Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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Examples of Conservation Laws

Example 1: Traffic Flow

Modeling the density of cars (Bressan, 2009)

\( u(x, t) \) density of cars on a highway (e.g. number of cars per 100 meters).

\( s(u) \) mean (equilibrium) speed of the cars (depends on the density).
Change in number of cars in segment \([a, b]\) equals the difference between cars entering at \(a\) and leaving at \(b\) during time interval \([t_1, t_2]\):

\[
\int_a^b \left( u(x, t_2) - u(x, t_1) \right) \, dx = \int_{t_1}^{t_2} \left( J(a, t) - J(b, t) \right) \, dt
\]

\[
\int_a^b \left( \int_{t_1}^{t_2} u_t(x, t) \, dt \right) \, dx = -\int_{t_1}^{t_2} \left( \int_a^b J_x(x, t) \, dx \right) \, dt
\]

where \(J(x, t) = u(x, t)s(u(x, t))\) is the traffic flow (e.g. in cars per hour) at location \(x\) and time \(t\).
Then, \( \int_a^b \int_{t_1}^{t_2} (u_t + J_x) \, dt \, dx = 0 \) holds \( \forall (a, b, t_1, t_2) \)

Yields the conservation law:

\[
\begin{align*}
  u_t + [s(u) \, u]_x &= 0 \\
  D_t \rho + D_x J &= 0
\end{align*}
\]

\( \rho = u \) is the conserved density;
\[ J(u) = s(u) \, u \] is the associated flux.

A simple Lighthill-Whitham-Richards model:

\[
s(u) = s_{\text{max}} \left( 1 - \frac{u}{u_{\text{max}}} \right), \quad 0 \leq u \leq u_{\text{max}}
\]

\( s_{\text{max}} \) is posted road speed, \( u_{\text{max}} \) is the jam density.
Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]
\[ (\rho \mathbf{u})_t + \nabla \cdot (\mathbf{u} \otimes (\rho \mathbf{u})) + \nabla p = 0 \]
\[ E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0 \]

or, in components

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]
\[ (\rho u_i)_t + \nabla \cdot (\rho u_i \mathbf{u} + p \mathbf{e}_i) = 0 \quad (i = 1, 2, 3) \]
\[ E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0 \]

Express conservation of mass, momentum, energy.
\( \otimes \) is the dyadic product.

\( \rho \) is the mass density.

\[ \mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \] is the velocity.

\( p \) is the pressure \( p(\rho, e) \).

\( E \) is the energy density per unit volume:

\[ E = \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e. \]

\( e \) is internal energy density per unit of mass (related to temperature).
Notation – Computations on the Jet Space

• Independent variables \( \mathbf{x} = (x, y, z) \)

• Dependent variables \( \mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)}) \)
  In examples: \( \mathbf{u} = (u, v, \theta, h, \ldots) \)

• Partial derivatives \( u_{kx} = \frac{\partial^k u}{\partial x^k}, \ u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}, \) etc.
  Examples: \( u_{xxxxxx} = u_5x = \frac{\partial^5 u}{\partial x^5} \)
  \( u_{xx\,yyyy} = u_2x\,4y = \frac{\partial^6 u}{\partial x^2 y^4} \)

• Differential functions
  Example: \( f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx} \)
• **Total derivatives:** $D_t, D_x, D_y, \ldots$

**Example:** Let $f = uvv_x + x^2u^3_xv_x + u_xv_{xx}$

Then

$$D_x f = \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}}$$

$$= 2xu^3_xv_x + u_x(uvv_x) + u_{xx}(3x^2u^2_xv_x + v_{xx}) + v_x(uvv_x) + v_{xx}(uv + x^2u^3_x) + v_{xxx}(u_x)$$

$$= 2xu^3_xv_x + uvv_x + 3x^2u^2_xv_xu_{xx} + u_{xx}v_{xx} + uvv_x + uvv_{xx} + x^2u^3_xv_{xx} + u_xv_{xxx}$$
Conservation Laws for Nonlinear PDEs

- System of evolution equations of order $M$

$$\mathbf{u}_t = F(\mathbf{u}^{(M)}(\mathbf{x}))$$

with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$D_t \rho + D_x J = 0$$

evaluated on the PDE.

