Continuous and Discrete Homotopy Operators with Applications in Integrability Testing

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Colloquium Talk
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Students: Adam Ringler, Ingo Kabirschke
Francis Martin & Kara Namanny

Talk dedicated to Ryan Sayers

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OUTLINE

Part I: Continuous Case
Integration by Parts on the Jet Space (by hand) + Mathematica Experiment
Exactness or Integrability Criterion: Continuous Euler Operator
Continuous Homotopy Operator
Application of Continuous Homotopy Operator
Demo of Mathematica software

Part II: Discrete Case
Inverting the Total Difference Operator (by hand)
Exactness or ‘Total Difference’ Criterion: Discrete Euler Operator
Discrete Homotopy Operator
Application of Discrete Homotopy Operator
Demo of Mathematica Software
Future Research
Problem Statement

For continuous case:

Given, for example,
\[ f = 3u' v^2 \sin(u) - u'^3 \sin(u) - 6v v' \cos(u) + 2u' u'' \cos(u) + 8v' v'' \]

Find \( F \) so that \( f = D_x F \) or \( F = \int f \, dx \).

Result:
\[ F = 4v'^2 + u'^2 \cos(u) - 3v^2 \cos(u) \]

Can this be done without integration by parts?
Can the problem be reduced to a single integral in one variable?

For discrete case:

Given, for example,
\[ f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

Find \( F_n \) so that \( f_n = \Delta F_n = F_{n+1} - F_n \) or \( F_n = \Delta^{-1} f_n \).

Result:
\[ F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1} \]

How can this be done algorithmically?
Can this be done in the same way as the continuous case?
Part I: Continuous Case
Integration by Parts on the Jet Space

• Given $f$ involving $u(x)$ and $v(x)$ and their derivatives

$$f = 3u'v^2\sin(u) - u^3\sin(u) - 6vv'\cos(u) + 2u'u''\cos(u) + 8v'v''$$

• Find $F$ so that $f = D_x F$ or $F = \int f \, dx$.

Integrate by parts (compute $F$ by hand)

$$8v''v' \longrightarrow 4v'^2$$

$$2u'u''\cos(u) \longrightarrow u'^2\cos(u)$$

$$-u^3\sin(u)$$

$$-6vv'\cos(u) \longrightarrow -3v^2\cos(u)$$

$$3u'v^2\sin(u)$$

• Integral:

$$F = 4v'^2 + u'^2\cos(u) - 3v^2\cos(u)$$

Remark: For simplicity we denote $f(u, u', u'', \cdots, u^{(m)})$ as $f(u)$. 
• Exactness Criterion:

**Continuous Euler Operator (variational derivative)**

**Definition (exactness):**

A function \( f(u) \) is exact, i.e. can be integrated fully, if there exists a function \( F(u) \), such that \( f(u) = D_x F(u) \) or equivalently \( F(u) = D_x^{-1} f(u) = \int_x f(u) \, dx \).

\( D_x \) is the (total) derivative with respect to \( x \).

**Theorem** (exactness or integrability test):

A necessary and sufficient condition for a function \( f \) to be exact, i.e. the derivative of another function, is that \( L^{(0)}(f) \equiv 0 \) where \( L^{(0)} \) is the continuous Euler operator (variational derivative) defined by

\[
L^{(0)} = \sum_{k=0}^{m_0} (-D_x)^k \frac{\partial}{\partial u^{(k)}}
\]

\[
= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u'} + D_x^2 \frac{\partial}{\partial u''} + \cdots + (-1)^{m_0} D_x^{m_0} \frac{\partial}{\partial u^{(m_0)}}
\]

where \( m_0 \) is the order (of \( f \)).

**Proof:**

See calculus of variations (derivation of Euler-Lagrange equations — the forgotten case!).
Example: Apply the continuous Euler operator to

\[ f(u) = 3u' v^2 \sin(u) - u'^3 \sin(u) - 6vv' \cos(u) + 2u' u'' \cos(u) + 8v'v'' \]

Here \( u = (u, v) \).

For component \( u \) (order 2):

\[
\mathcal{L}_u^{(0)}(f) = \frac{\partial}{\partial u}(f) - D_x \frac{\partial}{\partial u'}(f) + D_x^2 \frac{\partial}{\partial u''}(f)
\]
\[
= 3u' v^2 \cos(u) - u'^3 \cos(u) + 6v' \sin(u) - 2u' u'' \sin(u) - D_x[3v^2 \sin(u) - 3u'^2 \sin(u) + 2u'' \cos(u)] + D_x^2[2u' \cos(u)]
\]
\[
= 3u' v^2 \cos(u) - u'^3 \cos(u) + 6v' \sin(u) - 2u' u'' \sin(u) - [3u'v^2 \cos(u) + 6v' \sin(u) - 3u'^3 \cos(u) - 6u u'' \sin(u) - 2u' u'' \sin(u) + 2u''' \cos(u)]
\]
\[
+[-2u''' \cos(u) - 6u' u'' \sin(u) + 2u''' \cos(u)]
\]
\[
\equiv 0
\]

