{n|n-n+1}$$

$D^{-1}$ the **down-shift** (backward or left-shift) operator if

$$D^{-1}F_n = F_{n-1} = F_{n|n-n-1}$$

$\Delta = D - I$ is the total **difference operator**

$$\Delta F_n = (D - I)F_n = F_{n+1} - F_n$$

D (up-shift operator) corresponds the differential operator $D_x$

$$D_x F(x) \to \frac{F_{n+1} - F_n}{\Delta x} = \frac{\Delta F_n}{\Delta x} \quad \text{(set } \Delta x = 1)$$

For $k > 0$, $D^k = D \circ D \circ \cdots \circ D$ \quad (k times).

Similarly, $D^{-k} = D^{-1} \circ D^{-1} \circ \cdots \circ D^{-1}$.

Problem to be solved:

Continuous case:
Given $f$. Find $F$ so that $f = D_x F$ or $F = D_x^{-1} f = \int f \, dx$.

Discrete case:
Given $f_n$. Find $F_n$ so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$. 

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Inverting the Δ Operator

• Given \( f_n \) involving \( u_n \) and \( v_n \) and their shifts:
  \[
  f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n
  \]

• Find \( F_n \) so that \( f_n = \Delta F_n = F_{n+1} - F_n \) or \( F_n = \Delta^{-1} f_n \).
  Invert the Δ operator (compute \( F_n \) by hand)

<table>
<thead>
<tr>
<th>( f_n )</th>
<th>( F_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -v_n^2 )</td>
<td>( v_n^2 )</td>
</tr>
<tr>
<td>( v_{n+1}^2 )</td>
<td>( u_{n+1} v_{n+1} )</td>
</tr>
<tr>
<td>( -u_n u_{n+1} v_n )</td>
<td>( u_n u_{n+1} v_n )</td>
</tr>
<tr>
<td>( u_{n+1} u_{n+2} v_{n+1} )</td>
<td>( u_{n+1} v_{n+1} )</td>
</tr>
<tr>
<td>( -u_{n+1} v_n )</td>
<td>( u_{n+2} v_{n+1} )</td>
</tr>
<tr>
<td>( +u_{n+2} v_{n+1} )</td>
<td>( u_{n+2} v_{n+1} )</td>
</tr>
<tr>
<td>( -u_{n+2} v_{n+1} )</td>
<td>( u_{n+3} v_{n+2} )</td>
</tr>
</tbody>
</table>

• Result:
  \[
  F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}.
  \]

Remarks: We denote \( f(u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+p}) \) as \( f(u_n) \).
Assume that all negative shifts have been removed via up-shifting
Replace \( f_n = u_{n-2} v_n v_{n+3} \) by \( \tilde{f}_n = D^2 f_n = u_n v_{n+2} v_{n+5} \).
‘Total Difference’ Criterion:

Discrete Euler Operator (variational derivative)

Definition (exactness):

A function \( f_n(u_n) \) is exact, i.e. a total difference, if there exists a function \( F_n(u_n) \), such that \( f_n = \Delta F_n \) or equivalently \( F_n = \Delta^{-1} f_n \). D is the up-shift operator.

Theorem (exactness or total difference test):

A necessary and sufficient condition for a function \( f_n \) to be exact, i.e. a total difference, is that \( \mathcal{L}^{(0)}_{u_n}(f_n) \equiv 0 \), where \( \mathcal{L}^{(0)}_{u_n} \) is the discrete Euler operator (variational derivative) defined by

\[
\mathcal{L}^{(0)}_{u_n} = \sum_{k=0}^{m_0} D^{-k} \frac{\partial}{\partial u_{n+k}}
= \frac{\partial}{\partial u_n} + D^{-1}(\frac{\partial}{\partial u_{n+1}}) + D^{-2}(\frac{\partial}{\partial u_{n+2}}) + \cdots + D^{-m_0}(\frac{\partial}{\partial u_{n+m_0}})
= \frac{\partial}{\partial u_n}(\sum_{k=0}^{m_0} D^{-k})
\]

where \( m_0 \) is the highest forward shift (in \( f_n \)).
**Example:** Apply the discrete Euler operator to

\[ f_n(u_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

Here \( u_n = (u_n, v_n) \).

