Symbolic Computation of Conserved Densities of Nonlinear Evolution Equations and Differential-Difference Equations

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Montréal, Québec, Canada
Friday, May 23, 1997
Time: 14:00
• **Purpose**

Design and implement an algorithm to compute polynomial conservation laws for nonlinear systems of evolution equations, differential-difference equations, and dynamical systems

• **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics

  – For nonlinear PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability

  – Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures

  – Conservation laws can be used to test numerical integrators
PART I: Evolution Equations

- Conservation Laws for PDEs

Consider a single nonlinear evolution equation
\[ u_t = F(u, u_x, u_{xx}, \ldots, u_{nx}) \]
or a system of \( N \) nonlinear evolution equations
\[ \begin{align*}
\mathbf{u}_t &= F(\mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n)}) \\
\end{align*} \]
where \( \mathbf{u} = [u_1, \ldots, u_N]^T \), or, component-wise,
\[ u_{i,t} + F_i(u_j, u_j', u_j'', \ldots, u^{(n)}_j) = 0, \quad i, j = 1, 2, \ldots, N, \]
where
\[ u_{i,t} \overset{\text{def}}{=} \frac{\partial u_i}{\partial t}, \quad u_j^{(n)} = u_{j,xx} \overset{\text{def}}{=} \frac{\partial^n (u_j)}{\partial x^n} \]
All components of \( \mathbf{u} \) depend on \( x \) and \( t \).

Conservation law:
\[ D_t \rho + D_x J = 0 \]
\( \rho \) is the density, \( J \) is the flux

Both are polynomial in \( u, u_x, u_{2x}, \ldots, u_{nx} \)

Consequently
\[ P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant} \]
if \( J \) vanishes at infinity
• Example: Korteweg-de Vries (KdV) equation

\[ u_t + uu_x + u_{3x} = 0 \]

Conserved densities:

\[ \rho_1 = u, \quad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0 \]

\[ \rho_2 = u^2, \quad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - u_x^2)_x = 0 \]

\[ \rho_3 = u^3 - 3u_x^2, \quad \left( u^3 - 3u_x^2 \right)_t + \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_x^2 - 6u_xu_{3x} \right)_x = 0 \]

\[ \vdots \]

\[ \rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2, \quad \ldots \ldots \text{long} \ldots \ldots \]

\[ \vdots \]

Note: KdV equation is invariant under the scaling symmetry

\[(x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u)\]

\(u\) (resp. \(t\)) carry the weight of 2 (resp. 3) derivatives with respect to \(x\)

\[ u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3} \]
The Euler Operator (calculus of variations)

Useful tool to verify if an expression is a total derivative.

**Theorem:**

If
\[
f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)})
\]

then
\[
\mathcal{L}_y(f) \equiv 0
\]

if and only if
\[
f = D_x g
\]

where
\[
g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)})
\]

Notations:
\[
y = [y_1, \ldots, y_N]^T
\]
\[
\mathcal{L}_y(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T
\]
\[
0 = [0, \ldots, 0]^T
\]

($T$ for transpose)

and **Euler Operator**:
\[
\mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx} \left( \frac{\partial}{\partial y_i'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial}{\partial y_i''} \right) + \cdots + (-1)^n \frac{d^n}{dx^n} \left( \frac{\partial}{\partial y_i^{(n)}} \right)
\]
• **Key Steps of the Algorithm**

1. Determine weights (scaling properties) of variables & parameters
2. Construct the form of the density (building blocks)
3. Determine the unknown constant coefficients

• **Example:** For the KdV equation, compute the density of rank 6

(i) Take all the variables, except \( \frac{\partial}{\partial t} \), with positive weight.

Here, only \( u \) with \( w(u) = 2 \)

List all possible powers of \( u \), up to rank 6 : \([u, u^2, u^3]\)

Introduce \( x \) derivatives to ‘complete’ the rank

\( u \) has weight 2, introduce \( \frac{\partial^4}{\partial x^4} \)

\( u^2 \) has weight 4, introduce \( \frac{\partial^2}{\partial x^2} \)

\( u^3 \) has weight 6, no derivatives needed

(ii) Apply the derivatives

Remove terms that are total derivatives with respect to \( x \)

or total derivative up to terms kept earlier in the list

\([u_{4x}] \rightarrow [\ ] \text{ empty list}\)

\([u_x^2, uu_{2x}] \rightarrow [u_x^2] \text{ since } uu_{2x} = (uu_x)_x - u_x^2\)

\([u^3] \rightarrow [u^3]\)
Combine the ‘building blocks’

\[ \rho = c_1 u^3 + c_2 u_x^2 \]

(iii) Determine the coefficients \( c_1 \) and \( c_2 \)

1. Compute \( D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt} \)

2. Replace \( u_t \) by \( -(uu_x + u_{3x}) \) and \( u_{xt} \) by \( -(uu_x + u_{3x})_x \)

3. Integrate the result with respect to \( x \)
   Use the Euler operator (or carry out the integrations by parts)

\[
D_t \rho = -\left[ \frac{3}{4} c_1 u^4 - (3c_1 - c_2) uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x} \right]_x
- (3c_1 + c_2) u_x^3
\]