For component \( v \) (order 2):

\[
\mathcal{L}_v^{(0)}(f) = \frac{\partial}{\partial v}(f) - D_x \frac{\partial}{\partial v'}(f) + D_x^2 \frac{\partial}{\partial v''}(f)
\]
\[
= 6u' v \sin(u) - 6v' \cos(u) - D_x[-6v \cos(u) + 8v''] + D_x^2[8v']
\]
\[
= 6u' v \sin(u) - 6v' \cos(u) - [6u' v \sin(u) - 6v' \cos(u) + 8v''] + 8v'''
\]
\[
\equiv 0
\]

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Computation of the integral \( F \)

**Definition** (higher Euler operators):

The continuous higher Euler operators are defined by

\[
\mathcal{L}_u^{(i)} = \sum_{k=i}^{m_i} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial u^{(k)}}
\]

These Euler operators all terminate at some maximal order \( m_i \).

**Examples** (for component \( u \)):

\[
\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u'} + D^2_x \frac{\partial}{\partial u''} - D^3_x \frac{\partial}{\partial u^{(4)}} + \cdots + (-1)^{m_0} D^{m_0}_x \frac{\partial}{\partial u^{(m_0)}}
\]

\[
\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u'} - 2D_x \frac{\partial}{\partial u''} + 3D^2_x \frac{\partial}{\partial u^{(4)}} - 4D^3_x \frac{\partial}{\partial u^{(5)}} + \cdots + (-1)^{m_1} m_1 D^{m_1-1}_x \frac{\partial}{\partial u^{(m_1)}}
\]

\[
\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u''} - 3D_x \frac{\partial}{\partial u^{(4)}} + 6D^2_x \frac{\partial}{\partial u^{(5)}} - 10D^3_x \frac{\partial}{\partial u^{(6)}} + \cdots + (-1)^{m_2} \binom{m_2}{2} D^{m_2-2}_x \frac{\partial}{\partial u^{(m_2)}}
\]

\[
\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u^{(4)}} - 4D_x \frac{\partial}{\partial u^{(5)}} + 10D^2_x \frac{\partial}{\partial u^{(6)}} - 20D^3_x \frac{\partial}{\partial u^{(7)}} + \cdots + (-1)^{m_3} \binom{m_3}{3} D^{m_3-3}_x \frac{\partial}{\partial u^{(m_3)}}
\]

Similar formulae for component \( \mathcal{L}_v^{(i)} \)
**Definition** (homotopy operator):  
The continuous homotopy operator is defined by  
\[ \mathcal{H}(u) = \int_0^1 \sum_{r=1}^N f_r(u)[\lambda u] \frac{d\lambda}{\lambda} \]

where  
\[ f_r(u) = \sum_{i=0}^{p_r} D_i x [u_r \mathcal{L}_{u_r}^{(i+1)}] \]

\( p_r \) is the maximum order of \( u_r \) in \( f \)

\( N \) is the number of dependent variables

\( f_r(u)[\lambda u] \) means that in \( f_r(u) \) one replaces \( u \to \lambda u \), \( u' \to \lambda u' \), etc.

**Example:**

For a two-component system \((N = 2)\) where \( u = (u, v) \):

\[ \mathcal{H}(u) = \int_0^1 \{ f_1(u)[\lambda u] + f_2(u)[\lambda u] \} \frac{d\lambda}{\lambda} \]

with  
\[ f_1(u) = \sum_{i=0}^{p_1} D_i x [u \mathcal{L}_u^{(i+1)}] \]

and  
\[ f_2(u) = \sum_{i=0}^{p_2} D_i x [v \mathcal{L}_v^{(i+1)}] \]

**Theorem** (integration via homotopy operator):

Given an integrable function \( f \)

\[ F = D_x^{-1} f = \int f \, dx = \mathcal{H}(u)(f) \]

**Proof:** Olver’s book ‘Applications of Lie Groups to Differential Equations’, p. 372. Proof is given in terms of differential forms.

Work of Henri Poincaré (1854-1912), George de Rham (1950), and Ian Anderson & Tom Duchamp (1980).
Example: Apply the continuous homotopy operator to integrate

\[ f(u) = 3u'v^2 \sin(u) - u'^3 \sin(u) - 6vv' \cos(u) + 2u'u'' \cos(u) + 8v'v'' \]

For component \( u \) (order 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}_u^{(i+1)}(f) )</th>
<th>( D_x^i \left( u \mathcal{L}_u^{(i+1)}(f) \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\partial}{\partial u'} f - 2D_x \left( \frac{\partial}{\partial u''} f \right) )</td>
<td>( 3v^2 \sin u - 3u'^2 \sin u + 2u'' \cos u )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{\partial}{\partial u''} f = 2u' \cos u )</td>
<td>( -2D_x \left( 2u' \cos u \right) )</td>
</tr>
</tbody>
</table>

