Symbolic Computation of Travelling Wave Solutions of Nonlinear Partial Differential and Differential-Difference Equations

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, Colorado, U.S.A.
whereman@mines.edu
http://www.mines.edu/fs_home/whereman/

Centre de Recherches Mathématiques
Acknowledgements

Collaborators:

Ünal Göktaş (Wolfram Research, Inc.)

Graduate student: Douglas Baldwin

(University of Colorado – Boulder)

Several undergraduate students.

Research supported in part by NSF under Grants DMS-9912293 & CCR-9901929

This presentation was made in TeXpower
“Make Things as Simple as Possible but No Simpler”

Albert Einstein
Outline

• Purpose and Motivation

• Typical Examples of ODEs, PDEs, and DDEs

  Part I: Tanh Method for PDEs

• Review of Tanh Algorithm

• Typical Example

• Other Types of Functions

• Demo PDESpecialSolutions.m
Part II: Tanh Method for DDEs (Lattices)

- Algorithm for Tanh Solutions
- Typical Examples
- Table of Results
- Demo DDESpecialSolutions.m
- Conclusions and Future Research
- Research Papers and Software
• Develop and implement various methods to find exact solutions of nonlinear PDEs and DDEs: direct methods, Lie symmetry methods, similarity methods, etc.

• Fully automate the hyperbolic and elliptic function methods to compute travelling solutions of nonlinear PDEs.

• Fully automate the tanh method to compute travelling wave solutions of nonlinear DDEs (lattices).

• Class of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations, and lattices.
Examples PDEs: Korteweg-de Vries, Boussinesq, and Kuramoto-Sivashinsky equations. Fisher and FitzHugh-Nagumo equations.

Examples ODEs: Duffing and nonlinear oscillator equations.

Examples DDEs: Volterra, Toda, and Ablowitz-Ladik lattices.
• **PDEs:** Solutions of tanh (kink) or sech (pulse) type *model* solitary waves in fluids, plasmas, circuits, optical fibers, bio-genetics, etc.

**DDEs:** discretizations of PDEs, lattice theory, queing and network problems, solid state and quantum physics.

• **Benchmark** solutions for numerical PDE and DDE solvers.
• **Research aspect:** Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations and lattices.

• **Educational aspect:** Software as course ware for courses in nonlinear PDEs and DDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software. REU projects of NSF. Extreme Programming!

• **Users** scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.
Typical Examples of ODEs and PDEs

- The Duffing equation

\[ u'' + u + \alpha u^3 = 0 \]

Solutions in terms of elliptic functions

\[ u(x) = \pm \frac{\sqrt{c_1^2 - 1}}{\sqrt{\alpha}} \text{cn}(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}), \]

and

\[ u(x) = \pm \frac{\sqrt{2(c_1^2 - 1)}}{\sqrt{\alpha}} \text{sn}(c_1 x + \Delta; \frac{1 - c_1^2}{c_1^2}). \]
The Korteweg-de Vries (KdV) equation

\[ u_t + 6\alpha uu_x + u_{3x} = 0. \]

Solitary wave solution

\[ u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \Delta], \]

or, equivalently,

\[ u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \operatorname{sech}^2 [c_1 x + c_2 t + \Delta]. \]

Cnoidal wave solution

\[ u(x, t) = \frac{4c_1^3(1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \operatorname{cn}^2 (c_1 x + c_2 t + \Delta; m), \]

modulus \( m \).
• The modified Korteweg-de Vries (mKdV) equation

\[ u_t + \alpha u^2 u_x + u_{3x} = 0. \]

Solitary wave solution

\[ u(x, t) = \pm \sqrt{\frac{6}{\alpha}} c_1 \text{sech} \left[ c_1 x - c_1^3 t + \Delta \right]. \]

• Three-dimensional modified Korteweg-de Vries equation

\[ u_t + 6u^2 u_x + u_{xyz} = 0. \]

Solitary wave solution

\[ u(x, y, z, t) = \pm \sqrt{c_2 c_3} \text{sech} \left[ c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta \right]. \]
• The Fisher equation

\[ u_t - u_{xx} - u (1 - u) = 0. \]

Solitary wave solution

\[ u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \xi + \frac{1}{4} \tanh^2 \xi, \]

with

\[ \xi = \pm \frac{1}{2\sqrt{6}} x \pm \frac{5}{12} t + \Delta. \]
• The generalized Kuramoto-Sivashinski equation

\[ u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0. \]