4. The non-integrable (last) term must vanish. Thus, \( c_1 = -\frac{1}{3} c_2 \).
   Set \( c_2 = -3 \), hence, \( c_1 = 1 \)

Result:

\[ \rho = u^3 - 3 u_x^2 \]

Expression […] yields

\[ J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x} \]
• **Example: Boussinesq equation**

\[
    u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0
\]

with nonzero parameter \( \alpha \). Can be written as

\[
    u_t + v_x = 0 \\
    v_t + u_x - 3uu_x - \alpha u_{3x} = 0
\]

The terms \( u_x \) and \( \alpha u_{3x} \) are not uniform in rank

Introduce auxiliary parameter \( \beta \) with weight. Replace the system by

\[
    u_t + v_x = 0 \\
    v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0
\]

The system is invariant under the scaling symmetry

\[
    (x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta)
\]

Hence,

\[
    w(u) = 2, \ w(\beta) = 2, \ w(v) = 3 \quad \text{and} \quad w\left(\frac{\partial}{\partial t}\right) = 2
\]

or

\[
    u \sim \beta \sim \frac{\partial^2}{\partial x^2}, \quad v \sim \frac{\partial^3}{\partial x^3}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}
\]

Form \( \rho \) of rank 6

\[
    \rho = c_1 \beta^2 u + c_2 \beta u^2 + c_3 u^3 + c_4 v^2 + c_5 u_x v + c_6 u_x^2
\]

Compute the \( c_i \). At the end set \( \beta = 1 \)

\[
    \rho = u^2 - u^3 + v^2 + \alpha u_x^2
\]

which is no longer uniform in rank!
• Application: A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters, and \( u \sim \frac{\partial^2}{\partial x^2} \)

Special cases:

\[
\begin{align*}
\alpha &= 30 \quad \beta = 20 \quad \gamma = 10 \quad \text{Lax} \\
\alpha &= 5 \quad \beta = 5 \quad \gamma = 5 \quad \text{Sawada – Kotera} \\
\alpha &= 20 \quad \beta = 25 \quad \gamma = 10 \quad \text{Kaup – Kupershmidt} \\
\alpha &= 2 \quad \beta = 6 \quad \gamma = 3 \quad \text{Ito}
\end{align*}
\]

Under what conditions for the parameters \( \alpha, \beta \) and \( \gamma \) does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \)  \( \text{(Lax and Ito cases)} \)
  \[ \rho = u^2 \]
– Rank 6:
Condition:

\[ 10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2 \]

(Lax, SK, and KK cases)

\[ \rho = u^3 + \frac{15}{(-2\beta + \gamma)} u_x^2 \]

– Rank 8:

1. \( \beta = 2\gamma \) (Lax and Ito cases)

\[ \rho = u^4 - \frac{6\gamma}{\alpha} uu_x^2 + \frac{6}{\alpha} u_{2x}^2 \]

2. \( \alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45} \) (SK, KK and Ito cases)

\[ \rho = u^4 - \frac{135}{2\beta + \gamma} uu_x^2 + \frac{675}{(2\beta + \gamma)^2} u_{2x}^2 \]

– Rank 10:

Condition:

\[ \beta = 2\gamma \]

and

\[ 10\alpha = 3\gamma^2 \]

(Lax case)

\[ \rho = u^5 - \frac{50}{\gamma} u^2 u_x^2 + \frac{100}{\gamma^2} uu_{2x}^2 - \frac{500}{7\gamma^3} u_{3x}^2 \]
What are the necessary conditions for the parameters $\alpha$, $\beta$ and $\gamma$ for this equation to admit infinitely many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If
  \[\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}\]
  or
  \[\beta = 2\gamma\]
  then there is a conserved density of rank 8

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)
• More Examples

• Nonlinear Schrödinger Equation

\[ iq_t - q_{2x} + 2|q|^2 q = 0 \]

Program can not handle complex equations
Replace by

\[ u_t - v_{2x} + 2v(u^2 + v^2) = 0 \]
\[ v_t + u_{2x} - 2u(u^2 + v^2) = 0 \]

where \( q = u + iv \)

Scaling properties

\[ u \sim v \sim \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2} \]

First seven conserved densities:

\[ \rho_1 = u^2 + v^2 \]
\[ \rho_2 = vu_x \]
\[ \rho_3 = u^4 + 2u^2v^2 + v^4 + u_x^2 + v_x^2 \]
\[ \rho_4 = u^2vu_x + \frac{1}{3}v^3u_x - \frac{1}{6}vu_{3x} \]
\[ \rho_5 = \frac{1}{2} u^6 - \frac{3}{2} u^4 v^2 - \frac{3}{2} u^2 v^4 - \frac{1}{2} v^6 - \frac{5}{2} u^2 u_x^2 - \frac{1}{2} v^2 u_x^2 \]

\[ -\frac{3}{2} u^2 v_x^2 - \frac{5}{2} v^2 v_x^2 + u v^2 u_{2x} - \frac{1}{4} u_{2x}^2 - \frac{1}{4} v_{2x}^2 \]

\[ \rho_6 = -\frac{3}{4} u^4 v u_x - \frac{1}{2} u^2 v^3 u_x - \frac{3}{20} v^5 u_x + \frac{1}{4} v u_x^3 - \frac{1}{4} v u_x v_x^2 \]