Hence, \( f_1(u)(f) = 3uv^2 \sin(u) - uu'^2 \sin(u) + 2u'^2 \cos(u) \)

For component \( v \) (order 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}_v^{(i+1)}(f) )</th>
<th>( D_x^i \left[ v \mathcal{L}_v^{(i+1)}(f) \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( -6v \cos(u) + 8v'' - 2D_x [8v'] )</td>
<td>( -6v^2 \cos(u) - 8vv'' )</td>
</tr>
<tr>
<td>1</td>
<td>( 8v' )</td>
<td>( D_x [8v'] = 8v'^2 + 8vv'' )</td>
</tr>
</tbody>
</table>

Hence, \( f_2(u)(f) = -6v^2 \cos(u) + 8v'^2 \)

The homotopy operator leads to an integral for (one) variable \( \lambda \). (Use standard integration by parts to work the integral).

\[
F(u) = \int_0^1 \left\{ f_1(u)(f)[\lambda u] + f_2(u)(f)[\lambda u] \right\} \frac{d\lambda}{\lambda}
\]

\[
= \int_0^1 \left[ 3\lambda^2 uv^2 \sin(\lambda u) - \lambda^2 uu'^2 \sin(\lambda u) + 2\lambda u'^2 \cos(\lambda u) 
- 6\lambda v^2 \cos(\lambda u) + 8\lambda v'^2 \right] d\lambda
\]

\[
= 4v'^2 + u'^2 \cos(u) - 3v^2 \cos(u)
\]
Application: Conserved densities and fluxes for PDEs with transcendental nonlinearities

Definition (conservation law):
\[ D_t \rho + D_x J = 0 \quad \text{(on PDE)} \]

conserved density \( \rho \) and flux \( J \).

Example: Sine-Gordon system (type \( u_t = F \))

\[ u_t = v \]
\[ v_t = u_{xx} + \alpha \sin(u) \]

has scaling symmetry

\[(t, x, u, v, \alpha) \rightarrow (\lambda^{-1}t, \lambda^{-1}x, \lambda^0 u, \lambda v, \lambda^2 \alpha)\]

In terms of weights:

\[ w(D_x) = 1, \ w(D_t) = 1, \ w(u) = 0, \ w(v) = 1, \ w(\alpha) = 2 \]

Conserved densities and fluxes

\[ \rho^{(1)} = 2\alpha \cos(u) + v^2 + u_x^2 \]
\[ J^{(1)} = -2u_x v \]
\[ \rho^{(2)} = u_x v \]
\[ J^{(2)} = -\left[ \frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha \cos(u) \right] \]
\[ \rho^{(3)} = 12 \cos(u) v u_x + 2v^3 u_x + 2v u_x^3 - 16v_x u_{2x} \]
\[ \rho^{(4)} = 2 \cos^2(u) - 2 \sin^2(u) + v^4 + 6v^2 u_x^2 + u_x^4 + 4 \cos(u) v^2 \]
\[ + 20 \cos(u) u_x^2 - 16v_x^2 - 16u_{2x}^2. \]

are all scaling invariant!

Remark: \( J^{(3)} \) and \( J^{(4)} \) are not shown (too long).
• Algorithm for Conserved Densities and Fluxes

**Example:** Density and flux of rank 2 for sine-Gordon system

**Step 1:** Construct the form of the density

\[ \rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)u_xv \]

where \( h_i(u) \) are unknown functions.

**Step 2:** Determine the functions \( h_i \)

Compute

\[ E = D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[F] \quad \text{(on PDE)} \]

\[ = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m_1} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t + \sum_{k=0}^{m_2} \frac{\partial \rho}{\partial v_{kx}} D_x^k v_t \]

Since \( E = D_t \rho = -D_x J \), the expression \( E \) must be integrable.

Require that \( L_u^{(0)}(E) \equiv 0 \) and \( L_v^{(0)}(E) \equiv 0 \).

Solve the system of linear mixed system (algebraic eqs. and ODEs):

\[
egin{align*}
    h_2(u) - h_3(u) &= 0 \\
    h_2'(u) &= 0 \\
    h_3'(u) &= 0 \\
    h_4'(u) &= 0 \\
    h_2''(u) &= 0 \\
    h_4''(u) &= 0 \\
    2h_2'(u) - h_3'(u) &= 0 \\
    2h_2''(u) - h_3''(u) &= 0 \\
    h_1'(u) + 2\sin(u)h_2(u) &= 0 \\
    h_1''(u) + 2\sin(u)h_2'(u) + 2\cos(u)h_2(u) &= 0
\end{align*}
\]
Solution:

\[
\begin{align*}
    h_1(u) &= 2c_1 \cos(u) + c_3 \\
    h_2(u) &= h_3(u) = c_1 \\
    h_4(u) &= c_2
\end{align*}
\]

(with arbitrary constants \(c_i\)).