Conserved density $\rho$ and flux $J$. 
• Conservation law in (2+1)-dimensions

$$D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 = 0$$

evaluated on the PDE.

Conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2)$.

• Conservation law in (3+1)-dimensions

$$D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 = 0$$

evaluated on the PDE.

Conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2, J_3)$. 
Reasons for Computing Conservation Laws

• Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).

• Testing of complete integrability and application of Inverse Scattering Transform.

• Testing of numerical integrators.

• Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ...).

• Verify the closure of a model.
Examples of PDEs with Conservation Laws

Example 1: KdV Equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad u_t + uu_x + u_{xxx} = 0
\]

shallow water waves, ion-acoustic waves in plasmas

Diederik Korteweg    Gustav de Vries
Dilation Symmetry

\[ u_t + uu_x + u_{xxx} = 0 \]

has dilation (scaling) symmetry \((x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)\)

\(\lambda\) is an arbitrary parameter.

Notion of weight: \(W(x) = -1\), thus, \(W(D_x) = 1\).

\[ W(t) = -3, \text{ hence, } W(D_t) = 3. \]

\[ W(u) = 2. \]

Notion of rank (total weight of a monomial).

Examples: \(\text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6. \)

\[ \text{Rank}(u^3u_{xx}) = 10. \]
Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

\[ D_t \rho + D_x J = 0 \]

is uniform in rank.

Hence,

\[ \text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank} J + \text{Rank}(D_x) \]
• First six (of infinitely many) conservation laws:

\[ D_t(u) + D_x\left(\frac{1}{2}u^2 + u_{xx}\right) = 0 \]

\[ D_t(u^2) + D_x\left(\frac{2}{3}u^3 - u_x^2 + 2uu_{xx}\right) = 0 \]

\[ D_t\left(u^3 - 3u_x^2\right) + D_x\left(\frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{xx} + 3u_x^2 - 6u_xu_{xxx}\right) = 0 \]

\[ D_t\left(u^4 - 12uu_x^2 + \frac{36}{5}u_{xx}^2\right) + D_x\left(\frac{4}{5}u^5 - 18uu_x^2 + 4u^3u_{xx} + 12u_x^2u_{xx} + \frac{96}{5}uu_{xx}^2 - 24uu_xu_{xxx} - \frac{36}{5}u_{xxx}^2 + \frac{72}{5}u_{xx}u_{4x}\right) = 0 \]
\[
D_t \left( u^5 - 30 u^2 u_x + 36 uu_{xx} - \frac{108}{7} u_{xxx} \right) \\
+ D_x \left( \frac{5}{6} u^6 - 40 u^3 u_x^2 - \ldots - \frac{216}{7} u_{xxx} u_{5x} \right) = 0
\]

\[
D_t \left( u^6 - 60 u^3 u_x^2 - 30 u_x^4 + 108 u^2 u_{xx} \\
+ \frac{720}{7} u_{xx}^3 - \frac{648}{7} uu_{xxx}^2 + \frac{216}{7} u_{4x}^2 \right) \\
+ D_x \left( \frac{6}{7} u^7 - 75 u^4 u_x^2 - \ldots + \frac{432}{7} u_{4x} u_{6x} \right) = 0
\]

- **Third conservation law:** Gerald Whitham, 1965
- **Fourth and fifth:** Norman Zabusky, 1965-66
- **Seventh (sixth thru tenth):** Robert Miura, 1966
Robert Miura
Conservation law explicitly dependent on $t$ and $x$:

$$D_t \left( tu^2 - 2xu \right) + D_x \left( \frac{2}{3} tu^3 - xu^2 + 2u_x - tu^2 + 2tu u_{xx} - 2xu_{xx} \right) = 0$$
First five: IBM 7094 computer with FORMAC (1966) → storage space problem!
First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)

→ large integers problem!
Example 2:  

The Zakharov-Kuznetsov Equation

\[ u_t + \alpha uu_x + \beta (u_{xx} + u_{yy})_x = 0 \]

models ion-sound solitons in a low pressure uniform magnetized plasma.