\[ + u v u_x u_{2x} + \frac{1}{4} u^2 v u_{3x} + \frac{1}{12} v^3 u_{3x} - \frac{1}{40} v u_{5x} \]

\[ \rho_7 = \frac{5}{4} u^8 + 5 u^6 v^2 + \frac{15}{2} u^4 v^4 + 5 u^2 v^6 + \frac{5}{4} v^8 + \frac{35}{2} u^4 u_x^2 \]

\[ -5 u^2 v^2 u_x^2 + \frac{5}{2} v^4 u_x^2 - \frac{7}{4} v^4 + \frac{15}{2} u^4 v_x^2 + 25 u^2 v^2 v_x^2 \]

\[ + \frac{35}{2} v^4 v_x^2 - \frac{5}{2} u^2 v_x^2 - \frac{7}{4} v_x^4 - 10 u^3 v^2 u_{2x} - 5 u v^4 u_{2x} \]

\[ -5 u v_x^2 u_{2x} + \frac{7}{2} u^2 u_{2x}^2 + \frac{1}{2} v^2 u_{2x}^2 + \frac{5}{2} u^2 v_{2x}^2 \]

\[ + \frac{7}{2} v^2 v_{2x}^2 - v^2 u_x u_{3x} + \frac{1}{4} u_{3x}^2 + \frac{1}{4} v_{3x}^2 + u v^2 u_{4x} \]
• The Ito system

\[ u_t - u_{3x} - 6uu_x - 2vu_x = 0 \]
\[ v_t - 2u_xv - 2uv_x = 0 \]

\[ u \sim \frac{\partial^2}{\partial x^2}, \quad v \sim \frac{\partial^2}{\partial x^2} \]

\[ \rho_1 = c_1u + c_2v \]
\[ \rho_2 = u^2 + v^2 \]
\[ \rho_3 = 2u^3 + 2uv^2 - u_x^2 \]
\[ \rho_4 = 5u^4 + 6u^2v^2 + v^4 - 10uu_x^2 + 2v^2u_{2x} + u_{2x}^2 \]
\[ \rho_5 = 14u^5 + 20u^3v^2 + 6uv^4 - 70u^2u_x^2 + 10v^2u_x^2 - 4v^2v_x^2 + 20uv^2u_{2x} + 14uu_{2x}^2 - u_{3x}^2 + 2v^2u_{4x} \]

and more conservation laws
• The dispersiveless long-wave system

\[ \begin{align*}
    u_t + vu_x + uv_x &= 0 \\
    v_t + u_x + vv_x &= 0
\end{align*} \]

\[ u \sim 2v \quad w(v) \text{ is free} \]

choose \[ u \sim \frac{\partial}{\partial x} \] and \[ 2v \sim \frac{\partial}{\partial x} \]

\[ \begin{align*}
    \rho_1 &= v \\
    \rho_2 &= u \\
    \rho_3 &= uv \\
    \rho_4 &= u^2 + uv^2 \\
    \rho_5 &= 3u^2v + uv^3 \\
    \rho_6 &= \frac{1}{3}u^3 + u^2v^2 + \frac{1}{6}uv^4 \\
    \rho_7 &= u^3v + u^2v^3 + \frac{1}{10}uv^5 \\
    \rho_8 &= \frac{1}{3}u^4 + 2u^3v^2 + u^2v^4 + \frac{1}{15}uv^6
\end{align*} \]

and more

Always the same set irrespective the choice of weights
A generalized Schamel equation

\[ n^2 u_t + (n + 1)(n + 2)u^2 u_x + u_{3x} = 0 \]

where \( n \) is a positive integer

\[ \rho_1 = u, \quad \rho_2 = u^2 \]

\[ \rho_3 = \frac{1}{2}u_x^2 - \frac{n^2}{2}u^{2+\frac{2}{n}} \]

For \( n \neq 1, 2 \) no further conservation laws.
• Three-Component Korteweg-de Vries Equation

\[ u_t - 6uu_x + 2vv_x + 2ww_x - u_3x = 0 \]
\[ v_t - 2vu_x - 2uv_x = 0 \]
\[ w_t - 2wu_x - 2uw_x = 0 \]

Scaling properties

\[ u \sim v \sim w \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3} \]

First five densities:

\[ \rho_1 = c_1 u + c_2 v + c_3 w \]
\[ \rho_2 = u^2 - v^2 - w^2 \]
\[ \rho_3 = -2u^3 + 2uv^2 + 2uw^2 + u_x^2 \]
\[ \rho_4 = -\frac{5}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 + 3u^2w^2 - v^2w^2 - \frac{1}{2}w^4 + 5uu_x^2 + v^2u_{2x} + w^2u_{2x} - \frac{1}{2}u_{2x}^2 \]
\[ \rho_5 = -\frac{7}{10}u^5 + u^3v^2 - \frac{3}{10}uv^4 + u^3w^2 - \frac{3}{5}uv^2w^2 - \frac{3}{10}uw^4 + \frac{7}{2}u^2u_x^2 + \frac{1}{2}v^2u_x^2 + \frac{1}{2}w^2u_x^2 + \frac{1}{2}v^2v_x^2 - \frac{1}{5}w^2v_x^2 + \frac{1}{5}w^2w_x^2 + uv^2u_{2x} + uw^2u_{2x} - \frac{7}{10}uu_{2x}^2 - \frac{1}{5}vw^2v_{2x} + \frac{1}{20}u_{3x}^2 + \frac{1}{10}v^2u_{4x} + \frac{1}{10}w^2u_{4x} \]
The Deconinck-Meuris-Verheest equation