Substitute in \(\rho\)

\[
\rho = 2c_1 \alpha \cos(u) + c_1 v^2 + c_1 u_x^2 + c_2 u_x v + \alpha c_3
\]

**Step 3: Compute the flux \(J\)**

First, compute

\[
E = D_t \rho = c_1(-2\alpha u_t \sin u + 2v v_t + 2u_x u_{xt}) + c_2(u_{xt} v + u_x v_t)
\]

\[
= c_1(-2\alpha v \sin u + 2v (u_{2x} + \alpha \sin u) + 2u_x v_x)
\]

\[
+ c_2(v_x v + u_x (u_{2x} + \alpha \sin u))
\]

\[
= c_1(2u_{2x} v + 2u_x v_x + c_2(v v_x + u_x u_{2x} + \alpha u_x \sin u)
\]

Since \(E = D_t \rho = -D_x J\), one must integrate.

Apply the homotopy operator for each component of \(\mathbf{u} = (u, v)\).

For component \(u\) (order 2):

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\mathcal{L}^{(i+1)}u(-E))</th>
<th>(D_x^i (u \mathcal{L}^{(i+1)}u(-E)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(2c_1 v_x + c_2(u_{2x} - \alpha \sin u))</td>
<td>(2c_1 u v_x + c_2(u u_{2x} - \alpha u \sin u))</td>
</tr>
<tr>
<td>1</td>
<td>(-2c_1 v - c_2 u_x)</td>
<td>(-2c_1(u_x v + uv_x) - c_2(u_x^2 + uu_{2x}))</td>
</tr>
</tbody>
</table>

Hence, \(f_1(u)(f) = 2c_1 u_x v - c_2(u_x^2 + \alpha u \sin u)\)
For component $v$ (order 1):

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\mathcal{L}_v^{(i+1)}(-E)$</th>
<th>$D_x^i (v\mathcal{L}_v^{(i+1)}(-E))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-2c_1u_x - c_2v$</td>
<td>$-2c_1u_xv - c_2v^2$</td>
</tr>
</tbody>
</table>

Hence, $f_2(u)(f) = -2c_1u_xv - c_2v^2$

The homotopy operator leads to an integral for (one) variable $\lambda$:

$$J(u) = \int_0^1 (f_1(u)(f)[\lambda u] + f_2(u)(f)[\lambda u]) \frac{d\lambda}{\lambda}$$

$$= -\int_0^1 \left(4c_1\lambda u_x v + c_2(\lambda u_x^2 + \alpha u \sin(\lambda u) + \lambda v^2)\right) d\lambda$$

$$= -2c_1u_xv - c_2\left(\frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha \cos u\right)$$

Split the density and flux in independent pieces (for $c_1$ and $c_2$):

$$\rho(1) = 2\alpha \cos u + v^2 + u_x^2$$

$$\rho(2) = u_x v$$

$$J(1) = -u_x v$$

$$J(2) = -\frac{1}{2}v^2 - \frac{1}{2}u_x^2 + \alpha \cos u$$

**Remark:** Computation of $J(3)$ and $J(4)$ requires integration with the homotopy operator!
Computer Demos

(1) Use continuous homotopy operator to integrate
\[ f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v'' \]

(2) Compute densities of rank 8 and fluxes for 5th-order Korteweg-de Vries equation with three parameters:
\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]
(\(\alpha, \beta, \gamma\) are nonzero constant parameters).

(3) Compute density of rank 4 and flux for sine-Gordon system:
\[ u_t = v \\
\] \[ v_t = u_{xx} + \alpha \sin(u) \]
## Analogy PDEs and DDEs

<table>
<thead>
<tr>
<th>System</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_t = F(u, u_x, u_{2x}, \ldots)$</td>
<td>$\dot{u}<em>n = F(\ldots, u</em>{n-1}, u_n, u_{n+1}, \ldots)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conservation Law</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_t \rho + D_x J = 0$</td>
<td>$\dot{\rho}<em>n + J</em>{n+1} - J_n = 0$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_t G = F'(u)[G]$</td>
<td>$D_t G = F'(u_n)[G]$</td>
<td></td>
</tr>
<tr>
<td>$= \frac{\partial}{\partial \epsilon} F(u + \epsilon G)</td>
<td>_{\epsilon = 0}$</td>
<td>$= \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Recursion Operator</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_t R + [R, F'(u)] = 0$</td>
<td>$D_t R + [R, F'(u_n)] = 0$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 1: Conservation Laws and Symmetries

<table>
<thead>
<tr>
<th>Equation</th>
<th>KdV Equation</th>
<th>Volterra Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_t = 6uu_x + u_{3x}$</td>
<td>$\dot{u}<em>n = u_n (u</em>{n+1} - u_{n-1})$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Densities</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = u$, $\rho = u^2$, $\rho = u^3 - \frac{1}{2} u_x^2$</td>
<td>$\rho_n = u_n$, $\rho_n = u_n (\frac{1}{2} u_n + u_{n+1})$, $\rho_n = \frac{1}{3} u_n^3 + u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symmetries</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = u_x$, $G = 6uu_x + u_{3x}$, $G = 30u^2u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}$</td>
<td>$G = u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$, $-u_{n-1} u_{n-2} (u_{n-2} + u_{n-1} + u_n)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Recursion Operator</th>
<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1}$</td>
<td>$\mathcal{R} = u_n (I + D)(u_n D - D^{-1} u_n)$, $(D - I)^{-1} \frac{1}{u_n}$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: Prototypical Examples