For component \( u_n \) (highest shift 3):

\[
\mathcal{L}^{(0)}_{u_n}(f_n) = \frac{\partial}{\partial u_n} \left[ I + D^{-1} + D^{-2} + D^{-3} \right] (f_n) \\
= \left[ -u_{n+1} v_n \right] + \left[ -u_{n-1} v_{n-1} + u_{n+1} v_n - v_{n-1} \right] + \left[ u_{n-1} v_{n-1} \right] + \left[ v_{n-1} \right] \\
\equiv 0
\]

For component \( v_n \) (highest shift 2):

\[
\mathcal{L}^{(0)}_{v_n}(f_n) = \frac{\partial}{\partial v_n} \left[ I + D^{-1} + D^{-2} \right] (f_n) \\
= \left[ -u_n u_{n+1} - 2v_n - u_{n+1} \right] + \left[ u_n u_{n+1} + 2v_n \right] + \left[ u_{n+1} \right] \\
\equiv 0
\]
• Computation of $F_n$

**Definition** (higher Euler operators):

The discrete higher Euler operators are defined by

$$\mathcal{L}^{(i)}_{u_n} = \frac{\partial}{\partial u_n} \left( \sum_{k=i}^{m_i} \binom{k}{i} D^{-k} \right)$$

These Euler operators all terminate at some maximal shifts $m_i$.

**Examples** (for component $u_n$):

$$\mathcal{L}^{(0)}_{u_n} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots + D^{-m_0})$$

$$\mathcal{L}^{(1)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots + m_1 D^{-m_1})$$

$$\mathcal{L}^{(2)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots + \frac{1}{2} m_2 (m_2 - 1) D^{-m_2})$$

$$\mathcal{L}^{(3)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots + \binom{m_3}{3} D^{-m_3})$$

Similar formulae for $\mathcal{L}^{(i)}_{v_n}$. 

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• **Definition** (homotopy operator):

The discrete homotopy operator is defined by

\[
\mathcal{H} \underline{u}_n = \int_0^1 \sum_{r=1}^N f_{r,n}(\underline{u}_n)[\lambda \underline{u}_n] \frac{d\lambda}{\lambda}
\]

where

\[
f_{r,n}(\underline{u}_n) = \sum_{i=0}^{p_r} (D - I)^i [u_{r,n} \mathcal{L}^{(i+1)}_{u_{r,n}}]
\]

\(p_r\) is the maximum shift of \(u_{r,n}\) in \(f_n\)

\(N\) is the number of dependent variables

\(f_{r,n}(\underline{u}_n)[\lambda \underline{u}_n]\) means that in \(f_{r,n}(\underline{u}_n)\) one replaces \(\underline{u}_n \rightarrow \lambda \underline{u}_n\), \(\underline{u}_{n+1} \rightarrow \lambda \underline{u}_{n+1}\), etc.

**Example:**

For a two-component system \((N = 2)\) where \(\underline{u}_n = (u_n, v_n)\):

\[
\mathcal{H} \underline{u}_n = \int_0^1 \{f_{1,n}(\underline{u}_n)[\lambda \underline{u}_n] + f_{2,n}(\underline{u}_n)[\lambda \underline{u}_n]\} \frac{d\lambda}{\lambda}
\]

with

\[
f_{1,n}(\underline{u}_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}^{(i+1)}_{u_n}]
\]

and

\[
f_{2,n}(\underline{u}_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}^{(i+1)}_{v_n}]
\]

**Theorem (total difference via homotopy operator):**

Given a function \(f_n\) which is a total difference, then

\[
F_n = \Delta^{-1} f_n = \mathcal{H}_{\underline{u}_n}(f_n)
\]

**Proof:** Recent work by Mansfield and Hydon on discrete variational bi-complexes. Proof is given in terms of differential forms.