Consider the modified vector derivative NLS equation:
\[
\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_\perp \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + e_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0
\]

Replace the vector equation by
\[
\begin{align*}
\frac{d}{dt} u + \left(u(u^2 + v^2) + \beta u - v_x\right)_x &= 0 \\
\frac{d}{dt} v + \left(v(u^2 + v^2) + u_x\right)_x &= 0
\end{align*}
\]

\(u\) and \(v\) denote the components of \(\mathbf{B}_\perp\) parallel and perpendicular to \(\mathbf{B}_\perp^0\) and \(\beta = \alpha B^2_{\perp 0}\)

\[
\begin{align*}
\beta &\sim \frac{\partial}{\partial x}, \\
u^2 &\sim \frac{\partial}{\partial x}, \\
v^2 &\sim \frac{\partial}{\partial x}
\end{align*}
\]

First 6 conserved densities

\[
\begin{align*}
\rho_1 &= c_1 u + c_2 v \\
\rho_2 &= u^2 + v^2 \\
\rho_3 &= \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_v x + \beta u^2 \\
\rho_4 &= \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)
\end{align*}
\]
\[ \rho_5 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{2x} - u_{2x} v_x) + \frac{4}{5}(u u_x + v v_x)^2 \]
\[ + \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(u v_x - u_x v) \]
\[ + \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4 \]
\[ \rho_6 = \frac{7}{16}(u^2 + v^2)^5 + \frac{1}{2}(u_{2x}^2 + v_{2x}^2) \]
\[ - \frac{5}{2}(u^2 + v^2)(u_x v_{2x} - u_{2x} v_x) + 5(u^2 + v^2)(u u_x + v v_x)^2 \]
\[ + \frac{15}{4}(u^2 + v^2)(u_x^2 + v_x^2) - \frac{35}{16}(u^2 + v^2)^3(u v_x - u_x v) \]
\[ + \frac{\beta}{8}(5u^8 + 10u^6 v^2 - 10u^2 v^6 - 5v^8 + 20u^2 u_x^2 \]
\[ - 12u^5 v_x + 60u v^4 v_x - 20v^2 v_x^2) \]
\[ + \frac{\beta^2}{4}(u^6 + v^6) \]
• Conservation Laws for DDEs

Consider a system of DDEs, continuous in time, discretized in space

\[ \dot{u}_n = F(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots) \]

\( u_n \) and \( F \) are vector dynamical variables

*Conservation law:*

\[ \dot{\rho}_n = J_n - J_{n+1} \]

\( \rho_n \) is the density, \( J_n \) is the flux

Both are polynomials in \( u_n \) and its shifts

\[ \frac{d}{dt} \left( \sum_n \rho_n \right) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

If \( J_n \) is bounded for all \( n \), with suitable boundary conditions

\[ \sum_n \rho_n = \text{constant} \]
• **Definitions**

Define: \(D\) **shift-down** operator, \(U\) **shift-up** operator

\[
Dm = m|_{n \rightarrow n-1}, \quad Um = m|_{n \rightarrow n+1}
\]

For example,

\[
Du_{n+2}v_n = u_{n+1}v_{n-1}, \quad Uu_{n-2}v_{n-1} = u_{n-1}v_n
\]

Compositions of \(D\) and \(U\) define an **equivalence relation**

All shifted monomials are **equivalent**, e.g.

\[
u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}
\]

Use **equivalence criterion**: If two monomials, \(m_1\) and \(m_2\), are equivalent, \(m_1 \equiv m_2\), then

\[
m_1 = m_2 + [M_n - M_{n+1}]
\]

for some polynomial \(M_n\)

For example, \(u_{n-2}u_n \equiv u_{n-1}u_{n+1}\) since

\[
u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}]
\]

with \(M_n = u_{n-2}u_n\)
Main representative of an equivalence class; the monomial with label \( n \) on \( u \) (or \( v \))

For example, \( u_nu_{n+2} \) is the main representative of the class with elements \( u_{n-1}u_{n+1}, u_{n+1}u_{n+3}, \) etc.

Use lexicographical ordering to resolve conflicts

For example, \( u_nv_{n+2} \) (not \( u_{n-2}v_n \)) is the main representative of the class with elements \( u_{n-3}v_{n-1}, u_{n+2}v_{n+4}, \) etc.

- **Algorithm: Toda Lattice**

\[
\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1})
\]

Simplest conservation law (by hand):

\[
\dot{u}_n = \dot{\rho}_n = v_{n-1} - v_n = J_n - J_{n+1} \quad \text{with} \quad J_n = v_{n-1}
\]

*First pair:*

\[
\rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}
\]

*Second pair:*

\[
\rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_nv_{n-1}
\]
Key observation: The DDE

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n (u_n - u_{n+1}) \]

and the two pairs

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1} \]
\[ \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1} \]

are invariant under the scaling symmetry

\[ (t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n) \]

Dimensional analysis:
\( u_n \) corresponds to one derivative with respect to \( t \)

For short, \( u_n \sim \frac{d}{dt} \), and similarly, \( v_n \sim \frac{d^2}{dt^2} \)

Our algorithm exploits this symmetry to find conserved densities:

1. Determining the weights
2. Constructing the form of density
3. Determining the unknown coefficients
• Step 1: Determine the weights

The weight, \( w \), of a variable is equal to the number of derivatives with respect to \( t \) the variable carries.