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Part II: Discrete Case

Definitions (shift and total difference operators):
D is the up-shift (forward or right-shift) operator if for $F_n$

$$DF_n = F_{n+1} = F_{n|_{n=n+1}}$$

$D^{-1}$ the down-shift (backward or left-shift) operator if

$$D^{-1}F_n = F_{n-1} = F_{n|_{n=n-1}}$$

$\Delta = D - I$ is the total difference operator

$$\Delta F_n = (D - I)F_n = F_{n+1} - F_n$$

D (up-shift operator) corresponds the differential operator $D_x$

$$D_xF(x) \rightarrow \frac{F_{n+1} - F_n}{\Delta x} = \frac{DF_n}{\Delta x} \quad \text{(set } \Delta x = 1)$$

For $k > 0$, $D^k = D \circ D \circ \cdots \circ D$ \hspace{1pt} ($k$ times).

Similarly, $D^{-k} = D^{-1} \circ D^{-1} \circ \cdots \circ D^{-1}$.

Problem to be solved:

Continuous case:
Given $f$. Find $F$ so that $f = D_xF$ or $F = D_x^{-1}f = \int f \, dx$.

Discrete case:
Given $f_n$. Find $F_n$ so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1}f_n$. 
Inverting the $\Delta$ Operator

- Given $f_n$ involving $u_n$ and $v_n$ and their shifts:
  \[ f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

- Find $F_n$ so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

  Invert the $\Delta$ operator (compute $F_n$ by hand)
  
  \[ \begin{align*}
  -v_n^2 & \quad \longrightarrow \quad v_n^2 \\
  v_{n+1}^2 & \\
  -u_n u_{n+1} v_n & \quad \longrightarrow \quad u_n u_{n+1} v_n \\
  u_{n+1} u_{n+2} v_{n+1} & \\
  -u_{n+1} v_n & \quad \longrightarrow \quad u_{n+1} v_n \\
  +u_{n+2} v_{n+1} & \\
  -u_{n+2} v_{n+1} & \quad \longrightarrow \quad u_{n+2} v_{n+1} \\
  u_{n+3} v_{n+2} &
  \end{align*} \]

- Result:
  \[ F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}. \]

**Remarks:** We denote $f(u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+p})$ as $f(u_n)$. Assume that all negative shifts have been removed via up-shifting

Replace $f_n = u_{n-2} v_n v_{n+3}$ by $\tilde{f}_n = D^2 f_n = u_n v_{n+2} v_{n+5}$.
• ‘Total Difference’ Criterion:

**Discrete Euler Operator (variational derivative)**

**Definition (exactness):**

A function \( f_n(u_n) \) is exact, i.e. a total difference, if there exists a function \( F_n(u_n) \), such that \( f_n = \Delta F_n \) or equivalently \( F_n = \Delta^{-1} f_n \).

D is the up-shift operator.

**Theorem (exactness or total difference test):**

A necessary and sufficient condition for a function \( f_n \) to be exact, i.e. a total difference, is that \( L^{(0)}(0)u_n(f_n) \equiv 0 \), where \( L^{(0)}(0) \) is the discrete Euler operator (variational derivative) defined by

\[
L^{(0)}_{u_n} = \sum_{k=0}^{m_0} D^{-k} \frac{\partial}{\partial u_{n+k}}
\]

\[
= \frac{\partial}{\partial u_n} + D^{-1}(\frac{\partial}{\partial u_{n+1}}) + D^{-2}(\frac{\partial}{\partial u_{n+2}}) + \cdots + D^{-m_0}(\frac{\partial}{\partial u_{n+m_0}})
\]

\[
= \frac{\partial}{\partial u_n}(\sum_{k=0}^{m_0} D^{-k})
\]

\[
L^{(0)}_{u_n} = \frac{\partial}{\partial u_n}(I + D^{-1} + D^{-2} + \cdots + D^{-m_0})
\]

where \( m_0 \) is the highest forward shift (in \( f_n \)).
**Example:** Apply the discrete Euler operator to

\[ f_n(u_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

Here \( u_n = (u_n, v_n) \).