Solitary wave solutions
(ignoring symmetry \( u \rightarrow -u, x \rightarrow -x, \sigma \rightarrow -\sigma \)):

For \( \sigma = 4 \)

\[ u(x, t) = 9 - 2c_2 - 15 \tanh \xi \left( 1 + \tanh \xi - \tanh^2 \xi \right), \]

with \( \xi = \frac{x}{2} + c_2 t + \Delta. \)
For \( \sigma = \frac{12}{\sqrt{47}} \)

\[
u(x, t) = \frac{45 \mp 4418c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh \xi
- \frac{45}{47\sqrt{47}} \tanh^2 \xi \pm \frac{15}{47\sqrt{47}} \tanh^3 \xi,
\]

with \( \xi = \pm \frac{1}{2\sqrt{47}} x + c_2 t + \Delta. \)
For $\sigma = 16/\sqrt{73}$

$$u(x, t) = 2 \frac{(30 \mp 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \tanh \xi$$
$$- \frac{60}{73\sqrt{73}} \tanh^2 \xi \pm \frac{15}{73\sqrt{73}} \tanh^3 \xi,$$

with $\xi = \pm \frac{1}{2\sqrt{73}} x + c_2 t + \Delta.$

For $\sigma = 0$

$$u(x, t) = -2\sqrt{\frac{19}{11}} c_2 - \frac{135}{19} \sqrt{\frac{11}{19}} \tanh \xi + \frac{165}{19} \sqrt{\frac{11}{19}} \tanh^3 \xi,$$

with $\xi = \frac{1}{2} \sqrt{\frac{11}{19}} x + c_2 t + \Delta.$
• The Boussinesq (wave) equation

\[
  u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0,
\]

or written as a first-order system (\(v\) auxiliary variable):

\[
  \begin{align*}
    u_t + v_x &= 0, \\
    v_t + u_x - 3uu_x - \alpha u_{3x} &= 0.
  \end{align*}
\]

Solitary wave solution

\[
  \begin{align*}
    u(x, t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta], \\
    v(x, t) &= b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta].
  \end{align*}
\]
• sine-Gordon equation (light cone coordinates)

\[ \Phi_{xt} = \sin \Phi. \]

Set \( u = \Phi_x, \ v = \cos(\Phi) - 1, \)

\[ u_{xt} - u - u v = 0, \]

\[ u_t^2 + 2v + v^2 = 0. \]

Solitary wave solution (kink)

\[ u = \pm \frac{1}{\sqrt{-c}} \text{sech}\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right], \]

\[ v = 1 - 2 \text{sech}^2\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right]. \]
Solution:

\[
\Phi(x, t) = \int u(x, t) \, dx
\]

\[
= \pm 4 \arctan \left[ \exp \left( \frac{1}{\sqrt{-c}} (x - ct) + \Delta \right) \right].
\]
Typical Examples of DDEs (lattices)

- The Volterra lattice

\[
\begin{align*}
\dot{u}_n &= u_n(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n).
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= -c_1 \coth(d_1) + c_1 \tanh [d_1 n + c_1 t + \delta], \\
v_n(t) &= -c_1 \coth(d_1) - c_1 \tanh [d_1 n + c_1 t + \delta].
\end{align*}
\]
The Toda lattice

\[ \ddot{u}_n = (1 + \dot{u}_n) (u_{n-1} - 2u_n + u_{n+1}) . \]

Travelling wave solution:

\[ u_n(t) = a_{10} \pm \sinh(d_1) \tanh \left[ d_1 n \pm \sinh(d_1) t + \delta \right] . \]
The Ablowitz-Ladik lattice

\[ \dot{u}_n(t) = (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \]
\[ \dot{v}_n(t) = -(\alpha + u_n v_n)(v_{n+1} + v_{n-1}) + 2\alpha v_n. \]

Travelling wave solution:

\[ u_n(t) = \frac{\alpha \sinh^2(d_1)}{a_{21}} \left( \pm 1 - \tanh \left[ d_1 n + 2\alpha t \sinh^2(d_1) + \delta \right] \right), \]
\[ v_n(t) = a_{21} (\pm 1 + \tanh \left[ d_1 n + 2\alpha \sinh^2(d_1) t + \delta \right]). \]
• 2D Toda lattice

\[ \frac{\partial^2 u_n}{\partial x \partial t}(x, t) = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}). \]

Travelling wave solution:

\[ u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[ d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right]. \]
Algorithm for Tanh Solutions of PDEs

System of nonlinear PDEs of order \( m \)

\[ \Delta(u(x), u'(x), u''(x), \ldots u^{(m)}(x)) = 0. \]

Dependent variable \( u \) has \( M \) components \( u_i \) (or \( u, v, w, \ldots \)).