Weights are positive, rational, and independent of \( n \).

Set \( w(\frac{d}{dt}) = 1 \).

For the Toda lattice: \( w(u_n) = 1 \), and \( w(v_n) = 2 \).

The rank of a monomial is the total weight of the monomial, in terms of derivatives with respect to \( t \).

In each equation of the Toda lattice, all the terms are uniform in rank.

Requiring uniformity in rank for each equation, allows one to compute the weights of the dependent variables.

Indeed

\[
  w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n)
\]

yields

\[
  w(u_n) = 1, \quad w(v_n) = 2
\]

which is consistent with the scaling symmetry.
• Step 2: Construct the form of the density

For example, compute the form of the density of rank 3
List all monomials in $u_n$ and $v_n$ of rank 3 or less:

$$
\mathcal{G} = \{u_n^3, u_n^2, u_nv_n, u_n, v_n\}
$$

Next, for each monomial in $\mathcal{G}$, introduce enough $t$-derivatives, so that each term exactly has weight 3. Using the equations of Toda lattice,

$$
\frac{d^0}{dt^0}(u_n^3) = u_n^3, \quad \frac{d^0}{dt}(u_nv_n) = u_nv_n,
\frac{d}{dt}(u_n^2) = 2u_nv_{n-1} - 2u_n v_n, \quad \frac{d}{dt}(v_n) = u_nv_n - u_{n+1}v_n,
\frac{d}{dt^2}(u_n) = u_{n-1}v_{n-1} - u_nv_{n-1} - u_nv_n + u_{n+1}v_n
$$

Gather the resulting terms in a set

$$
\mathcal{H} = \{u_n^3, u_nv_{n-1}, u_nv_n, u_{n-1}v_{n-1}, u_{n+1}v_n\}
$$

Identify members that belong to the same equivalence classes and replace them by the main representatives.

For example, since $u_nv_{n-1} \equiv u_{n+1}v_n$ both are replaced by $u_nv_{n-1}$. $\mathcal{H}$ is replaced by

$$
\mathcal{I} = \{u_n^3, u_nv_{n-1}, u_nv_n\}
$$

containing the building blocks of the density.

Form a linear combination of the monomials in $\mathcal{I}$

$$
\rho_n = c_1 u_n^3 + c_2 u_nv_{n-1} + c_3 u_nv_n
$$

with constant coefficients $c_i$
• **Step 3: Determine the unknown coefficients**

Require that the conservation law holds

Compute $\dot{\rho}_n$

Use the equations to remove $u_n, \dot{v}_n$, etc.

Group the terms

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_{n-1}v_n$$
$$+ c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2 - c_3u_nu_{n+1}v_n - c_3v_n^2$$

Use the equivalence criterion to modify $\dot{\rho}_n$.

Replace $u_{n-1}u_nv_{n-1}$ by $u_nu_{n+1}v_n + [u_{n-1}u_nv_{n-1} - u_nv_{n+1}v_n]$.

The goal is to introduce the main representatives. Therefore,

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n$$
$$+(c_3 - c_2)v_nv_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_nv_{n+1}]$$
$$+ c_2u_nu_{n+1}v_n + [c_2u_{n-1}u_nv_{n-1} - c_2u_nv_{n+1}v_n]$$
$$+ c_2v_n^2 + [c_2v_{n-1}^2 - c_2v_n^2] - c_3u_nu_{n+1}v_n - c_3v_n^2$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom. Rearrange the latter terms so that they match the pattern $[J_n - J_{n+1}]$. Hence,

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n$$
$$+(c_3 - c_2)v_nv_{n+1} + (c_2 - c_3)u_nv_{n+1}v_n + (c_2 - c_3)v_n^2$$
$$+ \{[(c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2]$$
$$- \{(c_3 - c_2)v_nv_{n+1} + c_2u_nv_{n+1}v_n + c_2v_n^2]\}$$
The terms inside the square brackets determine:

\[ J_n = (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2 \]

The terms outside the square brackets must all vanish, thus

\[ S = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\} \]

The solution is \(3c_1 = c_2 = c_3\). Choose \(c_1 = \frac{1}{3}\), thus \(c_2 = c_3 = 1\)

\[ \rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2 \]

Analogously, conserved densities of rank \(\leq 5\):

\[ \rho_n^{(1)} = u_n, \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n \]

\[ \rho_n^{(3)} = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \]

\[ \rho_n^{(4)} = \frac{1}{4}u_n^4 + u_n^2(v_{n-1} + v_n) + u_nu_{n+1}v_n + \frac{1}{2}v_n^2 + v_nv_{n+1} \]

\[ \rho_n^{(5)} = \frac{1}{5}u_n^5 + u_n^3(v_{n-1} + v_n) + u_nu_{n+1}v_n(u_n + u_{n+1}) + u_nv_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_nv_n(v_{n-1} + v_n + v_{n+1}) \]
Application: A parameterized Toda lattice

\[ \dot{u}_n = \alpha v_{n-1} - v_n, \quad \dot{v}_n = v_n (\beta u_n - u_{n+1}) \]

\(\alpha\) and \(\beta\) are nonzero parameters. The system is integrable if \(\alpha = \beta = 1\).