Proof based on calculus: Deconinck and Hereman.
Higher Euler Operators Side by Side

Continuous Case  (for component $u$)

\[
\mathcal{L}^{(0)}_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \cdots \\
\mathcal{L}^{(1)}_u = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \cdots \\
\mathcal{L}^{(2)}_u = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \cdots \\
\mathcal{L}^{(3)}_u = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \cdots 
\]

Discrete Case  (for component $u_n$)

\[
\mathcal{L}^{(0)}_{u_n} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots) \\
\mathcal{L}^{(1)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots) \\
\mathcal{L}^{(2)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots) \\
\mathcal{L}^{(3)}_{u_n} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots) 
\]
Homotopy Operators Side by Side

**Continuous Case** (for components $u$ and $v$)

\[ \mathcal{H}_u = \int_0^1 \{ f_1(u)[\lambda u] + f_2(u)[\lambda u] \} \frac{d\lambda}{\lambda} \]

with

\[ f_1(u) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}] \]

and

\[ f_2(u) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}] \]

**Discrete Case** (for components $u_n$ and $v_n$)

\[ \mathcal{H}_{u_n} = \int_0^1 \{ f_{1,n}(u_n)[\lambda u_n] + f_{2,n}(u_n)[\lambda u_n] \} \frac{d\lambda}{\lambda} \]

with

\[ f_{1,n}(u_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}] \]

and

\[ f_{2,n}(u_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}] \]
Example: Apply the discrete homotopy operator to

\[ f_n(u_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

For component \( u_n \) (highest shift 3):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}_{u_n}^{(i+1)}(f_n) )</th>
<th>( (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}(f_n)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_{n-1} v_{n-1} + u_{n+1} v_n + 2v_{n-1} )</td>
<td>( u_{n-1} u_n v_{n-1} + u_{n+1} v_n + 2u_n v_{n-1} )</td>
</tr>
<tr>
<td>1</td>
<td>( u_{n-1} v_{n-1} + 3v_n )</td>
<td>( u_n u_{n+1} v_n + 3u_{n+1} v_n - u_{n-1} u_n v_{n-1} - 3u_nv_{n-1} )</td>
</tr>
<tr>
<td>2</td>
<td>( v_{n-1} )</td>
<td>( u_{n+2} v_{n+1} - u_{n+1} v_n - u_{n+1} v_n + u_n v_{n-1} )</td>
</tr>
</tbody>
</table>

Hence, \( f_{1,n}(u_n)(f_n) = 2u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1} \)

For component \( v_n \) (highest shift 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}_{v_n}^{(i+1)}(f_n) )</th>
<th>( (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}(f_n)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_n u_{n+1} + 2v_n + 2u_{n+1} )</td>
<td>( u_n u_{n+1} v_n + 2v_n^2 + 2u_{n+1} v_n )</td>
</tr>
<tr>
<td>1</td>
<td>( u_{n+1} )</td>
<td>( u_{n+2} v_{n+1} - u_{n+1} v_n )</td>
</tr>
</tbody>
</table>

Hence, \( f_{2,n}(u_n)(f_n) = u_n u_{n+1} v_n + 2v_n^2 + u_{n+1} v_n + u_{n+2} v_{n+1} \)

The homotopy operator leads to an integral for variable \( \lambda \). (Use standard integration by parts to work the integral).

\[
F_n(u_n) = \mathcal{H}_{u_n}(f_n) = \int_0^1 \{ f_{1,n}(u_n)(f_n)[\lambda u_n] + f_{2,n}(u_n)(f_n)[\lambda u_n] \} \frac{d\lambda}{\lambda}
\]

\[
= \int_0^1 [2\lambda v_n^2 + 3\lambda^2 u_n u_{n+1} v_n + 2\lambda u_{n+1} v_n + 2\lambda u_{n+2} v_{n+1}] \, d\lambda
\]

\[
= v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}
\]
• Application: Conserved densities and fluxes for DDEs

**Definition** (conservation law):

\[ D_t \rho_n + \Delta J_n = D_t \rho_n + J_{n+1} - J_n = 0 \quad \text{(on DDE)} \]

conserved density \( \rho_n \) and flux \( J_n \).