- Conservation laws:

\[
D_t(u) + D_x \left( \frac{\alpha}{2} u^2 + \beta u_{xx} \right) + D_y \left( \beta u_{xy} \right) = 0
\]

\[
D_t(u^2) + D_x \left( \frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy}) \right) + D_y \left( -2\beta u_x u_y \right) = 0
\]
• More conservation laws (ZK equation):

\[
\begin{align*}
\mathcal{D}_t \left( u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + \mathcal{D}_x \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) 
+ \frac{3\beta^2}{\alpha} (u_x^2 - u_y^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xx} + u_{yy})) \right) 
+ \mathcal{D}_y \left( 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) &= 0 \\
\mathcal{D}_t \left( tu^2 - \frac{2}{\alpha} xu \right) + \mathcal{D}_x \left( t \left( \frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) 
- x (u^2 + \frac{2\beta}{\alpha} u_{xx}) + \frac{2\beta}{\alpha} u_x \right) + \mathcal{D}_y \left( - 2\beta (tu_x u_y + \frac{1}{\alpha} xu_{xy}) \right) &= 0
\end{align*}
\]
Methods for Computing Conservation Laws

• Use the Lax pair $L$ and $A$, satisfying $[L, A] = 0$. If $L = D_x + U$, $A = D_t + V$ then $V_x - U_t + [U, V] = 0$.

  $\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

• Use Noether’s theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, and many others).

• Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Wolf, etc.) require solving ODEs (or PDEs).
Proposed Algorithmic Method

• Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.

• Compute $D_t \rho$ with total derivative operator.

• Use variational derivative (Euler operator) to express exactness.

• Solve a (parametrized) linear system to find the undetermined coefficients.

• Use the homotopy operator to compute the flux (invert $D_x$ or $\text{Div}$).
• Work with linearly independent pieces in finite dimensional spaces.

• Use linear algebra, calculus, and variational calculus (algorithmic).

• Implement the algorithm in Mathematica.
Tools from the Calculus of Variations

• Definition:
A differential function $f$ is a exact iff $f = \text{Div } F$.
Special case (1D): $f = D_x F$.

• Question: How can one test that $f = \text{Div } F$?

• Theorem (exactness test):
$f = \text{Div } F$ iff $\mathcal{L}_{u(j)(x)} f \equiv 0, \quad j = 1, 2, \ldots, N$.
$N$ is the number of dependent variables.

The Euler operator annihilates divergences
• Euler operator in 1D (variable $u(x)$):

$$
\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-D_x)^k \frac{\partial}{\partial u_{kx}} \\
= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots
$$

• Euler operator in 2D (variable $u(x, y)$):

$$
\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx} \ell y} \\
= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots
$$
Theorem (integration by parts):

- In 1D: If $f$ is exact then

$$F = D_x^{-1}f = \int f \, dx = \mathcal{H}_{u(x)}f$$

- In 2D: If $f$ is a divergence then

$$F = \text{Div}^{-1}f = (\mathcal{H}_{u(x,y)}^{(x)}f, \mathcal{H}_{u(x,y)}^{(y)}f)$$

The homotopy operator inverts total derivatives and divergences!
• Homotopy Operator in 1D (variable $x$):

$$
\mathcal{H}_{u(x)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

with integrand

$$
I_{u(j)} f = \sum_{k=1}^{M_x(j)} \left( \sum_{i=0}^{k-1} u_{i.x}^{(j)} (-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{k.x}^{(j)}}
$$

$(I_{u(j)} f)[\lambda u]$ means that in $I_{u(j)} f$ one replaces

$u \rightarrow \lambda u, \ u_x \rightarrow \lambda u_x, \ etc.$

More general: $u \rightarrow \lambda(u - u_0) + u_0$

$u_x \rightarrow \lambda(u_x - u_{x0}) + u_{x0} \ etc.$
• Homotopy Operator in 2D (variables $x$ and $y$):

$$
\mathcal{H}^{(x)}_{u(x,y)} f = \int_0^1 \sum_{j=1}^N (I^{(x)}_{u(j)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

$$
\mathcal{H}^{(y)}_{u(x,y)} f = \int_0^1 \sum_{j=1}^N (I^{(y)}_{u(j)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

where for dependent variable $u(x, y)$

$$
I^{(x)}_{u} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{(i+j)}{i} \frac{(k+\ell-i-j-1)}{k-i-1} \right) \frac{\partial f}{\partial u_{kx \ell y}}
$$

$$
(-D_x)^{k-i-1} (-D_y)^{\ell-j} \left( \frac{\partial f}{\partial u_{kx \ell y}} \right)
$$
Application 1: The KdV Equation

\[ u_t + uu_x + u_{xxx} = 0 \]