Compute the compatibility conditions for \(\alpha\) and \(\beta\), so that there is a conserved densities of, say, rank 3.

In this case, we have \(S\):

\[ \{3\alpha c_1 - c_2 = 0, \beta c_3 - 3c_1 = 0, \alpha c_3 - c_2 = 0, \beta c_2 - c_3 = 0, \alpha c_2 - c_3 = 0\} \]

A non-trivial solution \(3c_1 = c_2 = c_3\) will exist iff \(\alpha = \beta = 1\).

Analogously, the parameterized Toda lattice has density

\[ \rho_n^{(1)} = u_n \text{ of rank 1 if } \alpha = 1 \]

and density

\[ \rho_n^{(2)} = \frac{\beta}{2} u_n^2 + v_n \text{ of rank 2 if } \alpha \beta = 1 \]

Only when \(\alpha = \beta = 1\) will the parameterized system have conserved densities of rank \(\geq 3\).
Example: Nonlinear Schrödinger (NLS) equation

Ablowitz and Ladik discretization of the NLS equation:

\[
i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1})
\]

where \( u_n^* \) is the complex conjugate of \( u_n \).

Treat \( u_n \) and \( v_n = u_n^* \) as independent variables, add the complex conjugate equation, and absorb \( i \) in the scale on \( t \)

\[
\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1})
\]

\[
\dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1})
\]

Since \( v_n = u_n^* \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Circumvent this problem by introducing an auxiliary parameter \( \alpha \) with weight,

\[
\dot{u}_n = \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1})
\]

\[
\dot{v}_n = -\alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1})
\]

Uniformity in rank requires that

\[
w(u_n) + 1 = w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n)
\]

\[
w(v_n) + 1 = w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n)
\]

which yields

\[
w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1
\]
Uniformity in rank is essential for the first two steps of the algorithm. After Step 2, you can already set $\alpha = 1$.

The computations now proceed as in the previous examples

Conserved densities:

$$\rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}$$

$$\rho_n^{(2)} = c_1 \left( \frac{1}{2} u_n^2 v_{n-1} v_{n} + u_n u_{n+1} v_{n} v_{n-1} + u_n v_{n-2} \right) + c_2 \left( \frac{1}{2} u_n^2 v_{n+1} v_{n} + u_n u_{n+1} v_{n+2} + u_n v_{n+2} \right)$$

$$\rho_n^{(3)} = c_1 \left[ \frac{1}{3} u_n^3 v_{n-1} + u_n u_{n+1} v_{n-1} v_{n} (u_n v_{n-1} + u_{n+1} v_{n} + u_{n+2} v_{n+1}) ight.$$
$+ u_n v_{n-1} (u_n v_{n-2} + u_{n+1} v_{n-1})$
$+ u_n v_{n} (u_{n+1} v_{n-2} + u_{n+2} v_{n-1}) + u_n v_{n-3} \left. \right] + c_2 \left[ \frac{1}{3} u_n^3 v_{n+1} + u_n u_{n+1} v_{n+1} v_{n+2} (u_n v_{n+1} + u_{n+1} v_{n+2} + u_{n+2} v_{n+3}) ight.$$
$+ u_n v_{n+2} (u_n v_{n+1} + u_{n+1} v_{n+2})$
$+ u_n v_{n+3} (u_{n+1} v_{n+1} + u_{n+2} v_{n+2}) + u_n v_{n+3} \right]$
• **Scope and Limitations of Algorithm & Software**

- Systems must be polynomial in dependent variables
- Only two independent variables \((x \text{ and } t)\) are allowed
- No terms should *explicitly* depend on \(x\) and \(t\)
- Program only computes polynomial conserved densities; only polynomials in the dependent variables and their derivatives; no explicit dependencies on \(x\) and \(t\)
- No limit on the number of evolution equations and DDEs.
  In practice: time and memory constraints
- Input systems may have (nonzero) parameters.
  Program computes the compatibility conditions for parameters such that densities (of a given rank) exist
- Systems can also have parameters with (unknown) weight.
  Allows one to test systems with equations of non-uniform rank
- For systems where one or more of the weights are free,
  the program prompts the user to enter values for the free weights
- Negative weights are not allowed
- Fractional weights and ranks are permitted
- Form of \(\rho\) can be given in the data file (testing purposes)
• **Conserved Densities Software**


- Conserved densities in **DELiA** by Bocharov (Pascal, 1990)

- Conserved densities and formal symmetries **FS** by Gerdt and Zharkov (Reduce, 1993), Mikhailov and Yamilov (MuMath, 1990)

- Conserved densities by Roelofs, Sanders and Wang (Reduce 1994, Maple 1995, Form 1995-present)

- Conserved densities **condens.m** by Hereman and Göktaş (Mathematica, 1996)

- Conservation laws, based on **CRACK** by Wolf (Reduce, 1995)