Independent variable \( x \) has \( N \) components \( x_j \) (or \( x, y, z, \ldots, t \)).
Step T1

• Seek solution $u(x) = U(T)$, with

\[ T = \tanh \xi = \tanh \left[ \sum_{j}^{N} c_j x_j + \delta \right]. \]

• Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, all derivatives of $T$ are polynomial in $T$. For example, $T'' = -2T(1 - T^2)$, etc.

• Repeatedly apply the operator rule

\[
\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d\bullet}{dT} = c_j (1 - T^2) \frac{d\bullet}{dT}
\]
 Produces a nonlinear system of ODEs

$$\Delta(T, U(T), U'(T), U''(T), \ldots, U^{(m)}(T)) = 0.$$  

Compare with ultra-spherical (linear) ODE

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0,$$

with integer $n \geq 0$ and $\alpha$ real.

Includes:

* Legendre equation ($\alpha = \frac{1}{2}$),
* ODE for Chebyshev polynomials of type I ($\alpha = 0$),
* ODE for Chebyshev polynomials of type II ($\alpha = 1$).
• **Example:** For the Boussinesq system

\[ u_t + v_x = 0, \]
\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \]

after cancelling common factors \(1 - T^2\),

\[ c_2 U' + c_1 V' = 0, \]
\[ c_2 V' + c_1 U' - 3c_1 UU' + \alpha c_1^3 \left[ 2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2 U''' \right] = 0. \]
Step T2

- Seek polynomial solutions

\[ U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j. \]

Determine the highest exponents \( M_i \geq 1. \)

Substitute \( U_i(T) = T^{M_i} \) into the LHS of ODE.

Gives polynomial \( P(T). \)

For every \( P_i \) consider all possible balances of the highest exponents in \( T. \)

Solve the resulting linear system(s) for the unknowns \( M_i. \)
Example: Balance highest exponents for the Boussinesq system

\[ M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1. \]

So, \( M_1 = M_2 = 2. \)

Hence,

\[ U(T) = a_{10} + a_{11}T + a_{12}T^2, \]
\[ V(T) = a_{20} + a_{21}T + a_{22}T^2. \]
Step T3

• Derive algebraic system for the unknown coefficients $a_{ij}$ by setting to zero the coefficients of the power terms in $T$.

• Example: Algebraic system for Boussinesq case

\[
\begin{align*}
    a_{11}c_1(3a_{12} + 2\alpha c_1^2) &= 0, \\
    a_{12}c_1(a_{12} + 4\alpha c_1^2) &= 0, \\
    a_{21}c_1 + a_{11}c_2 &= 0, \\
    a_{22}c_1 + a_{12}c_2 &= 0, \\
    a_{11}c_1 - 3a_{10}a_{11}c_1 + 2\alpha a_{11}c_1^3 + a_{21}c_2 &= 0, \\
    6a_{10}a_{12}c_1 + 16\alpha a_{12}c_1^3 + 2a_{22}c_2 &= 0.
\end{align*}
\]
Step T4

• Solve the nonlinear algebraic system with parameters.

• Example: Solution for Boussinesq system

\[ a_{10} = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0, \]
\[ a_{12} = -4\alpha c_1^2, \quad a_{20} = \text{free}, \]
\[ a_{21} = 0, \quad a_{22} = 4\alpha c_1 c_2. \]
Step T5

• Return to the original variables.
Test the final solution(s) of PDE.
Reject trivial solutions.

