14th IMACS WORLD CONGRESS

SYMBOLIC METHODS TO FIND
EXACT SOLUTIONS OF NONLINEAR
PARTIAL DIFFERENTIAL EQUATIONS

Willy Hereman & Ameina Nuseir

Dept. of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887, USA

Georgia Institute of Technology
Atlanta, Georgia
July 11-15, 1994
I. INTRODUCTION

• Goals:
  - easy construction of exact solutions
  - solitary wave solutions and solitons
  - investigate integrability

• Technique:
  - simplified version of Hirota’s method
  - make method applicable to equations that have no bilinear form

• Implementation:
  - MACSYMA & MATHEMATICA
  - other symbolic manipulation programs

• Applications:
  - reaction-diffusion equations
  - 5th order evolution equations
II. SIMPLIFIED VERSION OF HIROTA’S METHOD

Hirota’s method requires:

- a clever change of dependent variable
- the introduction of a bilinear differential operator
- a perturbation-like expansion

**Example:**

The Korteweg-de Vries equation

\[ u_t + 6uu_x + u_{3x} = 0 \]

Substitute the Laurent expansion

\[ u(x, t) = f(x, t)^\alpha \sum_{k=0}^{\infty} u_k(x, t) f(x, t)^k \]

with \( u_0(t, x) \neq 0, \alpha \) negative integer
\( u_k(t, x) \) analytic in a neighborhood of the singular non-characteristic manifold \( f(t, x) = 0 \)

Determine \( \alpha = -2 \) (leading order behavior)
Truncate expansion at constant level term

\[ u(x, t) = f(x, t)^{-2} [ u_0 + u_1 f(x, t) + u_2(x, t) f(x, t)^2 ] \]
or with explicit forms of \( u_0, u_1 \) and with \( u_2 = 0 \)

\[ u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} = 2 \frac{(f f_{xx} - f_x^2)}{f^2} \]

Integrate with respect to \( x \)

\[ f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0 \]

Could be written in \textit{bilinear form}

\[ (D_x D_t + D_x^4) (f \cdot f) = 0 \]

via Hirota’s bilinear operator

\[ D_x^m D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x, t) g(x', t') \big|_{x'=x, t'=t} \]

Make Hirota’s technique applicable to equations that \textit{cannot} be written in bilinear form

Leave Hirota’s bilinear operators out
Write in general $\mathcal{N}(f, f) = 0$, with

$$\mathcal{N}(f, g) = (\mathcal{I} f) \left( \frac{\partial^2 g}{\partial x \partial t} + \frac{\partial^4 g}{\partial x^4} \right) - \frac{\partial f}{\partial x} \frac{\partial g}{\partial t} - 4 \frac{\partial f}{\partial x} \frac{\partial^3 g}{\partial x^3} + 3 \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial x^2}$$

$\mathcal{I}$ is the identity operator

Seek formal solution (book keeping parameter $\epsilon$)

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t)$$

Perturbation scheme (equate powers in $\epsilon$ to zero)

$$O(\epsilon^1) : \mathcal{L} f^{(1)} = 0$$
$$O(\epsilon^2) : \mathcal{L} f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)})$$
$$O(\epsilon^3) : \mathcal{L} f^{(3)} = -\mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)})$$

$$O(\epsilon^n) : \mathcal{L} f^{(n)} = - \sum_{j=1}^{n-1} \mathcal{N}(f^{(j)}, f^{(n-j)})$$

where $\mathcal{L}$ denotes the linear differential operator

$$\mathcal{L} \bullet = \frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^4 \bullet}{\partial x^4}$$

N-soliton solution is then generated by
\[ f^{(1)} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp (k_i x - \omega_i t + \delta_i) \]

with constant \( k_i, \omega_i \) and \( \delta_i \)

Determine the dispersion law (from level \( \epsilon^1 \))

\[ P(k_i, \omega_i) = -\omega_i k_i + k_i^4 = 0 \]