• **Step 1:** Compute the dilation symmetry

Set \((x, t, u) \rightarrow \left( \frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u \right) = (\tilde{x}, \tilde{t}, \tilde{u})\)

Apply change of variables (chain rule)

\[ \lambda^{-(a+b)} \tilde{u}_\tilde{t} + \lambda^{-(2b+1)} \tilde{u} \tilde{u}_\tilde{x} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0 \]

Solve \(a + b = 2b + 1 = b + 3\).

**Solution:** \(a = 3\) and \(b = 2\)

\((x, t, u) \rightarrow \left( \frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u \right)\)
Compute the density of selected \textbf{rank}, say, 6.

\textbf{Step 2: Determine the form of the density}

List powers of $u$, up to rank 6: $[u, u^2, u^3]$

Differentiate with respect to $x$ to increase the rank

- $u$ has weight 2 $\rightarrow$ apply $D^4_x$
- $u^2$ has weight 4 $\rightarrow$ apply $D^2_x$
- $u^3$ has weight 6 $\rightarrow$ no derivatives needed
Apply the $D_x$ derivatives

Remove total and highest derivative terms:

\[
D_x^4 u \rightarrow \{u_{4x}\} \rightarrow \text{empty list}
\]

\[
D_x^2 u^2 \rightarrow \{u_x^2, uu_{xx}\} \rightarrow \{u_x^2\}
\]

\[
\text{since } uu_{xx} = (uu_x)_x - u_x^2
\]

\[
D_x^0 u^3 \rightarrow \{u^3\} \rightarrow \{u^3\}
\]

Linearly combine the “building blocks”

Candidate density: \[
\rho = c_1 u^3 + c_2 u_x^2
\]
• **Step 3: Compute the coefficients** $c_i$ Compute

$$D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t]$$

$$= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t$$

$$= (3c_1 u^2 I + 2c_2 u_x D_x)u_t$$

Substitute $u_t$ by $-(uu_x + u_{xxx})$

$$E = -D_t \rho = (3c_1 u^2 I + 2c_2 u_x D_x)(uu_x + u_{xxx})$$

$$= 3c_1 u^3 u_x + 2c_2 u^3_x + 2c_2 uu_x u_{xx}$$

$$+ 3c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x}$$
Apply the Euler operator (variational derivative)

\[ \mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}} \]

Here, \( E \) has order \( m = 4 \), thus

\[ \mathcal{L}_u(x) E = \frac{\partial E}{\partial u} - D_x \frac{\partial E}{\partial u_x} + D_x^2 \frac{\partial E}{\partial u_{xx}} - D_x^3 \frac{\partial E}{\partial u_{3x}} + D_x^4 \frac{\partial E}{\partial u_{4x}} \]

\[ = -6(3c_1 + c_2)u_x u_{xx} \]

This term must vanish!

So, \( c_1 = -\frac{1}{3} c_2 \). Set \( c_2 = -3 \), then \( c_1 = 1 \)

Hence, the final form density is \( \rho = u^3 - 3u_x^2 \)
• **Step 4: Compute the flux** $J$

**Method 1: Integrate by parts (simple cases)**

Now,

$$E = 3u^3 u_x + 3u^2 u_{xxx} - 6u^3_x - 6uu_x u_{xx} - 6u_x u_{xxxx}$$

Integration of $D_x J = E$ yields the flux

$$J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx}$$
Method 2: Use the homotopy operator

\[ J = D_x^{-1} E = \int E \, dx = \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \]

with integrand

\[ I_u E = \sum_{k=1}^M \left( \sum_{i=0}^{k-1} u_{ix} (-D_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}} \]
Here \( M = 4 \), thus

\[
I_u E = (uI)(\frac{\partial E}{\partial u_x}) + (u_x I - uD_x)(\frac{\partial E}{\partial u_{xx}})
\]