• Example: Solitary wave solution for Boussinesq system

\[
\begin{align*}
    u(x, t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \delta], \\
v(x, t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \delta].
\end{align*}
\]
<table>
<thead>
<tr>
<th>Case</th>
<th>ODE</th>
<th>Chain Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>tanh(ξ)</td>
<td>( y' = 1 - y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{d T} )</td>
</tr>
<tr>
<td>sech(ξ)</td>
<td>( y' = -y \sqrt{1 - y^2} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = -c_j S \sqrt{1 - S^2} \frac{d \bullet}{d S} )</td>
</tr>
<tr>
<td>tan(ξ)</td>
<td>( y' = 1 + y^2 )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j (1 + \tan^2) \frac{d \bullet}{d \tan} )</td>
</tr>
<tr>
<td>exp(ξ)</td>
<td>( y' = y )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j E \frac{d \bullet}{d E} )</td>
</tr>
<tr>
<td>cn(ξ; m)</td>
<td>( y' = -\sqrt{(1-y^2)(1-m+my^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j )</td>
</tr>
<tr>
<td>sn(ξ; m)</td>
<td>( y' = \sqrt{(1-y^2)(1-my^2)} )</td>
<td>( \frac{\partial \bullet}{\partial x_j} = c_j )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \sqrt{(1-CN^2)(1-m+mCN^2)} \frac{d \bullet}{d CN} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \sqrt{(1-SN^2)(1-mSN^2)} \frac{d \bullet}{d SN} )</td>
</tr>
</tbody>
</table>
Given: System of nonlinear PDEs of order $m$

$$\Delta(u(x), u'(x), u''(x), \cdots u^{(m)}(x)) = 0.$$}

Dependent variable $\mathbf{u}$ has $M$ components $u_i$ (or $u, v, w, \ldots$).

Independent variable $\mathbf{x}$ has $N$ components $x_j$ (or $x, y, z, \ldots, t$).
Step CN1

• Seek solution $u(x) = U(CN)$, with

$$CN = \text{cn}(\xi; m) = \text{cn}\left(\sum_{j}^{N} c_{j}x_{j} + \Delta\right); m).$$

with modulus $m$.

• Observe $\text{cn}'(\xi; m) = -\text{sn}(\xi; m)\text{dn}(\xi; m)$.

Using

$$\text{sn}^{2}(\xi; m) = 1 - \text{cn}^{2}(\xi; m), \quad \text{dn}^{2}(\xi; m) = 1 - m + m \text{cn}^{2}(\xi; m),$$

one has

$$CN' = -\sqrt{(1 - CN^{2})(1 - m + m CN^{2})}.$$
Repeatedly apply the operator rule

\[
\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{d CN} \frac{d CN}{d \xi} \frac{\partial \xi}{\partial x_j} = -c_j \sqrt{(1-CN^2)(1-m+mCN^2)} \frac{d \bullet}{d CN},
\]

produces a nonlinear ODE:

\[
\Delta(CN, U(CN), U'(CN), U''(CN), \ldots, U^{(m)}(CN)) = 0.
\]

Example: The KdV equation

\[
\frac{\partial u}{\partial t} + \alpha uu_x + u_{xxx} = 0,
\]

transforms into

\[
\left( c_1^3 (1 - 2m + 6mCN^2) - c_2 - \alpha c_1 U_1 \right) U_1' + 3c_1^3 CN(1 - 2m + 2mCN^2)U_1'' - c_1^3 (1-CN^2)(1-m+mCN^2)U_1''' = 0.
\]
Step CN2

- Seek polynomial solutions

\[ U_i(CN) = \sum_{j=0}^{M_i} a_{ij} CN^j. \]

Determine the highest exponents \( M_i \geq 1 \).

- Example: For KdV case: \( M_1 = 2 \). Thus,

\[ U_1(CN) = a_{10} + a_{11} CN + a_{12} CN^2. \]
Step CN3

- Derive the algebraic system for the coefficients $a_{ij}$.
- **Example**: Algebraic system for KdV case

\[-3 a_{11} c_1 (\alpha a_{12} - 2m c_1^2) = 0,\]
\[-2 a_{12} c_1 (\alpha a_{12} - 12m c_1^2) = 0,\]
\[-a_{11} (\alpha a_{10} c_1 - c_1^3 + 2m c_1^3 + c_2) = 0,\]
\[-\alpha a_{11}^2 c_1 - a_{12} (2\alpha a_{10} c_1 - 16m c_1^3 - 8c_1^3 + 2c_2) = 0.\]

**Note**: modulus $m$ is extra parameter.
Step CN4

- Solve the nonlinear algebraic system with parameters.