- Conservation laws by Hickman (Maple, 1996)


- Conserved densities **diffdens.m** by Göktaş and Hereman (Mathematica, 1997)
• **Conclusions and Further Research**

– Two *Mathematica* programs are available:
  * `condens.m` for evolution equations (PDEs)
  * `diffdens.m` for differential-difference equations (DDEs)
– Usefulness
  *
  * Testing models for integrability
  *
  * Study of classes of nonlinear PDEs or DDEs
– Comparison with other programs
  *
  * Parameter analysis is possible
  *
  * Not restricted to uniform rank equations
  *
  * Not restricted to evolution equations provided that
    one can write the equation(s) as a system of evolution equations
– Future work
  *
  * Generalization towards broader classes of equations (e.g. \( u_{xt} \))
  *
  * Generalization towards more space variables (e.g. KP equation)
  *
  * Conservation laws with time and space dependent coefficients
* Exploit other symmetries in the hope to find conserved densities of non-polynomial form

* Constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)

- Research supported in part by NSF under Grant CCR-9625421
- In collaboration with Ünal Göktaş and Grant Erdmann
- Papers submitted to: J. Symb. Comp., and Phys. Lett. A
- Software: available via FTP, ftp site mines.edu

in subdirectory pub/papers/math_cs_dept/software/condens

or via the Internet

URL: http://www.mines.edu/fs_home/whereman/
<table>
<thead>
<tr>
<th>Density</th>
<th>Sawada-Kotera equation</th>
<th>Lax equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho_1)</td>
<td>(u)</td>
<td>(u)</td>
</tr>
<tr>
<td>(\rho_2)</td>
<td>----</td>
<td>(\frac{1}{2}u^2)</td>
</tr>
<tr>
<td>(\rho_3)</td>
<td>(\frac{1}{3}u^3 - u_x^2)</td>
<td>(\frac{1}{3}u^3 - \frac{1}{6}u_x^2)</td>
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<tr>
<td>(\rho_4)</td>
<td>(\frac{1}{4}u^4 - \frac{9}{4}u u_x^2 + \frac{3}{4}u_x^2)</td>
<td>(\frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{72}u_x^2)</td>
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<tr>
<td>(\rho_5)</td>
<td>----</td>
<td>(\frac{1}{6}u^5 - u^2u_x^2 + \frac{1}{5}uu_x^2 - \frac{1}{70}u_x^3^2)</td>
</tr>
<tr>
<td>(\rho_6)</td>
<td>(\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_x^2)</td>
<td>(\frac{1}{6}u^6 - \frac{5}{3}u^3u_x^2 - \frac{5}{36}u_x^4 + \frac{1}{2}uu_x^2)</td>
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<tr>
<td></td>
<td>+2u_x^3 - \frac{21}{8}uu_x^3^2 + \frac{3}{8}u_4^2</td>
<td>+\frac{5}{63}u_x^3 - \frac{1}{14}uu_x^2 + \frac{1}{252}u_4^2</td>
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<tr>
<td>(\rho_7)</td>
<td>(\frac{1}{7}u^7 - \frac{9}{4}u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u_x^3u_x^2)</td>
<td>(\frac{1}{7}u^7 - \frac{5}{2}u^4u_x^2 - \frac{5}{6}uu_x^4 + u^3u_x^2)</td>
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<tr>
<td></td>
<td>+\frac{648}{35}u_x^2u_x^2 + \frac{489}{35}uu_x^2^3 - \frac{261}{35}u_x^2u_x^2\</td>
<td>+\frac{1}{2}uu_x^2u_x^2 + \frac{10}{21}uu_x^2^3 - \frac{3}{14}u^2u_x^3^2</td>
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<td>-\frac{268}{35}u_x^3u_x^2 + \frac{81}{35}uu_x^4^2 - \frac{9}{35}u_5^2</td>
<td>-\frac{5}{42}u_x^2u_x^3^2 + \frac{1}{42}uu_x^4^2 - \frac{1}{92}u_5^2</td>
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<tr>
<td>(\rho_8)</td>
<td>----</td>
<td>(\frac{1}{8}u^8 - \frac{7}{2}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_x^2)</td>
</tr>
<tr>
<td></td>
<td>+\frac{5}{2}uu_x^2u_x^2 + \frac{5}{3}u^2u_x^2^3 + \frac{7}{24}uu_x^4^2 + \frac{1}{2}u^3u_x^3^2</td>