Thus

\[ \omega_i = k_i^3 \quad i = 1, 2, ..., N \]

Compute RHS at level \( \epsilon^2 \)

\[ - \sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j) f_i f_j = \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i - k_j)^2 f_i f_j \]

Note: terms in \( f_i^2 \) drop out

Form of \( f^{(2)} \) is determined

\[ f^{(2)} = \sum_{1 \leq i < j \leq N} a_{ij} f_i f_j \]

\[ = \sum_{1 \leq i < j \leq N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)] \]
Compute LHS at level $\epsilon^2$

$$\mathcal{L} f^{(2)} = \sum_{1 \leq i < j \leq N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} f_i f_j$$

$$= \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i + k_j)^2 a_{ij} f_i f_j$$

Equate LHS and RHS

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \quad 1 \leq i < j \leq N$$

Proceeding in a similar fashion with equation at order $\epsilon^3$

Example: for $N = 3$ (three soliton solution)

$$f_2 = a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3)$$

$$= a_{12} \exp [(k_1 + k_2)x - (\omega_1 + \omega_2)t + (\delta_1 + \delta_2)]$$

$$+ a_{13} \exp [(k_1 + k_3)x - (\omega_1 + \omega_3)t + (\delta_1 + \delta_3)]$$

$$+ a_{23} \exp [(k_2 + k_3)x - (\omega_2 + \omega_3)t + (\delta_2 + \delta_3)]$$

and

$$f_3 = b_{123} \exp(\theta_1 + \theta_2 + \theta_3)$$

$$= b_{123} \exp [(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t + (\delta_1 + \delta_2 + \delta_3)]$$
with
\[ b_{123} = a_{12} a_{13} a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2} \]

Note: \( f^{(3)} \) has no terms in \( f_i^2 f_j \) \((i, j = 1, ..., N, i \neq j)\)

Also: for \( N = 3 \), one has \( f^{(n)} = 0 \) for \( n > 3 \)

The expansion truncates (set \( \epsilon = 1 \))
\[ f = 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 + a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) + b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \]

Substitute the explicit form of \( f(x, t) \) back into
\[ u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} = 2 \frac{ff_{xx} - f_x^2}{f^2} \]

N-soliton solution for any \( N > 3 \) is constructed similarly
• Single soliton solution

\[ f = 1 + e^{\theta} \]
\[ \theta = kx - \omega t + \delta \]

\( k, \omega \) and \( \delta \) are constants

\[ P(k, -\omega) = -\omega k + k^4 = 0 \]

Substituting \( f \) into

\[ u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} \]
\[ = 2 \left( \frac{f_{xx}f - f_x^2}{f^2} \right) \]

Denote \( k = 2K \), to get the solitary wave solution

\[ u = 2K^2 \text{sech}^2 K(x - 4K^2 t + \delta) \]
• Two-soliton solution

\[ f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2} \]
\[ \theta_i = k_i x - \omega_i t + \delta_i \]

\[ P(k_i, -\omega_i) = 0 \quad \text{or} \quad \omega_i = k_i^3 \quad (i = 1, 2) \]

\[ a_{12} = -\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)} \]
\[ = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \]

Select

\[ e^{\delta_i} = \frac{c_i^2}{k_i} e^{k_i x - \omega_i t + \Delta_i} \]
\[ \tilde{f} = \frac{1}{4} f e^{-\frac{1}{2}(\tilde{\theta}_1 + \tilde{\theta}_2)} \]
\[ \tilde{\theta}_i = k_i x - \omega_i t + \Delta_i \]
\[ c_i^2 = \left( \frac{k_2 + k_1}{k_2 - k_1} \right) k_i \]
One obtains upon return to $u$

$$u(x, t) = \tilde{u}(x, t) = 2 \frac{\partial^2 \ln \tilde{f}(x, t)}{\partial x^2}$$

$$= \left( \frac{k_2^2 - k_1^2}{2} \right) \left( \frac{k_2^2 \cosech^2 \frac{\theta_2}{2} + k_1^2 \sech^2 \frac{\theta_1}{2}}{\left( k_2 \coth \frac{\theta_2}{2} - k_1 \tanh \frac{\theta_1}{2} \right)^2} \right)$$