\[
+ (u_{xx}I - u_x D_x + uD_x^2)(\frac{\partial E}{\partial u_{xxx}})
\]

\[
+ (u_{xxx}I - u_{xx} D_x + u_x D_x^2 - uD_x^3)(\frac{\partial E}{\partial u_{xxxx}})
\]

\[
= (uI)(3u^3 + 18u_x^2 - 6uu_{xx} - 6u_{xxxx})
\]

\[
+ (u_xI - uD_x)(-6uu_x)
\]

\[
+ (u_{xx}I - u_x D_x + uD_x^2)(3u^2)
\]

\[
+ (u_{xxx}I - u_{xx} D_x + u_x D_x^2 - uD_x^3)(-6u_x)
\]

\[
= 3u^4 - 18uu_x^2 + 9u^2u_{xx} + 6u_{xx}^2 - 12u_x u_{xxx}
\]

Note: correct terms but incorrect coefficients!
Finally,

\[ J = \mathcal{H}_{u(x)} E = \int_{0}^{1} (I_u E) \lambda u \frac{d\lambda}{\lambda} \]

\[ = \int_{0}^{1} \left( 3\lambda^3 u^4 - 18\lambda^2 uu_x^2 + 9\lambda^2 u^2 u_{xx} + 6\lambda u_{xx}^2 \right. \]

\[ \left. - 12\lambda uu_x u_{xxx} \right) d\lambda \]

\[ = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx} \]

**Final form of the flux:**

\[ J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx} \]
Application 2: Zakharov-Kuznetsov Equation

\[ u_t + \alpha uu_x + \beta (u_{xx} + u_{yy}) x = 0 \]

- **Step 1: Compute the dilation invariance**

ZK equation is invariant under scaling symmetry

\[(t, x, y, u) \rightarrow \left( \frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u \right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})\]

\(\lambda\) is an arbitrary parameter.

- Hence, the weights of the variables are

\[ W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1. \]
• A conservation law is invariant under the scaling symmetry of the PDE.

\[ W(u) = 2, \ W(D_t) = 3, \ W(D_x) = 1, \ W(D_y) = 1. \]

For example,

\[
\begin{align*}
D_t \left( u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right) \\
+ \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \\
+ D_y \left( 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0
\end{align*}
\]

\[ \text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8. \]

\[ \text{Rank } (\text{conservation law}) = 9. \]
Compute the density of selected rank, say, 6.

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

\[ \{ u^3, u_x^2, u_{xx}, u_y^2, uu_{yy}, u_xu_y, uu_{xy}, u_4x, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y} \} \]

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

\[ \rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y \]
Step 3: Compute the undetermined coefficients

Compute

\[ D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \]

\[ = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ell y}} D_x^k D_y^\ell u_t \]

\[ = \left( 3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t \]

Substitute \( u_t = -\left( \alpha uu_x + \beta (u_{xx} + u_{yy})_x \right) \).
\[ E = -D_t \rho = 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy}) x) \\
+ 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy}) x) x + 2c_3 u_y (\alpha u u_x \\
+ \beta (u_{xx} + u_{yy}) x) y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy}) x) x \\
+ u_x (\alpha u u_x + \beta (u_{xx} + u_{yy}) x) y) \]

Apply the Euler operator (variational derivative)

\[
\mathcal{L}_{u(x,y)} E = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}} \\
= -2 \left( (3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} \\
+ 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right) \equiv 0
\]
Solve a parameterized linear system for the $c_i$:

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2)$$
Step 4: Compute the flux

Use the homotopy operator to invert $\text{Div}$:

$$J = \text{Div}^{-1} E = \left( \mathcal{H}^{(x)}_{u(x,y)} E, \mathcal{H}^{(y)}_{u(x,y)} E \right)$$

where

$$\mathcal{H}^{(x)}_{u(x,y)} E = \int_0^1 (I^{(x)}_u E)[\lambda u] \frac{d\lambda}{\lambda}$$

with

$$\mathcal{I}^{(x)}_u E = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ixjy} \frac{(i+j) \binom{k+i+j-1}{i} \binom{k+i+j-1}{k-i-1}}{\binom{k+i+j-1}{k}} \right) \left( -D_x \right)^{k-i-1} \left( -D_y \right)^{\ell-j} \frac{\partial E}{\partial u_{kx\ell y}}$$