**Example** The Toda lattice (type \( \dot{u}_n = F \)):

\[
\begin{align*}
\dot{u}_n &= v_{n-1} - v_n \\
\dot{v}_n &= v_n(u_n - u_{n+1})
\end{align*}
\]

has scaling symmetry

\[(t, u_n, v_n) \rightarrow (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n).\]

In terms of weights:

\[ w(\frac{d}{dt}) = 1, \quad w(u_n) = w(u_{n+1}) = 1, \quad w(v_n) = w(v_{n-1}) = 2. \]

Conserved densities and fluxes

\[
\begin{align*}
\rho_n^{(0)} &= \ln(v_n) & J_n^{(0)} &= u_n \\
\rho_n^{(1)} &= u_n & J_n^{(1)} &= v_{n-1} \\
\rho_n^{(2)} &= \frac{1}{2} u_n^2 + v_n & J_n^{(2)} &= u_n v_{n-1} \\
\rho_n^{(3)} &= \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) & J_n^{(3)} &= u_{n-1} u_n v_{n-1} + v_{n-1}^2
\end{align*}
\]

are all scaling invariant!
• Algorithm for Conserved Densities and Fluxes

**Example:** Density of rank 3 for Toda system

**Step 1:** Construct the form of the density.

\[ \rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n \]

where \( c_i \) are unknown constants.

**Step 2:** Determine the constants \( c_i \).

Compute

\[ E_n = D_t \rho_n = \dfrac{\partial \rho_n}{\partial t} + \rho'_n(u_n)[F] \quad \text{(on DDE)} \]

\[ = (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1} v_n \]

\[ + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2 \]

Compute \( \tilde{E}_n = D E_n \) to remove negative shift \( n - 1 \).

Since \( \tilde{E}_n = -\Delta \tilde{J}_n \), the expression \( \tilde{E}_n \) must be a total difference.

Require

\[ \mathcal{L}^{(0)}_{u_n}(\tilde{E}_n) = \dfrac{\partial}{\partial u_n}(I + D^{-1} + D^{-2})(\tilde{E}_n) = \dfrac{\partial}{\partial u_n}(D + I + D^{-1})(E_n) \]

\[ = 2(3c_1 - c_2)u_n v_{n-1} + 2(c_3 - 3c_1)u_n v_n \]

\[ + (c_2 - c_3)u_{n-1} v_{n-1} + (c_2 - c_3)u_{n+1} v_n \equiv 0 \]

and

\[ \mathcal{L}^{(0)}_{v_n}(\tilde{E}_n) = \dfrac{\partial}{\partial v_n}(I + D^{-1})(\tilde{E}_n) = \dfrac{\partial}{\partial v_n}(D + I)(E_n) \]

\[ = (3c_1 - c_2)u_{n+1}^2 + (c_3 - c_2)v_{n+1} + (c_2 - c_3)u_n u_{n+1} \]

\[ + 2(c_2 - c_3)v_n + (c_3 - 3c_1)u_n^2 + (c_3 - c_2)v_{n-1} \equiv 0. \]
Solve the linear system
\[ S = \{ 3c_1 - c_2 = 0, \ c_3 - 3c_1 = 0, \ c_2 - c_3 = 0 \}. \]
Solution: \[ 3c_1 = c_2 = c_3 \] Choose \( c_1 = \frac{1}{3} \), and \( c_2 = c_3 = 1 \).
Substitute in \( \rho_n \)
\[ \rho_n = \frac{1}{3} u_n^3 + u_n (v_{n-1} + v_n) \]

**Step 3: Compute the flux \( J_n \).**
Start from \( -\tilde{E}_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 \)
Apply the discrete homotopy operator to \( f_n = -\tilde{E}_n \).