- For the general $N$-soliton solution

$$f = \sum_{\mu=0,1} \exp \left[ \sum_{i<j} A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i \theta_i \right]$$

$$a_{ij} = \exp A_{ij} = -\frac{P(k_i - k_j, -\omega_i + \omega_j)}{P(k_i + k_j, -\omega_i - \omega_j)}$$

Additional condition for $P(D_x, D_t)$

$$S[P, n] = \sum_{\sigma=\pm 1} P \left( \sum_{i=1}^n \sigma_i k_i, -\sum_{i=1}^n \sigma_i \omega_i \right)$$

$$\times \prod_{i<j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i \omega_i + \sigma_j \omega_j) \sigma_i \sigma_j = 0$$

for $n = 2, \ldots, N$
III. THE FISHER EQUATION WITH CONVECTION

\[ u_t + ku u_x - u_{xx} - u(1 - u) = 0 \]

\( k \) is the convection constant

Truncated Laurent expansion

\[ u(x, t) = -\frac{2}{k} \frac{\partial \ln f(x, t)}{\partial x} = -\frac{2}{k} \left( \frac{f_x}{f} \right) \]

transforms the PDE into

\[ \mathcal{N}(f, f) = f(f_{xxx} + f_x - f_{xt}) + f_x(f_t - f_{xx} + \frac{2}{k} f_x) = 0 \]

The nonlinear operator is given by

\[ \mathcal{N}(f, g) = (\mathcal{I} f) \left( \frac{\partial^3 g}{\partial x^3} + \frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial x \partial t} \right) + \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + \frac{2}{k} \frac{\partial g}{\partial x} \right) \]

Seek solution of type

\[ f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t) \]
Perturbation scheme:

\[ O(\epsilon^1) : \mathcal{L} f^{(1)} = 0 \]
\[ O(\epsilon^2) : \mathcal{L} f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) \]
\[ O(\epsilon^3) : \mathcal{L} f^{(3)} = -\mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) \]

\vdots

\[ O(\epsilon^n) : \mathcal{L} f^{(n)} = -\sum_{j=1}^{n-1} \mathcal{N}(f^{(j)}, f^{(n-j)}) \]

with

\[ \mathcal{L} \bullet = \frac{\partial^3 \bullet}{\partial x^3} + \frac{\partial \bullet}{\partial x} - \frac{\partial^2 \bullet}{\partial x \partial t} \]

Here

\[ \omega_i = -\left(1 + k_i^2\right) \quad i = 1, 2, \ldots, N \]

Consequently

\[ \mathcal{L} f^{(2)} = -\sum_{i=1}^{N} k_i \left(1 + \frac{2}{k_i} k_i\right) f_i^2 - \sum_{1 \leq i < j \leq N} \frac{4}{k_i k_j (k_i + k_j)} f_i f_j \]

Note presence of terms in \( f_i^2 \)

We need to set \( k_i = -\frac{k}{2} \)

N-soliton solution for \( N \geq 2 \) does no longer exist

Leads to the case \( N = 1 \)
\[ f(x, t) = 1 + \exp \theta = 1 + \exp \left[ -\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta \right] \]

and

\[ u(x, t) = \frac{\exp \theta}{c + \exp \theta} = \frac{\exp \left[ -\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta \right]}{c + \exp \left[ -\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta \right]} \]

Final solution in a more pleasing form

\[ u(x, t) = \frac{1 + \tanh \frac{1}{2} \theta}{(1 + c) + (1 - c) \tanh \frac{1}{2} \theta} \]
Fisher equation without convection \((k = 0)\)