Similar formulas for $\mathcal{H}^{(y)}_{u(x,y)} E$ and $\mathcal{I}^{(y)}_u E$. 
Let \( A = \alpha uu_x + \beta (u_{xxx} + u_{xyy}) \) so that
\[
E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y
\]
Then,
\[
J = \left( \mathcal{H}^{(x)}_{u(x,y)} E, \mathcal{H}^{(y)}_{u(x,y)} E \right)
\]
\[
= \left( \frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2) 
+ \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x \left( \frac{7}{2} u_{xyy} + 6u_{xxx} \right) 
- \frac{\beta^2}{\alpha} u_y \left( 4u_{xxy} + \frac{3}{2} u_{yyy} \right) + \frac{\beta^2}{\alpha} \left( 3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2 \right) 
+ \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta uu_x u_y 
- \frac{3\beta^2}{4\alpha} u (u_{x3y} + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x \left( 13u_{xxy} + 3u_{yyy} \right) 
- \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy}) \right)
\]
However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (\mathcal{D}_y \theta, -\mathcal{D}_x \theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left( 3u(u_{xxy} + u_{yyy}) + 10u_xu_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$$

$$= \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} \left( u_{xx}^2 - u_{yy}^2 \right) \right. \right.$$  

$$\left. - \frac{6\beta^2}{\alpha} \left( u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}) \right), \right.$$  

$$\left. 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right)$$
Software Demonstration

Software packages in *Mathematica*

Codes are available via the Internet:
URL: http://inside.mines.edu/~whereman/
Additional Examples

• Manakov-Santini system

\[ u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y = 0 \]
\[ v_{tx} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0 \]

• Conservation laws for Manakov-Santini system:

\[ D_t \left( f u_x v_x \right) + D_x \left( f (uu_x v_x - u_x v_x v_y - u_y v_y) \right) \]
\[ - f' y (u_t + uu_x - u_x v_y) \right) + D_y \left( f (u_x v_y + u_y v_x + u_x v_x^2) \right) \]
\[ + f' (u - yu_y - yu_x v_x) \right) = 0 \]

where \( f = f(t) \) is arbitrary.
Conservation laws – continued:

\[ D_t \left( f(2u + v_x^2 - yu_xv_x) \right) + D_x \left( f \left( u^2 + uv_x^2 + uyv \right) \\
- \left( v_y^2 - v_x^2v_y - y(uu_xv_x - uxv_xv_y - uyv_y) \right) \right) \\
- f' y(v_t + uv_x - v_xv_y) + (f' - 2fx)y^2(ut + uu_x - uxv_y) \right) \\
+ D_y \left( f(v_x^3 + 2v_xv_y - uxv - y(u_xv_x^2 + uxv_y + uyv_x) \right) \\
+ f'(v - y(2u + v_y + v_x^2)) + (f'y^2 - 2fx)(uxv_x + uy) \right) = 0 \]

where \( f = f(t) \) is arbitrary.

There are three additional conservation laws.
• (2+1)-dimensional Camassa-Holm equation

\[(\alpha u_t + \kappa u_x - u_{txx} + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0\]

Interchange \(t\) with \(y\)

\[(\alpha u_y + \kappa u_x - u_{xyy} + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{tt} = 0\]

Set \(v = u_t\) to get

\[u_t = v\]

\[v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta uu_{xx} + 2u_{xx}^2\]

\[+ 3u_x u_{xxx} + uu_{4x}\]
Conservation laws for the Camassa-Holm equation

\[ D_t (f u) + D_x \left( \frac{1}{\alpha} f \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \right) \\
+ \left( \frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) \left( \alpha u_t + \kappa u_x + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx} - u_{ttx} \right) + D_y \left( \left( \frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) u_y - f' y u \right) = 0 \]

\[ D_t (fyu) + D_x \left( \frac{1}{\alpha} f y \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \right) \\
+ y \left( \frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) \left( \alpha u_t + \kappa u_x + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx} - u_{ttx} \right) + D_y \left( y \left( \frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) u_y + \left( \frac{1}{\alpha} f x - \frac{1}{2} f' y^2 \right) u \right) = 0 \]

where \( f = f(t) \) is an arbitrary function.
Khoklov-Zabolotskaya equation describes e.g. sound waves in nonlinear media

\[(u_t - uu_x)_x - u_{yy} - u_{zz} = 0\]