- **Example:** Solution for KdV system

\[
a_{10} = \frac{4c_1^3 (1 - 2m) - c_2}{\alpha c_1},
\]

\[
a_{11} = 0,
\]

\[
a_{12} = \frac{12m c_1^2}{\alpha}.
\]
Step CN5

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Cnoidal solution for the KdV equation

\[ u(x, t) = \frac{4c_1^3(1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \, \text{cn}^2(c_1 x + c_2 t + \Delta; m). \]
NOTE: For Jacobi sn solutions, use

\[ cn^2(\xi; m) = 1 - \text{sn}^2(\xi; m), \]
\[ dn^2(\xi; m) = 1 - m \text{sn}^2(\xi; m), \]
\[ \text{sn}'(\xi; m) = \text{cn}(\xi; m) \text{dn}(\xi; m). \]

Hence,

\[ SN' = \sqrt{(1 - SN^2)(1 - m SN^2)}, \]

with \( SN = \text{sn}(\xi; m) \).

Chain rule:

\[ \frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dSN} \frac{dSN}{d\xi} \frac{\partial \xi}{\partial x_j} = c_j \sqrt{(1 - SN^2)(1 - m SN^2)} \frac{d \bullet}{dSN}. \]
Algorithm for Tanh Solutions of DDEs

Nonlinear DDEs of order $m$

$$\Delta(u_{n+p_1}(x), u_{n+p_2}(x), \cdots, u_{n+p_k}(x), u'_{n+p_1}(x), u'_{n+p_2}(x), \cdots, u'_{n+p_k}(x), \cdots, u^{(r)}_{n+p_1}(x), u^{(r)}_{n+p_2}(x), \cdots, u^{(r)}_{n+p_k}(x)) = 0.$$ 

Dependent variable $u_n$ has $M$ components $u_{i,n}$
(or $u_n, v_n, w_n, \cdots$)

Independent variable $x$ has $N$ components $x_i$
(or $t, x, y, \cdots$).

Shift vectors $p_i \in \mathbb{Z}^Q$.

$u^{(r)}(x)$ is collection of mixed derivatives of order $r$. 
Simplest case for independent variable \((t)\), and one lattice point \((n)\):

\[
\Delta\left(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots, \dot{u}_{n-1}, \dot{u}_n, \dot{u}_{n+1}, \ldots, u_{n-1}^{(r)}, u_n^{(r)}, u_{n+1}^{(r)}, \ldots\right) = 0.
\]

**Step D1**

- Seek solution \(u_n(x) = U_n(T_n)\), with \(T_n = \tanh(\xi_n)\),

\[
\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = d \cdot n + c \cdot x + \delta.
\]
Repeatedly apply the operator rule

\[ \frac{d\bullet}{dx_j} = \frac{\partial \xi_n}{\partial x_j} \frac{dT_n}{d\xi_n} \frac{d\bullet}{dT_n} = c_j (1 - T_n^2) \frac{d\bullet}{dT_n}, \]

transforms DDE into

\[ \Delta(U_{n+p_1}(T_n), \ldots, U_{n+p_k}(T_n), U'_{n+p_1}(T_n), \ldots, U'_{n+p_k}(T_n), \ldots, U^{(r)}_{n+p_1}(T_n), \ldots, U^{(r)}_{n+p_k}(T_n)) = 0. \]

Note: \( U_{n+p_s} \) is function of \( T_n \) not of \( T_{n+p_s} \).
• Example: Toda lattice

\[ \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}) \]

transforms into

\[ c_2^2(1 - T^2) \left[ 2U'_n - (1 - T^2)U''_n \right] \]
\[ + \left[ 1 + c_2(1 - T^2)U'_n \right] [U_{n-1} - 2U_n + U_{n+1}] = 0. \]
Step D2

• Seek polynomial solutions

\[ U_{i,n}(T_n) = \sum_{j=0}^{M_i} a_{ij} T_n^j. \]

Use

\[ \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \]

to deal with the shift:

\[ T_{n+p_s} = \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s}, \]

where

\[ \phi_s = p_s \cdot d = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q, \]
Substitute $U_{i,n}(T_n) = T_n^{M_i}$, and

$$U_{i,n+p_s}(T_n) = T_{n+p_s}^{M_i} = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i},$$

and balance the highest exponents in $T_n$ to determine $M_i$.

