Symbolic Computation of Lax Pairs of Nonlinear Systems of Partial Difference Equations using Multidimensional Consistency

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Outline

- Origins of nonlinear $P\Delta E$s
- Classification of integrable nonlinear $P\Delta E$s in 2D
- Lax pairs of nonlinear PDEs & gauge equivalence
- Lax pair of nonlinear $P\Delta E$s & gauge equivalence
- Examples of Lax pairs of nonlinear $P\Delta E$s
- Algorithmic computation of Lax pairs (Nijhoff 2001, Bobenko & Suris 2001)
- Software demonstration
- Conclusions and future work
- Addendum: Additional examples
Origins of nonlinear PΔEs

- full discretizations of completely integrable PDEs (Ablowitz, Ladik, Taha)
- fully discretized bilinear equations (Hirota)
- direct linearization of completely integrable PDEs (Quispel, Nijhoff)
- superposition principle (Bianchi permutability) for auto-Bäcklund transformations between solutions of a completely integrable PDE
- classification of multi-dimensionally consistent PΔEs (Adler, Bobenko, Suris)
• **Example:** discrete potential Korteweg-de Vries (pKdV) equation

\[
(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0
\]

• \(u\) is dependent variable or field (scalar case)
  
  \(n\) and \(m\) are lattice points

• \(p\) and \(q\) are parameters

• **Notation:**

\[
(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (x, x_1, x_2, x_{12})
\]

• Alternate notations (in the literature):

\[
(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (\tilde{u}, \hat{u}, \hat{\tilde{u}}, \hat{\hat{u}})
\]

\[
(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u_{00}, u_{10}, u_{01}, u_{11})
\]
• Example: discrete pKdV equation

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short: $$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$
Concept of Consistency Around the Cube

Superposition of auto-Bäcklund transformations between 4 solutions $x, x_1, x_2, x_3$ (3 parameters: $p, q, k$)
• Introduce a third lattice variable $\ell$

• View $u$ as dependent on three lattice points: $n, m, \ell$. So, $x = u_{n,m} \rightarrow x = u_{n,m,\ell}$

• Move in three directions:
  - $n \rightarrow n + 1$ over distance $p$
  - $m \rightarrow m + 1$ over distance $q$
  - $\ell \rightarrow \ell + 1$ over distance $k$ (spectral parameter)

• Require that the same PΔE holds on the front, bottom, and left faces of the cube

• Require consistency for the computation of $x_{123} = u_{n+1,m+1,\ell+1}$ (3 ways $\rightarrow$ same answer)
Example: discrete pKdV equation

⋆ Equation on front face of cube:

\[(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0\]

Solve for \(x_{12} = x - \frac{p^2 - q^2}{x_1 - x_2}\)

Compute \(x_{123}:\) \(x_{12} \rightarrow x_{123} = x_3 - \frac{p^2 - q^2}{x_1 - x_2}\)

⋆ Equation on floor of cube:

\[(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0\]

Solve for \(x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3}\)

Compute \(x_{123}:\) \(x_{13} \rightarrow x_{123} = x_2 - \frac{p^2 - k^2}{x_1 - x_2}\)
Equation on left face of cube:

\[(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0\]

Solve for \(x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3}\)

Compute \(x_{123} : \quad x_{23} \rightarrow x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}}\)

Verify that all three coincide:

\[x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}\]

Upon substitution of \(x_{12}, x_{13}, \text{ and } x_{23}\):

\[x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)}\]

Consistency around the cube is satisfied!
Tetrahedron property

\[ x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)} \]

is independent of \( x \). Connects \( x_{123} \) to \( x_1, x_2 \) and \( x_3 \).
Classification of 2D scalar nonlinear PΔEs

Adler, Bobenko, Suris (ABS) 2003, 2007

• Consider a family PΔEs: \( Q(x, x_1, x_2, x_{12}; p, q) = 0 \)

• Assumptions (ABS 2003):
  1. Affine linear
     \[
     Q(x, x_1, x_2, x_{12}; p, q) = a_1 xx_1 x_2 x_{12} + a_2 xx_1 x_2 + \ldots + a_{14} x_2 + a_{15} x_{12} + a_{16}
     \]

  2. Invariant under \( D_4 \) (symmetries of square)
     \[
     Q(x, x_1, x_2, x_{12}; p, q) = \epsilon Q(x, x_2, x_1, x_{12}; q, p)
     = \sigma Q(x_1, x, x_{12}, x_2; p, q)
     \]
     \( \epsilon, \sigma = \pm 1 \)

  3. Consistency around the cube
Result of the ABS Classification

• List H
  ▶ H1
  \[(x - x_{12})(x_1 - x_2) + q - p = 0\]

  ▶ H2
  \[(x - x_{12})(x_1 - x_2) + (q - p)(x + x_1 + x_2 + x_{12}) + q^2 - p^2 = 0\]