Conservation law:

\[D_t(fu) - D_x\left(\frac{1}{2} fu^2 + (fx + g)(u_t - uu_x)\right)\]
\[+ D_y \left((fx + g)u_y - (fy x + gy)u\right)\]
\[+ D_z \left((fx + g)u_z - (fzx + gz)u\right) = 0\]

under the constraints \(\Delta f = 0\) and \(\Delta g = f_t\)

where \(f = f(t, y, z)\) and \(g = g(t, y, z)\).
• Shallow water wave model (atmosphere)

\[ u_t + (u \cdot \nabla) u + 2 \Omega \times u + \nabla (\theta h) - \frac{1}{2} h \nabla \theta = 0 \]
\[ \theta_t + u \cdot (\nabla \theta) = 0 \]
\[ h_t + \nabla \cdot (uh) = 0 \]

where \( u(x, y, t), \theta(x, y, t) \) and \( h(x, y, t) \).

• In components:

\[ u_t + uu_x + vu_y - 2 \Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0 \]
\[ v_t + uv_x + vv_y + 2 \Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0 \]
\[ \theta_t + u \theta_x + v \theta_y = 0 \]
\[ h_t + hu_x + uh_x + hv_y + vh_y = 0 \]
• First few conservation laws of SWW model:

\[ \rho^{(1)} = h \]
\[ \rho^{(2)} = h \theta \]
\[ \rho^{(3)} = h \theta^2 \]
\[ \rho^{(4)} = h (u^2 + v^2 + h\theta) \]
\[ \rho^{(5)} = \theta (2\Omega + v_x - u_y) \]

\[ \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(2)} = h \theta \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(3)} = h \theta^2 \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(4)} = h \begin{pmatrix} u (u^2 + v^2 + 2h\theta) \\ v (v^2 + u^2 + 2h\theta) \end{pmatrix} \]
\[ \mathbf{J}^{(5)} = \frac{1}{2} \theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix} \]
More general conservation laws for SWW model:

\[ D_t \left( f(\theta)h \right) + D_x \left( f(\theta)hu \right) + D_y \left( f(\theta)hv \right) = 0 \]

\[ D_t \left( g(\theta)(2\Omega + v_x - u_x) \right) \]
\[ + D_x \left( \frac{1}{2} g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y) \right) \]
\[ + D_y \left( \frac{1}{2} g(\theta)(4\Omega v - 2u_yv + 2vv_x + h\theta_x) \right) = 0 \]

for any functions \( f(\theta) \) and \( g(\theta) \).
• Kadomtsev-Petviashvili (KP) equation

\[
(u_t + \alpha uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0
\]

parameter \( \alpha \in \mathbb{R} \) and \( \sigma^2 = \pm 1 \).

Equation be written as a conservation law

\[
D_t(u_x) + D_x(\alpha uu_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0.
\]

Exchange \( y \) and \( t \) and set \( u_t = \nu \)

\[
\begin{align*}
u_t &= \nu \\
v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha uu_{xx} + u_{xxxx})
\end{align*}
\]
• Examples of conservation laws for KP equation (explicitly dependent on $t, x, \text{and } y$)

$$D_t(xu_x) + D_x\left(3u^2 - uu_{xx} - 6xuu_x + xu_{xxx}\right) + D_y\left(\alpha xu_y\right) = 0$$

$$D_t(yu_x) + D_x\left(y(\alpha uu_x + uu_{xx})\right) + D_y\left(\sigma^2(yu_y - uu_{xy})\right) = 0$$

$$D_t\left(\sqrt{t}u\right) + D_x\left(\frac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}uu_x + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}uu_{xx} + \frac{\alpha \sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}uu_{xx}\right)$$