Truncated Laurent expansion reveals the transformation

\[
  u(x, t) = -6 \frac{\partial^2 \ln f(x, t)}{\partial x^2} + \frac{6 \partial \ln f(x, t)}{5 \partial t} \\
  = -6 \left( \frac{f f_xx - f_x^2}{f^2} \right) + \frac{6}{5} \left( \frac{f_t}{f} \right)
\]

The quadratic equation in \(f\) and its derivatives is

\[
  \mathcal{N}(f, f) = 5f (5f_{xxx} + 5f_xx + f_{tt} - 6f_{xxt} - f_t) + 75f_{xx}^2 \\
  - 100f_x f_{xxx} - 25f_x^2 + f_t^2 - 30f_t f_xx + 60f_x f_{xt} = 0
\]

Proceeding as above

\[
  u(x, t) = \frac{[1 - \tanh \frac{1}{2} \theta]^2}{[(1 + c) - (1 - d) \tanh \frac{1}{2} \theta]^2}
\]

with either \(d = c\) or \(d = c + 4\), \(c\) any constant, and

\[
  \theta = \frac{1}{\sqrt{6}}x - \frac{5}{6}t + \delta
\]

Note: that solution does not follow from the previous one for \(k \to 0\)
IV. THE FITZHUGH-NAGUMO EQUATION WITH CONVECTION

\[ u_t + kuu_x - u_{xx} - u(1-u)(a-u) = 0 \]

\( k \) is convection constant

Substitute the Laurent expansion

\[ u(x, t) = f(x, t)^\alpha \sum_{k=0}^{\infty} u_k(x, t)f(x, t)^k \]

Here, \( \alpha = -1 \), \( u_0 \) must satisfy

\[ u_0^2 - ku_0 f_x - 2f_x^2 = 0 \]

Resonances (\( u_r \) arbitrary)

\[ r = -1 \]

\[ r = 4 + k \left( \frac{u_0}{f_x} \right) = 2 + \left( \frac{u_0}{f_x} \right)^2 \]

For integer resonances

\[ u_0(x, t) = \sqrt{mf_x} \quad k = \frac{m - 2}{\sqrt{m}} \]

\( m \) positive integer
Substitute the truncated Laurent expansion

\[ u(x,t) = \sqrt{m} \frac{\partial \ln f(x,t)}{\partial x} + u_1(x,t) = \sqrt{m} \frac{f_x(x,t)}{f(x,t)} + u_1(x,t) \]

Collect power terms in \( f(x,t) \rightarrow \) overdetermined system for \( f(x,t) \) and \( u_1(x,t) \):

\[
\begin{align*}
    f_t - (1 + m) f_{xx} - \sqrt{m} \left[ \frac{2}{m} (1 + m) u_1 - 1 - a \right] f_x &= 0 \\
    f_{xt} - f_{xxx} + \frac{1}{\sqrt{m}} (m - 2) u_1 f_{xx} \\
    &+ \left[ 3 u_1^2 - 2(1 + a) u_1 + a + \frac{1}{\sqrt{m}} (m - 2) (u_1)_x \right] f_x &= 0 \\
    (u_1)_t + \frac{1}{\sqrt{m}} (m - 2) u_1 (u_1)_x - (u_1)_{xx} + u_1 (1 - u_1) (a - u_1) &= 0
\end{align*}
\]

Trivial solutions \( u_1 = 0, 1, \) or \( a \)

For \( u_1 = 0 \) (\( u_1 = 1 \) and \( u_1 = a \) are similar)
Seek solution of type
\[
f(x, t) = \sum_{i=1}^{N} \exp(k_i x - \omega_i t + \delta_i)
\]