For component \( u_n \) (highest shift 3):

\[
\mathcal{L}^{(0)}_{u_n}(f_n) = \frac{\partial}{\partial u_n} [I + D^{-1} + D^{-2} + D^{-3}](f_n) \\
= [-u_{n+1} v_n] + [-u_{n-1} v_{n-1} + u_{n+1} v_n - v_{n-1}] + [u_{n-1} v_{n-1}] + [v_{n-1}] \\
\equiv 0
\]

For component \( v_n \) (highest shift 2):

\[
\mathcal{L}^{(0)}_{v_n}(f_n) = \frac{\partial}{\partial v_n} [I + D^{-1} + D^{-2}](f_n) \\
= [-u_n u_{n+1} - 2v_n - u_{n+1}] + [u_n u_{n+1} + 2v_n] + [u_{n+1}] \\
\equiv 0
\]
**Computation of** \( F_n \)

**Definition** (higher Euler operators):

The discrete higher Euler operators are defined by

\[
\mathcal{L}_{u_n}^{(i)} = \frac{\partial}{\partial u_n} \left( \sum_{k=i}^{m_i} \binom{k}{i} D^{-k} \right)
\]

These Euler operators all terminate at some maximal shifts \( m_i \).

**Examples** (for component \( u_n \)):

\[
\begin{align*}
\mathcal{L}_{u_n}^{(0)} &= \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots + D^{-m_0}) \\
\mathcal{L}_{u_n}^{(1)} &= \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots + m_1 D^{-m_1}) \\
\mathcal{L}_{u_n}^{(2)} &= \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots + \frac{1}{2} m_2 (m_2 - 1) D^{-m_2}) \\
\mathcal{L}_{u_n}^{(3)} &= \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots + \binom{m_3}{3} D^{-m_3})
\end{align*}
\]

Similar formulae for \( \mathcal{L}_{v_n}^{(i)} \).
• **Definition** (homotopy operator):

The discrete homotopy operator is defined by

\[
\mathcal{H}(u_n) = \int_0^1 \sum_{r=1}^N f_{r,n}(u_n)[\lambda u_n] \frac{d\lambda}{\lambda}
\]

where

\[
f_{r,n}(u_n) = \sum_{i=0}^{p_r} (D - I)^i [u_{r,n} L_{u_{r,n}}^{(i+1)}]
\]

\(p_r\) is the maximum shift of \(u_{r,n}\) in \(f_n\)

\(N\) is the number of dependent variables

\(f_{r,n}(u_n)[\lambda u_n]\) means that in \(f_{r,n}(u_n)\) one replaces \(u_n \rightarrow \lambda u_n\), \(u_{n+1} \rightarrow \lambda u_{n+1}\), etc.

**Example:**

For a two-component system \((N = 2)\) where \(u_n = (u_n, v_n)\):

\[
\mathcal{H}(u_n) = \int_0^1 \{f_{1,n}(u_n)[\lambda u_n] + f_{2,n}(u_n)[\lambda u_n]\} \frac{d\lambda}{\lambda}
\]

with

\[
f_{1,n}(u_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n L_{u_n}^{(i+1)}]
\]

and

\[
f_{2,n}(u_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n L_{v_n}^{(i+1)}]
\]

**Theorem (total difference via homotopy operator):**

Given a function \(f_n\) which is a total difference, then

\[
F_n = \Delta^{-1} f_n = \mathcal{H}(u_n)(f_n)
\]

**Proof:** Recent work by Mansfield and Hydon on discrete variational bi-complexes. Proof is given in terms of differential forms.
Higher Euler Operators Side by Side

Continuous Case  (for component $u$)

\[ \mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \cdots \]

\[ \mathcal{L}_u^{(1)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \cdots \]

\[ \mathcal{L}_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \cdots \]

\[ \mathcal{L}_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \cdots \]

Discrete Case  (for component $u_n$)

\[ \mathcal{L}_{u_n}^{(0)} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(1)} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(2)} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots) \]

\[ \mathcal{L}_{u_n}^{(3)} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots) \]
Homotopy Operators Side by Side

**Continuous Case** (for components $u$ and $v$)

$$\mathcal{H}(u) = \int_0^1 \{ f_1(u)[\lambda u] + f_2(u)[\lambda u] \} \frac{d\lambda}{\lambda}$$

with

$$f_1(u) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]$$

and

$$f_2(u) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]$$

**Discrete Case** (for components $u_n$ and $v_n$)

$$\mathcal{H}(u_n) = \int_0^1 \{ f_{1,n}(u_n)[\lambda u_n] + f_{2,n}(u_n)[\lambda u_n] \} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(u_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(u_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$
**Example:** Apply the discrete homotopy operator to

\[ f_n(u_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \]

For component \( u_n \) (highest shift 3):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}^{(i+1)}_{u_n}(f_n) )</th>
<th>( (D - I)^i[u_n\mathcal{L}^{(i+1)}_{u_n}(f_n)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_{n-1}v_{n-1} + u_{n+1}v_n + 2v_{n-1} )</td>
<td>( u_{n-1}u_nv_{n-1} + u_nu_{n+1}v_n + 2u_nv_{n-1} )</td>
</tr>
<tr>
<td>1</td>
<td>( u_{n-1}v_{n-1} + 3v_{n-1} )</td>
<td>( u_nu_{n+1}v_n + 3u_{n+1}v_n - u_{n-1}u_nv_{n-1} - 3u_nv_{n-1} )</td>
</tr>
<tr>
<td>2</td>
<td>( v_{n-1} )</td>
<td>( u_{n+2}v_{n+1} - u_{n+1}v_n - u_{n+1}v_n + u_nv_{n-1} )</td>
</tr>
</tbody>
</table>