For component \( u_n \) (highest shift 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}^{(i+1)}(f_n) )</th>
<th>( (D - I)^i(u_n \mathcal{L}^{(i+1)}(\tilde{E}_n)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_{n-1} v_{n-1} + u_{n+1} v_n )</td>
<td>( u_n u_{n-1} v_{n-1} + u_n u_{n+1} v_n )</td>
</tr>
<tr>
<td>1</td>
<td>( u_{n-1} v_{n-1} )</td>
<td>( u_{n+1} u_n v_n - u_n u_{n-1} v_{n-1} )</td>
</tr>
</tbody>
</table>

Hence, \( f_{1,n}(u_n)(f_n) = 2u_n u_{n+1} v_n \)

For component \( v_n \) (highest shift 1):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}^{(i+1)}(f_n) )</th>
<th>( (D - I)^i(v_n \mathcal{L}^{(i+1)}(\tilde{E}_n)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_n u_{n+1} + 2v_n )</td>
<td>( v_n u_n u_{n+1} + 2v_n^2 )</td>
</tr>
</tbody>
</table>

Hence, \( f_{2,n}(u_n)(f_n) = u_n u_{n+1} v_n + 2v_n^2 \)

\[ \tilde{J}_n = \mathcal{H}_{u_n}(f_n) = \int_0^1 (f_{1,n}(u_n)(f_n)[\lambda u_n] + f_{2,n}(f_n)(u_n)[\lambda u_n]) \frac{d\lambda}{\lambda} \]
\[ = \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2) \ d\lambda \]
\[ = u_n u_{n+1} v_n + v_n^2. \]

Final Result:
\[ J_n = D^{-1} \tilde{J}_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2 \]
Computer Demos

(1) Use discrete homotopy operator to compute $F_n = \Delta^{-1} f_n$ for

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

(2) Compute density of rank 4 and flux for Toda system:

$$\dot{u}_n = v_{n-1} - v_n$$
$$\dot{v}_n = v_n(u_n - u_{n+1})$$

(3) Compute density of rank 2 for Ablowitz-Ladik system:

$$i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u^*_n u_n(u_{n+1} + u_{n-1})$$

($u^*_n$ is the complex conjugate of $u_n$).

This is an integrable discretization of the NLS equation:

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

Take equation and its complex conjugate.

Treat $u_n$ and $v_n = u^*_n$ as dependent variables. Absorb $i$ in $t$:

$$\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n(u_{n+1} + u_{n-1})$$
$$\dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n(v_{n+1} + v_{n-1}).$$
Future Research

• Generalize continuous homotopy operator in multi-dimensions \((x, y, z, \ldots)\).

• Problem (in three dimensions):
  
  Given \( E = \nabla \cdot J = J_x^{(1)} + J_y^{(2)} + J_z^{(3)}. \)

  Find \( J = (J^{(1)}, J^{(2)}, J^{(3)}). \)

• Application:
  
  Compute densities and fluxes of multi-dimensional systems of PDEs (in \( t, x, y, z \)).

• Generalize discrete homotopy operator in multi-dimensions \((n, m, \ldots)\).
Higher Euler Operators Side by Side

Continuous Case  (for component $u$)

\[ \mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \cdots \]

\[ \mathcal{L}_u^{(1)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \cdots \]

\[ \mathcal{L}_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \cdots \]

\[ \mathcal{L}_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \cdots \]

Discrete Case  (for component $u_n$)

\[ \mathcal{L}_{u_n}^{(0)} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(1)} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(2)} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(3)} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots) \]
Homotopy Operators Side by Side

**Continuous Case** (for components $u$ and $v$)

\[
\mathcal{H}_u = \int_0^1 \{ f_1(u)[\lambda u] + f_2(u)[\lambda u] \} \frac{d\lambda}{\lambda}
\]

with

\[
f_1(u) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]
\]

and

\[
f_2(u) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]
\]

**Discrete Case** (for components $u_n$ and $v_n$)

\[
\mathcal{H}_{u_n} = \int_0^1 \{ f_{1,n}(u_n)[\lambda u_n] + f_{2,n}(u_n)[\lambda u_n] \} \frac{d\lambda}{\lambda}
\]

with

\[
f_{1,n}(u_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]
\]

and

\[
f_{2,n}(u_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]
\]