$$+ D_y\left(x\sqrt{t}u_y + \frac{y^2 u_y^2}{4\sqrt{t}} - \frac{yu_{yy}}{2\sqrt{t}}\right) = 0$$
More general conservation laws for KP equation:

\[
D_t(fu) + D_x \left( f \left( \frac{\alpha}{2} u^2 + u_{xx} \right) \right) \\
+ \left( \frac{\sigma^2}{2} f' y^2 - fx \right) \left( u_t + \alpha uu_x + u_{3x} \right) \\
+ D_y \left( \left( \frac{1}{2} f' y^2 - \sigma^2 fx \right) u_y - f' y u \right) = 0
\]

\[
D_t(ftyu) + D_x \left( fyu \left( \frac{\alpha}{2} u^2 + u_{xx} \right) \right) \\
+ y \left( \frac{\sigma^2}{6} f' y^2 - fx \right) \left( u_t + \alpha uu_x + u_{3x} \right) \\
+ D_y \left( y \left( \frac{1}{6} f' y^2 - \sigma^2 fx \right) u_y + (\sigma^2 fx - \frac{1}{2} f' y^2) u \right) = 0
\]

where \( f(t) \) is arbitrary function.
• Potential KP equation

Replace $u$ by $u_x$ and integrate with respect to $x$.

$$u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0$$

• Examples of conservation laws
(not explicitly dependent on $x, y, t$):

$$D_t(u_x) + D_x \left( \frac{\alpha}{2} u_x^2 + u_{xxx} \right) + D_y \left( \sigma^2 u_y \right) = 0$$

$$D_t \left( u_x^2 \right) + D_x \left( \frac{2\alpha}{3} u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy} \right) + D_y \left( 2\sigma^2 u_x u_y \right) = 0$$
Conservation laws for pKP equation – continued:

\[
D_t \left( u_x u_y \right) + D_x \left( \alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy} \right) \\
+ D_y \left( \sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2 \right) = 0
\]

\[
D_t \left( 2\alpha uu_x u_{xx} + 3uu_4x - 3\sigma^2 u_y^2 \right) + D_x \left( 2\alpha uu_t u_x^2 + 3u_t^2 \\
- 2\alpha uu_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{t xx} - 3uu_t xxx \right) \\
+ D_y \left( 6\sigma^2 u_t u_y \right) = 0
\]

Various generalizations exist.
• Generalized Zakharov-Kuznetsov equation

\[ u_t + \alpha u^n u_x + \beta (u_{xx} + u_{yy})_x = 0 \]

where \( n \) is rational, \( n \neq 0 \).

Conservation laws:

\[ D_t(u) + D_x\left(\frac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\right) + D_y\left(\beta u_{xy}\right) = 0 \]

\[ D_t(u^2) + D_x\left(\frac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) + D_y\left(-2\beta u_x u_y\right) = 0 \]
• Third conservation law for gZK equation:

\[ D_t \left( u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha} (u_x^2 + u_y^2) \right) \]
\[ + D_x \left( \frac{(n+2)\alpha}{2(n+1)} u^{2(n+1)} + (n + 2)\beta u^{n+1} u_{xx} \right. \]
\[ - (n + 1)(n + 2)\beta u^n (u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha} (u_{xx}^2 - u_{yy}^2) \]
\[ - \frac{(n+1)(n+2)\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yy})) \right) \]
\[ + D_y \left( (n + 2)\beta u^{n+1} u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0. \]
Conclusions and Future Work

• The power of Euler and homotopy operators:
  ▶ Testing exactness
  ▶ Integration by parts: $D_x^{-1}$ and $\text{Div}^{-1}$

• Integration of non-exact expressions

Example: $f = u_x v + u v_x + u^2 u_{xx}$

\[
\int f \, dx = uv + \int u^2 u_{xx} \, dx
\]

• Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)
• Broader class of PDEs (beyond evolution type)

Example: short pulse equation (nonlinear optics)

\[ u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx} \]

with non-polynomial conservation law

\[ D_t \left( \sqrt{1 + 6u_x^2} \right) - D_x \left( 3u^2 \sqrt{1 + 6u_x^2} \right) = 0 \]

• Continue the implementation in Mathematica

• Software:  http://inside.mines.edu/~whereman
Thank You
Publications