Hence, \( f_{1,n}(u_n)(f_n) = 2u_nu_{n+1}v_n + u_{n+1}v_n + u_{n+2}v_{n+1} \)

For component \( v_n \) (highest shift 2):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \mathcal{L}^{(i+1)}_{v_n}(f_n) )</th>
<th>( (D - I)^i[v_n\mathcal{L}^{(i+1)}_{v_n}(f_n)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u_nu_{n+1} + 2v_n + 2u_{n+1} )</td>
<td>( u_nu_{n+1}v_n + 2v_n^2 + 2u_{n+1}v_n )</td>
</tr>
<tr>
<td>1</td>
<td>( u_{n+1} )</td>
<td>( u_{n+2}v_{n+1} - u_{n+1}v_n )</td>
</tr>
</tbody>
</table>

Hence, \( f_{2,n}(u_n)(f_n) = u_nu_{n+1}v_n + 2v_n^2 + u_{n+1}v_n + u_{n+2}v_{n+1} \)

The homotopy operator leads to an integral for (one) variable \( \lambda \).
(Use standard integration by parts to work the integral).

\[
F_n(u_n) = \int_0^1 \{ f_{1,n}(u_n)(f_n)[\lambda u_n] + f_{2,n}(u_n)(f_n)[\lambda u_n] \} \frac{d\lambda}{\lambda} = \int_0^1 [2\lambda v_n^2 + 3\lambda^2 u_nu_{n+1}v_n + 2\lambda u_{n+1}v_n + 2\lambda u_{n+2}v_{n+1}] d\lambda = v_n^2 + u_nu_{n+1}v_n + u_{n+1}v_n + u_{n+2}v_{n+1}
\]
• **Application: Conserved densities and fluxes for DDEs**

**Definition** (conservation law):

\[
D_t \rho_n + \Delta J_n = D_t \rho_n + J_{n+1} - J_n = 0 \quad \text{(on DDE)}
\]

conserved density \(\rho_n\) and flux \(J_n\).

**Example** The Toda lattice (type \(\dot{u}_n = F\)):

\[
\begin{align*}
\dot{u}_n &= v_{n-1} - v_n \\
\dot{v}_n &= v_n(u_n - u_{n+1})
\end{align*}
\]

has scaling symmetry

\[
(t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n).
\]

In terms of weights:

\[
w(\frac{d}{dt}) = 1, \ w(u_n) = w(u_{n+1}) = 1, \ w(v_n) = w(v_{n-1}) = 2.
\]

Conserved densities and fluxes

\[
\begin{align*}
\rho_n^{(0)} &= \ln(v_n) & J_n^{(0)} &= u_n \\
\rho_n^{(1)} &= u_n & J_n^{(1)} &= v_{n-1} \\
\rho_n^{(2)} &= \frac{1}{2} u_n^2 + v_n & J_n^{(2)} &= u_n v_{n-1} \\
\rho_n^{(3)} &= \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) & J_n^{(3)} &= u_{n-1} u_n v_{n-1} + v_{n-1}^2
\end{align*}
\]

are all scaling invariant!
Algorithm for Conserved Densities and Fluxes

**Example:** Density of rank 3 for Toda system

**Step 1:** Construct the form of the density.

\[ \rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n \]

where \( c_i \) are unknown constants.

**Step 2:** Determine the constants \( c_i \).

Compute

\[
\tilde{E}_n = D E_n \]

to remove negative shift \( n - 1 \).

Since \( \tilde{E}_n = -\Delta \tilde{J}_n \), the expression \( \tilde{E}_n \) must be a total difference.

Require

\[
\mathcal{L}^{(0)}_{u_n}(\tilde{E}_n) = \frac{\partial}{\partial u_n}(I + D^{-1} + D^{-2})(\tilde{E}_n) = \frac{\partial}{\partial u_n}(D + I + D^{-1})(E_n)
\]

\[
= 2(3c_1 - c_2)u_n v_{n-1} + 2(c_3 - 3c_1)u_n v_n
\]

\[
+ (c_2 - c_3)u_{n-1}v_{n-1} + (c_2 - c_3)u_{n+1}v_n \equiv 0
\]

and

\[
\mathcal{L}^{(0)}_{v_n}(\tilde{E}_n) = \frac{\partial}{\partial v_n}(I + D^{-1})(\tilde{E}_n) = \frac{\partial}{\partial v_n}(D + I)(E_n)
\]

\[
= (3c_1 - c_2)u_{n+1}^2 + (c_3 - c_2)v_{n+1} + (c_2 - c_3)u_n u_{n+1}
\]

\[
+ 2(c_2 - c_3)v_n + (c_3 - 3c_1)u_n^2 + (c_3 - c_2)v_{n-1} \equiv 0.
\]
Solve the linear system
\[ S = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}. \]

Solution: \(3c_1 = c_2 = c_3\) Choose \(c_1 = \frac{1}{3}\), and \(c_2 = c_3 = 1\).