Now \(N = 2\), and
\[
k_1 = \frac{1}{\sqrt{m}} \quad \omega_1 = \frac{am - 1}{m}
\]
\[
k_2 = \frac{a}{\sqrt{m}} \quad \omega_2 = \frac{a(m-a)}{m}
\]
Hence
\[
f = c + \exp\left[\frac{1}{\sqrt{m}} x + \frac{a(a-m)}{m} t + \delta_1\right] + \exp\left[\frac{a}{\sqrt{m}} x + \frac{(1-am)}{m} t + \delta_2\right]
\]

Returning to \(u(x, t)\)
\[
u(x, t) = \frac{\exp\left[\frac{1}{\sqrt{m}} x + \frac{(1-am)}{m} t + \delta_1\right] + a\exp\left[\frac{a}{\sqrt{m}} x + \frac{a(a-m)}{m} t + \delta_2\right]}{c + \exp\left[\frac{1}{\sqrt{m}} x + \frac{(1-am)}{m} t + \delta_1\right] + \exp\left[\frac{a}{\sqrt{m}} x + \frac{a(a-m)}{m} t + \delta_2\right]}
\]
This solution describes two coalescent wave fronts

Reduces for \(m = 2\) to solution of FHN equation without convection \((k = 0)\)
V. FIFTH-ORDER EVOLUTION EQUATIONS

Class of equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0 \]

Special cases:

- \( \alpha = 30 \), \( \beta = 20 \), \( \gamma = 10 \) \quad \text{Lax}
- \( \alpha = 5 \), \( \beta = 5 \), \( \gamma = 5 \) \quad \text{Sawada Kotera or Caudry–Dodd–Gibbon}
- \( \alpha = 20 \), \( \beta = 25 \), \( \gamma = 10 \) \quad \text{Kaup Kuperschmidt}
- \( \alpha = 2 \), \( \beta = 6 \), \( \gamma = 3 \) \quad \text{Ito}

Substitute the Laurent expansion

\[ u(x, t) = f(x, t)^{-2} \sum_{k=0}^{\infty} u_k(x, t) f(x, t)^k \]

Assume transformation of the form

\[ u(x, t) = K \frac{\partial^2 \ln f(x, t)}{\partial x^2} = K \frac{f f_{xx} - f_x^2}{f^2} \]
Case that leads to $N$-soliton solution

$$K = \frac{60}{\beta + \gamma} \quad \alpha = \frac{\gamma(\beta + \gamma)}{10}$$

Substitute and integrate with respect to $x$

$$(\beta + \gamma)f^2[f_{xt} + f_{6x}] - f[(\beta + \gamma)f_t f_x + 6(\beta + \gamma)f_x f_{5x}$$

$$(15(\beta - 3\gamma)f_{2x} f_{4x} - 20(\beta - 2\gamma)f_{3x}^2] + 30(\beta - \gamma)[f_x f_{4x} - 2f_x f_{2x} f_{3x} + f_{2x}^3] = 0$$

One solitary wave solution

$$f(x, t) = 1 + \exp(kx - \omega t + \delta) \quad \omega = k^5$$

Thus

$$u(x, t) = \frac{15k^2}{\beta + \gamma} \text{sech}^2 \frac{1}{2}(kx - k^5 t + \delta)$$

Two soliton solution requires

$$\beta = 2\gamma \quad \text{or} \quad \beta = \gamma$$
Case 1: LAX equation \((\beta = 2\gamma)\)

\[
f^2[f_{xt} + f_{6x}] - f[f_{t}f_{x} + 6f_{x}f_{5x} - 5f_{2x}f_{4x}] \\
+ 10[f_{x}^{2}f_{4x} - 2f_{x}f_{2x}f_{3x} + f_{2x}^{3}] = 0
\]

Note: This is a cubic equation!