  ▶ H3
  \[p(x x_1 + x_2 x_{12}) - q(x x_2 + x_1 x_{12}) + \delta(p^2 - q^2) = 0\]
List A

A1

\[ p(x + x_2)(x_1 + x_{12}) - q(x + x_1)(x_2 + x_{12}) - \delta^2 pq(p - q) = 0 \]

A2

\[ (q^2 - p^2)(xx_1x_2x_{12} + 1) + q(p^2 - 1)(xx_2 + x_1x_{12}) - p(q^2 - 1)(xx_1 + x_2x_{12}) = 0 \]
• List Q

▶ Q1

\[ p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q) = 0 \]

▶ Q2

\[ p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q) \]
\[ (x+x_1+x_2+x_{12})-pq(p-q)(p^2-pq+q^2) = 0 \]

▶ Q3

\[ (q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12}) \]
\[ -p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0 \]
Q4 (mother)  Hietarinta’s Parametrization

\[ \text{sn}(\alpha + \beta; k) \left( x_1 x_2 + x x_{12} \right) \]
\[ - \text{sn}(\alpha; k) \left( x x_1 + x_2 x_{12} \right) - \text{sn}(\beta; k) \left( x x_2 + x_1 x_{12} \right) \]
\[ + \text{sn}(\alpha; k) \text{sn}(\beta; k) \text{sn}(\alpha + \beta; k)(1 + k^2 x x_1 x_2 x_{12}) = 0 \]

where \( \text{sn}(\alpha; k) \) is the Jacobi elliptic sine function with modulus \( k \).

Other parameterizations (Adler, Nijhoff, Viallet) are given in the literature.
Systems of PDEs

Example: Schwarzian-Boussinesq System

\[
\begin{align*}
y x_1 - z_1 + z &= 0 \\
y x_2 - z_2 + z &= 0 \\
x y_{12}(y_1 - y_2) - y (px_1 y_2 - qx_2 y_1) &= 0
\end{align*}
\]

• System has three dependent variable \(x, y,\) and \(z\). Thus, \(x = (x, y, z)\).

• System has two single-edge equations and one full-face equation.

• System is consistent around the cube (CAC).
Peter D. Lax (1926-)

Refresher: Lax Pairs of Nonlinear PDEs

• Historical example: Korteweg-de Vries equation

\[ u_t + \alpha uu_x + u_{xxx} = 0 \quad \alpha \in \mathbb{R} \]

• Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

\[ \psi_{xx} + \left( \frac{1}{6} \alpha u - \lambda \right) \psi = 0 \]

\[ \psi_t + 4\psi_{xxx} + \alpha u\psi_x + \frac{1}{2} \alpha u_x \psi = 0 \]

\[ \psi \text{ is eigenfunction; } \lambda \text{ is constant eigenvalue} \]

(\(\lambda_t = 0\)) (isospectral)
Lax Pairs in Matrix Form (AKNS Scheme)

- Express compatibility of

\[ \begin{align*}
    D_x \Psi &= X \Psi \\
    D_t \Psi &= T \Psi
\end{align*} \]

where \( \Psi = \begin{bmatrix} \psi \\ \psi_x \end{bmatrix} \)

- Lax equation (zero-curvature equation):

\[ D_t X - D_x T + [X, T] \doteq 0 \]

with commutator \([X, T] = XT - TX\)

and where \( \doteq \) means “evaluated on the PDE”
• Example: Lax pair for the KdV equation

\[ X = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6} \alpha u & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} \frac{1}{6} \alpha u_x & -4\lambda - \frac{1}{3} \alpha u \\ -4\lambda^2 + \frac{1}{3} \alpha \lambda u + \frac{1}{18} \alpha^2 u^2 + \frac{1}{6} \alpha u_{2x} & -\frac{1}{6} \alpha u_x \end{bmatrix} \]

Substitution into the Lax equation yields

\[ D_t X - D_x T + [X, T] = -\frac{1}{6} \alpha \begin{bmatrix} 0 & 0 \\ u_t + \alpha uu_x + u_{3x} & 0 \end{bmatrix} \]
Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If \((X, T)\) is a Lax pair then so is \((X', T')\) with

\[
X' = GXG^{-1} + D_x(G)G^{-1} \\
T' = GTG^{-1} + D_t(G)G^{-1}
\]

\(G\) is arbitrary invertible matrix and \(\Phi = G\Psi\) where \(\Phi\) goes with \((X, T)\), i.e., \(D_x\Phi = X\Phi\) and \(D_t\Phi = T\Phi\).

