SOLITARY WAVE SOLUTIONS OF
COUPLED NONLINEAR EVOLUTION EQUATIONS
USING MACSYMA

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401
1. INTRODUCTION

• Construct solitary wave solutions by a direct method

• Applicable to:
  Single nonlinear evolution and wave equations
  Systems of nonlinear PDEs
  Nonlinear ODEs

• Goal: Exact solutions
  Single solitary wave or soliton solutions
  N-solitons
  Implicit solutions

• Method:
  Hirota’s direct method
  Rosales’ perturbation method
  Trace method
  Hereman et al real exponential approach

• Requirements:
  Based on physical principles
  Simple and straightforward
  Programmable in MACSYMA, REDUCE, MATHEMATICA, SCRATCHPAD II
2. EXAMPLES

• Korteweg-de Vries equation and generalizations

\[ u_t + au^nu_x + u_{xxx} = 0, \quad n \in \mathbb{N} \]

\[ u(x, t) = \left\{ \frac{c(n + 1)(n + 2)}{2a} \right\} \text{sech}^2 \left[ \frac{n}{2} \sqrt{c(x - ct) + \delta} \right] \]

• Burgers equation

\[ u_t + auu_x - u_{xx} = 0 \]

\[ u(x, t) = \frac{c}{a} \left\{ 1 - \tanh \left[ \frac{c}{2}(x - ct) + \delta \right] \right\} \]

• Fisher equation and generalizations

\[ u_t - u_{xx} - u(1 - u^n) = 0, \quad n \in \mathbb{N} \]

\[ u(x, t) = \left\{ \frac{1}{2} \left[ 1 - \tanh \left[ \frac{n}{2\sqrt{2n + 4}} (x - \frac{(n + 4)}{\sqrt{2n + 4}} t) + \delta \right] \right] \right\} \]
• Fitzhugh-Nagumo equation

\[ u_t - u_{xx} + u(1 - u)(a - u) = 0 \]

\[ u(x, t) = \frac{a}{2} \left\{ 1 + \tanh \left[ \frac{a}{2\sqrt{2}} (x - \frac{(2 - a)}{\sqrt{2}} t) + \delta \right] \right\} \]

• Kuramoto-Sivashinski equation

\[ u_t + uu_x + au_{xx} + bu_{xxxx} = 0 \]

\[ u(x, t) = c + \frac{165ak}{19} \left\{ \tanh^3 \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

\[ - \frac{135ak}{19} \left\{ \tanh \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

with \( k = \sqrt{\frac{11a}{19b}} \)

\[ u(x, t) = c - \frac{15ak}{19} \left\{ \tanh^3 \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

\[ + \frac{45ak}{19} \left\{ \tanh \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

with \( k = \sqrt{\frac{-a}{19b}} \)
• Harry Dym equation

\[ u_t + (1 - u)^3 u_{xxx} = 0 \]

\[ u(x, t) = \text{sech}^2 \left[ \frac{1}{2} \sqrt{c} \left[ x - ct + \delta(x, t) \right] \right] \]

\[ \delta(x, t) = \frac{2}{\sqrt{c}} \tanh \left[ \frac{\sqrt{c}}{2} \left[ x - ct + \delta(x, t) \right] \right] \]

• sine-Gordon equation

\[ u_{tt} - u_{xx} - \sin u = 0 \]

\[ u(x, t) = 4 \arctan \left\{ \exp \left[ \frac{1}{\sqrt{-c}} (x - ct) + \delta \right] \right\} \]
Coupled Korteweg-de Vries equations

\[ u_t - a(6uu_x + u_{xxx}) - 2b \nu v_x = 0, \]
\[ v_t + 3uv_x + v_{xxx} = 0 \]

\[ u(x, t) = 2c \sech^2 \left[ \sqrt{c}(x - ct) + \delta \right], \]
\[ v(x, t) = \pm c \sqrt{-\frac{2(4a + 1)}{b}} \sech \left[ \sqrt{c}(x - ct) + \delta \right], \]

\[ u(x, t) = c \sech^2 \left[ \frac{1}{2} \sqrt{c}(x - ct) + \delta \right] \]
\[ v(x, t) = \frac{3}{\sqrt{6|b|}} u(x, t) = \frac{3c}{\sqrt{6|b|}} \sech^2 \left[ \frac{1}{2} \sqrt{c}(x - ct) + \delta \right] \]
3. THE ALGORITHM

• Step 1: System of two coupled nonlinear PDEs

\[ F(u, v, u_t, u_x, v_t, v_x, u_{tx}, \ldots, u_{mx}, v_{nx}) = 0, \]
\[ G(u, v, u_t, u_x, v_t, v_x, u_{tx}, \ldots, u_{px}, v_{qx}) = 0, \quad (m, n, p, q \in \mathbb{N}) \]

where \( F \) and \( G \) are polynomials in their arguments and

\[ u_t = \frac{\partial u}{\partial t}, \quad u_{nx} = \frac{\partial^n u}{\partial x^n} \]