<td>-\frac{1}{4}u^2u_x^3^2 - \frac{5}{6}uu_x^2u_x^3^2 + \frac{1}{12}u_x^2u_x^4^2</td>
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<td></td>
<td>+\frac{7}{152}uu_x^2u_x^4^2 - \frac{1}{132}uu_x^5^2 + \frac{1}{3432}u_6^2</td>
<td>+\frac{7}{132}uu_x^2u_x^4^2 - \frac{1}{132}uu_x^5^2 + \frac{1}{3432}u_6^2</td>
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<td>$\frac{u^3}{3} - \frac{1}{8} u_x^2$</td>
<td>$\frac{u^2}{2}$</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>$\frac{u^4}{4} - \frac{9}{16} uu_x^2 + \frac{3}{64} u_{2x}^2$</td>
<td>$\frac{u^4}{4} - \frac{9}{4} uu_x^2 + \frac{3}{4} u_{2x}^2$</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>$\frac{u^6}{6} - \frac{35}{16} u^2 u_x^2 - \frac{31}{256} u_x^4 + \frac{51}{64} u^2 u_{2x}^2$</td>
<td>$\frac{u^6}{6} - \frac{35}{16} u^2 u_x^2 - \frac{31}{256} u_x^4 + \frac{51}{64} u^2 u_{2x}^2$</td>
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<td>$\rho_5$</td>
<td>$\frac{u^7}{7} - \frac{27}{8} u^4 u_x^2 - \frac{369}{320} uu_x^4 + \frac{69}{40} u^3 u_{2x}^2$</td>
<td>$\frac{u^7}{7} - \frac{27}{8} u^4 u_x^2 - \frac{369}{320} uu_x^4 + \frac{69}{40} u^3 u_{2x}^2$</td>
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<tr>
<td>$\rho_6$</td>
<td>$\frac{u^8}{8} - \frac{2619}{4480} u_x^2 u_{2x}^2 + \frac{2211}{2240} uu_x^4 - \frac{477}{1120} u^2 u_{3x}^2$</td>
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<tr>
<td>$\rho_7$</td>
<td>$\frac{u^9}{9} - \frac{171}{640} u_{2x} u_{3x}^2 + \frac{27}{560} uu_{4x}^2 - \frac{9}{4480} u_{5x}^2$</td>
<td>$\frac{u^9}{9} - \frac{171}{640} u_{2x} u_{3x}^2 + \frac{27}{560} uu_{4x}^2 - \frac{9}{4480} u_{5x}^2$</td>
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<td>$\frac{u^{10}}{10}$</td>
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<tr>
<td>$\rho_3$</td>
<td>$-u^3 + u_x^2$</td>
<td>$-2u^3 + u_x^2$</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>$3u^4 - 9uu_x^2 + u_{2x}^2$</td>
<td>$5u^4 - 10uu_x^2 + u_{2x}^2$</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>$-14u^6 + 70u^2u_x^2 - 14uu_{2x}^2 + u_{3x}^2$</td>
<td>$-14u^6 + 70u^2u_x^2 - 14uu_{2x}^2 + u_{3x}^2$</td>
</tr>
</tbody>
</table>
| $\rho_6$ | $-\frac{12}{7}u^6 + \frac{150}{7}u^3u_x^2 + \frac{17}{7}u^4 - \frac{48}{7}u^2u_{2x}^2$ 
$-\frac{16}{21}u_{2x}^3 + uu_{3x}^2 - \frac{1}{21}u_{4x}^2$ | $-\frac{7}{3}u^6 + \frac{70}{3}u^3u_x^2 + \frac{35}{9}u^4 - 7u^2u_{2x}^2$ 
$-\frac{10}{9}u_{2x}^3 + uu_{3x}^2 - \frac{1}{18}u_{4x}^2$ |
| $\rho_7$ | $\frac{5u^7 - 105u^4u_x^2 - 42uu_x^4 + \frac{133}{3}u^3u_{2x}^2}{-\frac{2}{3}u^7 + \frac{35}{3}u^4u_x^2 + \frac{35}{9}uu_x^4 - \frac{14}{3}u^3u_{2x}^2}$ 
$\frac{24u_x^2u_{2x}^2 + \frac{163}{9}uu_{2x}^3 - \frac{29}{3}u^2u_{3x}^2}{-\frac{7}{3}u_x^2u_{2x}^2 - \frac{20}{9}uu_{2x}^3 + u^2u_{3x}^2}$ 
$-\frac{32}{9}u_{2x}u_{3x}^2 + uu_{4x}^2 - \frac{1}{27}u_{5x}^2$ | $\frac{5u_{2x}u_{3x}^2 - \frac{1}{9}uu_{4x}^2 + \frac{1}{18}uu_{5x}^2}{u_{2x}u_{3x}^2 - \frac{1}{9}uu_{4x}^2 + \frac{1}{18}uu_{5x}^2}$ |
| $\rho_8$ | $\frac{3}{2}u^8 - 42u^5u_x^2 - 35u^2u_x^4 + 21u^4u_{2x}^2$ 
$+ 42uu_x^2u_{2x}^2 + 20uu_x^3 + \frac{7}{5}u_{2x}^4 - 6u^3u_{3x}^2$ 
$-3u_x^2u_{3x}^2 - 10uu_xu_{3x}^2 + u^2u_{4x}^2$ 
$+ \frac{7}{11}uu_{2x}u_{4x}^2 - \frac{1}{11}uu_{5x}^2 + \frac{1}{255}u_{6x}^2$ | $\frac{3}{2}u^8 - 42u^5u_x^2 - 35u^2u_x^4 + 21u^4u_{2x}^2$ 
$+ 42uu_x^2u_{2x}^2 + 20uu_x^3 + \frac{7}{5}u_{2x}^4 - 6u^3u_{3x}^2$ 
$-3u_x^2u_{3x}^2 - 10uu_xu_{3x}^2 + u^2u_{4x}^2$ 
$+ \frac{7}{11}uu_{2x}u_{4x}^2 - \frac{1}{11}uu_{5x}^2 + \frac{1}{255}u_{6x}^2$ |