**Note:** $U_{i,n+0}(T_n) = T_n^{M_i}$ is of degree $M_i$ in $T_n$.

$U_{i,n+p_s}(T_n) = \left[ \frac{T_n + \tanh \phi_s}{1 + T_n \tanh \phi_s} \right]^{M_i}$ is of degree zero in $T_n$. 
Example: Balance of exponents for Toda lattice

\[ 2M_1 + 1 = M_1 + 2. \]

So, \( M_1 = 1 \). Hence,

\[
U_n(T_n) = a_{10} + a_{11} T_n, \\
U_{n\pm1}(T(n \pm 1)) = a_{10} + a_{11} T(n \pm 1) \\
= a_{10} + a_{11} \frac{T_n \pm \tanh(d_1)}{1 \pm T_n \tanh(d_1)}. 
\]
Step D3

• **Determine the algebraic system for the unknown coefficients** $a_{ij}$ by setting to zero the coefficients of the powers in $T_n$.

• **Example**: Algebraic system for Toda lattice

\[
c_1^2 - \tanh^2(d_1) - a_{11} c_1 \tanh^2(d_1) = 0,
\]
\[
c_1 - a_{11} = 0.
\]
Step D4

- Solve the nonlinear algebraic system with parameters.
- **Example:** Solution of algebraic system for Toda lattice
  
  \[ a_{10} = \text{free}, \quad a_{11} = \pm \sinh(d_1), \quad c_1 = \pm \sinh(d_1). \]
Step D5

• Return to the original variables. Test solution(s) of DDE.
  Reject trivial ones.

• **Example:** Solitary wave solution for Toda lattice

\[ u_n(t) = a_{10} \pm \sinh(d_1) \tanh \left[ d_1 n \pm \sinh(d_1) t + \delta \right]. \]
Example: Relativistic Toda Lattice

\[
\begin{align*}
\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
\end{align*}
\]

Change of variables

\[
\begin{align*}
u_n(t) &= U_n(T_n), & v_n(t) &= V_n(T_n),
\end{align*}
\]

with

\[
T_n(t) = \tanh [d_1 n + c_1 t + \delta].
\]

gives

\[
\begin{align*}
c_1(1 - T^2)U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) &= 0, \\
c_1(1 - T^2)V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) &= 0.
\end{align*}
\]
Seek polynomial solutions

\[ U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \quad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j. \]

Balance the highest exponents in \( T_n \) to determine \( M_1 \) and \( M_2 \):

\[ M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2. \]

So, \( M_1 = M_2 = 1 \). Hence,

\[ U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n. \]
Algebraic system for $a_{ij}$:

$$-a_{11} c_1 + a_{21} \tanh(d_1) + \alpha a_{10} a_{21} \tanh(d_1) = 0,$$

$$a_{11} \tanh(d_1) (\alpha a_{21} + c_1) = 0,$$

$$-a_{21} c_1 + a_{11} a_{20} \tanh(d_1) + 2\alpha a_{20} a_{21} \tanh(d_1) = 0,$$

$$\tanh(d_1) (a_{11} a_{21} + 2\alpha a^2_{21} - a_{11} a_{20} \tanh(d_1)) = 0,$$

$$a_{21} \tanh^2(d_1) (c_1 - a_{11}) = 0.$$
Solution of the algebraic system

\[ a_{10} = -\frac{1}{\alpha} - c_1 \coth(d_1), \]
\[ a_{11} = c_1, \]
\[ a_{20} = \frac{c_1 \coth(d_1)}{\alpha}, \]
\[ a_{21} = -\frac{c_1}{\alpha}. \]

Solitary wave solution in original variables:

\[ u_n(t) = -\frac{1}{\alpha} - c_1 \coth(d_1) + c_1 \tanh [d_1 n + c_1 t + \Delta], \]
\[ v_n(t) = \frac{c_1 \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh [d_1 n + c_1 t + \Delta]. \]
Example: 2D Toda Lattice

2D Toda lattice:

\[
\frac{\partial^2 y_n}{\partial x \partial t} = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}),
\]

\(y_n(x, t)\) is displacement from equilibrium of the \(n\)-th unit mass under an exponential decaying interaction force between nearest neighbors.