• Step 2:

− Introduce the variable \( \xi = x - ct \), \((c \text{ is the constant velocity})\)
− Integrate the system of ODEs for \( \phi(\xi) \equiv u(x, t) \) and \( \psi(\xi) \equiv v(x, t) \). with respect to \( \xi \) to reduce the order
− Ignore integration constants and assume that the solutions \( \phi \) and \( \psi \) and their derivatives vanish at \( \xi = \pm \infty \)

• Step 3:

− Expand \( \phi \) and \( \psi \) in a power series

\[ \phi = \sum_{n=1}^{\infty} a_n g^n, \quad \psi = \sum_{n=1}^{\infty} b_n g^n \]

− \( g(\xi) = \exp(-K(c)\xi) \) solves the linear part of one of the equations
– Consider the dispersion laws $K(c)$ of the linearized equations
– Substitute the expansions into the full nonlinear system
– Use Cauchy’s rule for multiple series to rearrange the multiple sums
– Equate the coefficient of $g^n$ to get the coupled recursion relations for $a_n$ and $b_n$

**Step 4:**
– Assume that $a_n$ and $b_n$ are polynomials in $n$
– Determine their degrees $\delta_1$ and $\delta_2$
– Substitute

$$a_n = \sum_{j=0}^{\delta_1} A_j \, n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j \, n^j$$

into the recursion relations
– Compute the sums by using the formulae for
\[ S_k = \sum_{i=1}^{n} i^k, \quad (k = 0, 1, 2, \ldots) \]

– Examples:
\[ S_0 = n, \quad S_1 = \frac{(n + 1)n}{2}, \]
\[ S_2 = \frac{n(n + 1)(2n + 1)}{6}, \quad \text{etc.} \]

– Equate to zero the different coefficients of the polynomial in \( n \)
– Solve the algebraic (nonlinear) equations for the constant coefficients \( A_j \) and \( B_j \)

• Step 5:
– Find the closed forms for
\[ \phi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_1} A_j \ n^j \ g^n \equiv \sum_{j=0}^{\delta_1} A_j \ F_j(g), \]
\[ \psi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_2} B_j \ n^j \ g^n \equiv \sum_{j=0}^{\delta_2} B_j \ F_j(g) \]

with
\[ F_j(g) \equiv \sum_{n=1}^{\infty} n^j g^n \]
\[ F_{j+1}(g) = g F_j'(g), \quad j = 0, 1, 2, \ldots \]
- Examples

\[ F_0(g) = \frac{g}{1 - g}, \quad F_1(g) = \frac{g}{(1 - g)^2}, \]

\[ F_2(g) = \frac{g(1 + g)}{(1 - g)^3}, \quad \text{etc.} \]

- Return to the original variables \( x \) and \( t \) to obtain the travelling wave solution(s)
4. EXAMPLE: The Coupled KdV Equations

- **Step 1:** System of PDEs:

\[
\begin{align*}
    u_t - a(6uu_x + u_{3x}) - 2b v v_x &= 0, \\
    v_t + 3u u_x + v_{3x} &= 0, \quad a, b \in \mathbb{R}
\end{align*}
\]

- **Step 2:**
  - Introduce the variable \( \xi = x - ct \), \( c \) is the constant velocity
  - Integrate the system of ODEs for \( \phi(\xi) \equiv u(x, t) \) and \( \psi(\xi) \equiv v(x, t) \)

\[
\begin{align*}
    c\phi + 3a\phi^2 + \alpha\phi_2\xi + b\psi^2 &= 0, \\
    -c\psi_\xi + 3\phi\psi_\xi + \psi_3\xi &= 0
\end{align*}
\]

- **Step 3:**
  - Expand \( \phi \) and \( \psi \) in a power series

\[
\phi = \frac{c}{3} \sum_{n=1}^{\infty} a_n \ g^n, \quad \psi = \frac{c}{\sqrt{3|b|}} \sum_{n=1}^{\infty} b_n \ g^n
\]

\( g(\xi) = \exp(-K(c)\xi) \) solves the linear part of one of the equations in the system

- Consider the dispersion law \( K(c) = \sqrt{c} \) of the second equation
Substitute the expansions into the full nonlinear system
Use Cauchy’s rule for multiple series to rearrange the sums
Equate the coefficient of $g^n$ to get the coupled recursion relations
\[
(1 + a n^2) a_n + \sum_{l=1}^{n-1} (a a_l a_{n-l} + e b_l b_{n-l}) = 0,
\]
\[
n (n^2 - 1) b_n + \sum_{l=1}^{n-1} l b_l a_{n-l} = 0, \quad n \geq 2
\]
\[
(1 + a)a_1 = 0,
\]
with $b_1$ arbitrary, and $e = \pm 1$ if $|b| = \pm b$