Bilinear form consists of two equations

\[
(D_x D_\tau + D_4^{x})(f \cdot f) = 0
\]

\[
(D_x D_t + D_6^{x})(f \cdot f) + a(D_3^{x} D_\tau + D_6^{x})(f \cdot f) \\
- \frac{5 + 6a}{3}(D_2^{\tau} + D_\tau D_3^{x})(f \cdot f) = 0
\]

with auxiliary time variable \(\tau\)

Bilinear form is no longer needed

Write the cubic equation as

\[
f^2\mathcal{L}(f) + f\mathcal{N}_1(f, f) + \mathcal{N}_2(f, f, f) = 0
\]

with

\[
\mathcal{L}(f) = f_{xt} + f_{6x} \\
\mathcal{N}_1(f, g) = -(f_{t}g_{x} + 6f_{x}g_{5x} - 5f_{2x}g_{4x}) \\
\mathcal{N}_2(f, g, h) = 10(f_{x}g_{x}h_{4x} - 2f_{x}g_{2x}h_{3x} + f_{2x}g_{2x}h_{2x})
\]
Seek a solution of the form

\[ f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t) \]

Perturbation scheme (equate powers in \( \epsilon \) to zero)

\[ O(\epsilon^1) : \mathcal{L} f^{(1)} = 0 \]
\[ O(\epsilon^2) : \mathcal{L} f^{(2)} = -2 f^{(1)} \mathcal{L} f^{(1)} - \mathcal{N}_1(f^{(1)}, f^{(1)}) \]
\[ O(\epsilon^3) : \mathcal{L} f^{(3)} = -2 f^{(1)} \mathcal{L} f^{(2)} - 2 f^{(2)} \mathcal{L} f^{(1)} - f^{(1)} \mathcal{L} f^{(1)} \]
\[ - \mathcal{N}_1(f^{(1)}, f^{(2)}) - \mathcal{N}_1(f^{(2)}, f^{(1)}) \]
\[ - f^{(1)} \mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \]

\[ \vdots \]

N-soliton solution is then generated by

\[ f^{(1)} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - \omega_i t + \delta_i) \]

with dispersion law (from level \( \epsilon^1 \))

\[ P(k_i, \omega_i) = -\omega_i k_i + k_i^6 = 0 \]

or

\[ \omega_i = k_i^5 \quad i = 1, 2, \ldots, N \]
Then

\[ \mathcal{L} f^{(2)} = \sum_{1 \leq i < j \leq N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} f_i f_j \]

\[ = \sum_{1 \leq i < j \leq N} 5k_i k_j (k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2) a_{ij} f_i f_j \]

must balance

\[-\mathcal{N}_1(f^{(1)}, f^{(1)}) = \sum_{1 \leq i < j \leq N} 5k_i k_j (k_i - k_j)^2 (k_i^2 + k_i k_j + k_j^2) f_i f_j \]

Hence

\[ a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \]

N-soliton solution in analogous way
Case 2: SK or CDG equation ($\beta = \gamma$)

\[ \mathcal{N}(f, f) = f[f_{xt} + f_{6x}] - f_t f_x - 6 f_x f_{5x} + 15 f_{2x} f_{4x} - 10 f_{3x}^2 = 0 \]

Note: This is a quadratic equation!

Bilinear representation

\[ (D_x D_t + D_x^6) (f \cdot f) = 0 \]

Easy to find but not needed!

Seek solution of the form (3-soliton)

\[
\begin{align*}
    f &= 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 \\
    &\quad + a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\
    &\quad + b_{123} \exp(\theta_1 + \theta_2 + \theta_3)
\end{align*}
\]

Solve the perturbation scheme

Here

\[
\begin{align*}
    a_{ij} &= \frac{(k_i - k_j)^2 (k_j^2 - k_i k_j + k_i^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)} = \frac{(k_i - k_j)^3 (k_i^3 + k_j^3)}{(k_i + k_j)^3 (k_i^3 - k_j^3)} \\
    b_{123} &= a_{12} a_{13} a_{23}
\end{align*}
\]

N-soliton solution in similar way