Set

\[
\frac{\partial u_n}{\partial t} = \exp(y_{n-1} - y_n) - 1. \quad (*)
\]

Then,

\[
\exp(y_{n-1} - y_n) = \frac{\partial u_n}{\partial t} + 1,
\]
and the 2D-Toda lattice becomes

\[
\frac{\partial^2 y_n}{\partial x \partial t} = \frac{\partial u_n}{\partial t} + 1 - \left( \frac{\partial u_{n+1}}{\partial t} + 1 \right) = \frac{\partial u_n}{\partial t} - \frac{\partial u_{n+1}}{\partial t}.
\]

Integrate with respect to \( t \) to get

\[
\frac{\partial y_n}{\partial x} = u_n - u_{n+1}.
\]
Differentiate (*) with respect to $x$ and

\[
\frac{\partial^2 u_n}{\partial x \partial t} = \frac{\partial}{\partial x} \left( \exp(y_{n-1} - y_n) - 1 \right)
\]

\[
= \exp(y_{n-1} - y_n) \left( \frac{\partial y_{n-1}}{\partial x} - \frac{\partial y_n}{\partial x} \right),
\]

\[
= \left( \frac{\partial u_n}{\partial t} + 1 \right) \left[ (u_{n-1} - u_n) - (u_n - u_{n+1}) \right],
\]

\[
= \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}).
\]
So, the 2D Toda lattice is written in polynomial form:

\[
\frac{\partial^2 u_n}{\partial x \partial t}(x, t) = \left( \frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}).
\]

Travelling wave solution:

\[
u_n(x, t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[ d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right].
\]
The Ablowitz-Ladik lattice:

\[
\begin{align*}
\dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\
\dot{v}_n(t) &= - (\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n.
\end{align*}
\]

Travelling wave solution:

\[
\begin{align*}
u_n(t) &= \frac{\alpha \sinh^2(d_1)}{a_{21}} \left( \pm 1 - \tanh \left[ d_1 n + 2\alpha t \sinh^2(d_1) + \delta \right] \right) \\
v_n(t) &= a_{21} \left( \pm 1 + \tanh \left[ d_1 n + 2\alpha \sinh^2(d_1)t + \delta \right] \right).
\end{align*}
\]
Solving Nonlinear Parameterized Systems

• Assumptions
  ▶ All $c_i \neq 0$ and $d_i \neq 0$ (and modulus $m \neq 0$).
  ▶ Parameters ($\alpha, \beta, \gamma, ...$). Otherwise the maximal exponents $M_i$ may change.
  ▶ All $M_i \geq 1$.
  ▶ All $a_i M_i \neq 0$. Highest power terms in $U_i$ must be present, except in mixed sech-tanh-method.
  ▶ Solve for $a_{ij}$, then $c_i, \tanh(d_i)$, and $m$.
    Then find conditions on parameters.
• **Strategy** followed by hand

▶ Solve all linear equations in $a_{ij}$ first (branching).
Start with the ones without parameters.
Capture constraints in the process.

▶ Solve linear equations in $c_i, \tanh(d_i), m$ if they are free of $a_{ij}$.

▶ Solve linear equations in parameters if they free of $a_{ij}, c_i, \tanh(d_i), m$.

▶ Solve quasi-linear equations for $a_{ij}, c_i, \tanh(d_i), m$. 
Solve quadratic equations for $a_{ij}, c_i, \tanh(d_i), m$.

Eliminate cubic terms for $a_{ij}, c_i, \tanh(d_i), m$, without solving.

Show remaining equations, if any.

 Alternatives

 Use (adapted) Gröbner bases techniques.

 Use Ritt-Wu characteristic sets method.

 Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$. 
• Other applications (of the nonlinear algebraic solver)

Computation of conservation laws, symmetries, first integrals, etc. leading to linear parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).
Demonstration and Future Work

- Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and tanh function for DDEs.
- Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry methods, etc.
• Consider other types of explicit solutions involving
  
  ▶ other hyperbolic and elliptic functions $\sinh$, $\cosh$, $\text{dn}$, ....
  
  ▶ complex exponentials combined with $\text{sech}$, $\tanh$. 
Papers and Software

Papers


Software

- D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller,

**PDESpecialSolutions.m**: A Mathematica program for the symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for systems of nonlinear partial differential equations (2001-2006).

Available on the Internet

**URL**: http://www.mines.edu/fs_home/whereman/
D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller,


URL: http://www.mines.edu/fs_home/whereman/