• CASE 1: $a \neq -1$ then $a_1 = 0$ thus, $a_{2n-1} = 0,$ $b_{2n} = 0,$ $(n = 1, 2, ...)$

Shift the labels in the recurrence relations
\[
(1 + 4a n^2) a_{2n} + \sum_{l=1}^{n-1} a a_{2l} a_{2(n-l)} + e \sum_{l=1}^{n} b_{2l-1} b_{2n-2l+1} = 0,
\]
\[
4n (n - 1)(2n - 1) b_{2n-1} + \sum_{l=1}^{n-1} (2l - 1) b_{2l-1} a_{2(n-l)} = 0, \quad n \geq 2
\]
Step 4:

- Assume that $a_{2n}$ and $b_{2n-1}$ polynomials in $n$ and determine their degrees $\delta_1 = 1$ and $\delta_2 = 0$
- Substitute $a_{2n} = A_1 n + A_0$; $b_{2n-1} = B_0$, $(n = 1, 2, ...)$ into the recursion relations
- Compute the sums by using the formulae for $S_k = \sum_{i=1}^{n} i^k$
- Equate to zero the different coefficients of the polynomial in $n$ of degree 3
- Solve the algebraic (nonlinear) equations for the constant coefficients $A_1$, $A_0$ and $B_0$
- Solution (with MACSYMA):

$$a_{2n} = 24 n (-1)^{n+1} a_0^n,$$
$$b_{2n-1} = (-1)^{n-1} b_1 a_0^{n-1}, \quad n = 1, 2, ...$$

with $a_0 = -eb_1^2/24(4a + 1) > 0$
- Remark: $b$ and $4a + 1$ must have opposite signs
Step 5:

- Find the closed forms for $\phi$ and $\psi$
- Use $F_0(g) = \frac{g}{1-g}$ and $F_1(g) = \frac{g}{(1-g)^2}$ to get

$$
\phi = 8c \sum_{n=1}^{\infty} (-1)^{n+1} n (a_0 g^2)^n = \frac{8 c a_0 g^2}{(1 + a_0 g^2)^2} 
$$

$$
\psi = \frac{c}{\sqrt{3|b|}} \sum_{n=0}^{\infty} (-1)^n b_1 a_0^n g^{2n+1} = \frac{c b_1 g}{\sqrt{3|b|}(1 + a_0 g^2)} 
$$

- Return to the variables $x$ and $t$

$$
\begin{align*}
    u(x,t) &= 2c \operatorname{sech}^2\left[\sqrt{c}(x-ct) + \delta\right], \\
    v(x,t) &= \pm c \sqrt{- \frac{2(4a+1)}{b}} \operatorname{sech}\left[\sqrt{c}(x-ct) + \delta\right],
\end{align*}
$$

with $\delta = \frac{1}{2} \ln |24(4a+1)/b_1^2|$

CASE 2: $a = -1$ then $a_1$ and $b_1$ are arbitrary, take $e = 1$

- Solution of the recursion relations (with MACSYMA):

$$
\begin{align*}
    a_n &= 12 n (-1)^{n+1} a_0^n, \\
    b_n^2 &= \frac{a_n^2}{2} = 72 n^2 a_0^{2n}, \quad n = 1, 2, ... 
\end{align*}
$$

with $a_0 = a_1/12$. 
Return to the original variables

\[
  u(x, t) = c \sech^2 \left[ \frac{1}{2} \sqrt{c} (x - ct) + \delta \right],
\]

\[
  v(x, t) = \frac{3}{\sqrt{6|b|}} u(x, t) = \frac{3c}{\sqrt{6|b|}} \sech^2 \left[ \frac{1}{2} \sqrt{c} (x - ct) + \delta \right],
\]

with \( \delta = \frac{1}{2} \ln |12/a_1| \).

Observe that for \( v(x, t) = \frac{3}{\sqrt{6b}} u(x, t) \) both equations reduce to the KdV equation

\[
  u_t + 3 uu_x + u_{3x} = 0
\]
• Sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x, t) = \arctan \left\{ \exp \left[ \frac{(1 - c)}{2\sqrt{c}} (x - \frac{(c + 1)}{c - 1} t) + \delta \right] \right\}$$

• Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b \nu \nu_x = 0, \quad v_t + 3u\nu_x + v_{xxx} = 0$$

$$u(x, t) = 2c \sech^2 \left[ \sqrt{c} (x - ct) + \delta \right],$$

$$v(x, t) = \pm c \sqrt{\frac{-2(4a + 1)}{b}} \sech \left[ \sqrt{c} (x - ct) + \delta \right],$$

$$u(x, t) = c \sech^2 \left[ \frac{1}{2} \sqrt{c} (x - ct) + \delta \right]$$

$$v(x, t) = \frac{3}{\sqrt{6|b|}} u(x, t) = \frac{3c}{\sqrt{6|b|}} \sech^2 \left[ \frac{1}{2} \sqrt{c} (x - ct) + \delta \right]$$