Substitute in \(\rho_n\)
\[ \rho_n = \frac{1}{3} u_n^3 + u_n (v_{n-1} + v_n) \]

**Step 3: Compute the flux \(J_n\).**

Start from \(-\tilde{E}_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2\)

Apply the discrete homotopy operator!

For component \(u_n\) (highest shift 2):

<table>
<thead>
<tr>
<th>(i)</th>
<th>(L_{u_n}^{(i+1)}(-E_n))</th>
<th>((D - I)^i(u_n L_{u_n}^{(i+1)}(-E_n)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(u_{n-1} v_{n-1} + u_{n+1} v_n)</td>
<td>(u_n u_{n-1} v_{n-1} + u_n u_{n+1} v_n)</td>
</tr>
<tr>
<td>1</td>
<td>(u_{n-1} v_{n-1})</td>
<td>(u_{n+1} u_n v_n - u_n u_{n-1} v_{n-1})</td>
</tr>
</tbody>
</table>

Hence, \(\tilde{j}_{1,n}(u_n) = 2 u_n u_{n+1} v_n\)

For component \(v_n\) (highest shift 1):

<table>
<thead>
<tr>
<th>(i)</th>
<th>(L_{v_n}^{(i+1)}(-E_n))</th>
<th>((D - I)^i(v_n L_{v_n}^{(i+1)}(-E_n)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(u_n u_{n+1} + 2 v_n)</td>
<td>(v_n u_n u_{n+1} + 2 v_n^2)</td>
</tr>
</tbody>
</table>

Hence, \(\tilde{j}_{2,n}(u_n) = u_n u_{n+1} v_n + 2 v_n^2\)

\[
\tilde{J}_n = \int_0^1 (\tilde{j}_{1,n}(u_n)[\lambda u_n] + \tilde{j}_{2,n}(u_n)[\lambda u_n]) \frac{d\lambda}{\lambda} = \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2) \, d\lambda = u_n u_{n+1} v_n + v_n^2.
\]

Final Result:
\[
J_n = D^{-1}\tilde{J}_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2
\]
Computer Demos

(1) Use discrete homotopy operator to compute $F_n = \Delta^{-1} f_n$ for

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

(2) Compute density of rank 4 and flux for Toda system:

$$\dot{u}_n = v_{n-1} - v_n$$
$$\dot{v}_n = v_n(u_n - u_{n+1})$$

(3) Compute density of rank 2 for Ablowitz-Ladik system:

$$i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n(u_{n+1} + u_{n-1})$$

($u_n^*$ is the complex conjugate of $u_n$).

This is an integrable discretization of the NLS equation:

$$i u_t + u_{xx} + \kappa u^2 u^* = 0$$

Take equation and its complex conjugate.

Treat $u_n$ and $v_n = u_n^*$ as dependent variables. Absorb $i$ in $t$:

$$\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n(u_{n+1} + u_{n-1})$$
$$\dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n(v_{n+1} + v_{n-1}).$$
Future Research

- Generalize continuous homotopy operator in multi-dimensions $(x, y, z, ...)$.

- Problem (in three dimensions):
  
  Given $E = \nabla \cdot \mathbf{J} = J_x^{(1)} + J_y^{(2)} + J_z^{(3)}$. 
  
  Find $\mathbf{J} = (J^{(1)}, J^{(2)}, J^{(3)})$.

- Application:
  
  Compute densities and fluxes of multi-dimensional systems of PDEs (in $t, x, y, z$).

- Generalize discrete homotopy operator in multi-dimensions $(n, m, ...)$.
Higher Euler Operators Side by Side

**Continuous Case** (for component \( u \))

\[
L_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \cdots
\]

\[
L_u^{(1)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \cdots
\]

\[
L_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \cdots
\]

\[
L_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \cdots
\]

**Discrete Case** (for component \( u_n \))

\[
L_{u_n}^{(0)} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \cdots)
\]

\[
L_{u_n}^{(1)} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots)
\]

\[
L_{u_n}^{(2)} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots)
\]

\[
L_{u_n}^{(3)} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots)
\]
Homotopy Operators Side by Side

**Continuous Case** (for components $u$ and $v$)

\[
\mathcal{H}(u) = \int_0^1 \left\{ f_1(u)[\lambda u] + f_2(u)[\lambda u] \right\} \frac{d\lambda}{\lambda}
\]

with

\[
f_1(u) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]
\]

and

\[
f_2(u) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]
\]

**Discrete Case** (for components $u_n$ and $v_n$)

\[
\mathcal{H}(u_n) = \int_0^1 \left\{ f_{1,n}(u_n)[\lambda u_n] + f_{2,n}(u_n)[\lambda u_n] \right\} \frac{d\lambda}{\lambda}
\]

with

\[
f_{1,n}(u_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]
\]

and

\[
f_{2,n}(u_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]
\]