Thus,

\[
D_tX - D_xT + [X, T] = 0
\]
• Example: For the KdV equation

\[ X = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6} \alpha u & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} -ik & \frac{1}{6} \alpha u \\ -1 & ik \end{bmatrix} \]

Here,\n
\[ X = GXG^{-1} \text{ and } T = GTG^{-1} \]

with\n
\[ G = \begin{bmatrix} -ik & 1 \\ -1 & 0 \end{bmatrix} \]

where \( \lambda = -k^2 \)
Reasons to Compute a Lax Pair

• Compatible linear system is the starting point for application of the IST and the Riemann-Hilbert method for boundary value problems
• Confirm the complete integrability of the PDE
• Zero-curvature representation of the PDE
• Compute conservation laws of the PDE
• Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method
Lax Pair of Nonlinear PΔEs

• Replace the nonlinear PΔE by

\[ \psi_1 = L \psi \quad (\text{recall } \psi_1 = \psi_{n+1,m}) \]
\[ \psi_2 = M \psi \quad (\text{recall } \psi_2 = \psi_{n,m+1}) \]

For scalar PΔEs, \( L, M \) are \( 2 \times 2 \) matrices;
\[ x_3 = \frac{f}{F} \quad \text{and} \quad \psi = \begin{bmatrix} F \\ f \end{bmatrix} \]

For systems of PΔEs, \( L, M \) are \( N \times N \) matrices;
\[ x_3 = \frac{f}{F}, y_3 = \frac{g}{F}, z_3 = \frac{h}{F}, \text{ etc., } \psi = [F \ f \ g \ h \ \ldots]^T \]

where \( T \) is transpose.
Express compatibility:

\[ \psi_{12} = L_2 \psi_2 = L_2 M \psi \]
\[ \psi_{12} = M_1 \psi_1 = M_1 L \psi \]

Lax equation:

\[ L_2 M - M_1 L = 0 \]
Equivalence under Gauge Transformations

- Lax pairs of the same size are equivalent under a gauge transformation

If \((L, M)\) is a Lax pair then so is \((L', M')\) with

\[
L' = G_1 L G^{-1}
\]
\[
M' = G_2 M G^{-1}
\]

where \(G\) is an invertible matrix, \(\phi = G \psi\)
goes with \((L', M')\) i.e., \(\phi_1 = L \phi, \ \phi_2 = M \phi\).

Proof: Trivial verification that

\[
(L_2 M - M_1 L) \phi \overset{\cdot}{=} 0 \iff (L_2 M - M_1 L) \psi \overset{\cdot}{=} 0
\]
Examples of Lax Pairs of PΔEs

• **Example 1:** Discrete pKdV Equation

\[
(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0
\]

• **Lax pair:** \(x_3 = \frac{f}{F}, \quad \psi = [F \ f]^T\)

\[
L = tL_c = t \begin{bmatrix} 1 & -x_1 \\ x & p^2 - k^2 - xx_1 \end{bmatrix}
\]

\[
M = sM_c = s \begin{bmatrix} 1 & -x_2 \\ x & q^2 - k^2 - xx_2 \end{bmatrix}
\]

with \(t = s = 1\) or \(t = \frac{1}{\sqrt{\text{Det}L_c}} = \frac{1}{\sqrt{k^2 - p^2}}\)

and \(s = \frac{1}{\sqrt{\text{Det}M_c}} = \frac{1}{\sqrt{k^2 - q^2}}\). Here, \(t \frac{t_2}{t} \frac{s}{s_1} = 1\).
• **Example 2: Schwarzian-Boussinesq System**

\[
\begin{align*}
  y x_1 - z_1 + z &= 0 \\
  y x_2 - z_2 + z &= 0 \\
  x y_{12}(y_1 - y_2) - y (p x_1 y_2 - q x_2 y_1) &= 0
\end{align*}
\]

• **Lax pair:** \[ \psi = [F \ g \ h]^T \]

\[
L = tL_c = t \begin{bmatrix}
  y_1 & -1 & 0 \\
  \frac{k z y_1}{x} & \frac{p y x_1}{x} & -\frac{k y_1}{x} \\
  0 & -z_1 & y_1
\end{bmatrix}
\]
\[ M = sM_c = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{k z y_2}{x} & \frac{q y x_2}{x} & -\frac{k y_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix} \]

with \( t = s = \frac{1}{y} \), or \( t = \frac{1}{y_1} \) and \( s = \frac{1}{y_2} \),

or \( t = 3 \sqrt{\frac{x}{y_1^2 y x_1}} \) and \( s = 3 \sqrt{\frac{x}{y_2^2 y x_2}} \).

Here, \( \frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2} \).
Lax Pair Algorithm for Scalar PΔEs

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to PΔEs that are consistent around the cube

Example: Discrete pKdV Equation

• Step 1: Verify the consistency around the cube

Use the equation on floor

\[(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0\]

to compute \[x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3} = \frac{x_3x - xx_1 + p^2 - k^2}{x_3 - x_1}\]

Use the equation on left face

\[(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0\]

to compute \[x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3} = \frac{x_3x - xx_2 + q^2 - k^2}{x_3 - x_2}\]
Step 2: Homogenization

★ Numerator and denominator of

$$x_{13} = \frac{x_3x-xx_1+p^2-k^2}{x_3-x_1} \quad \text{and} \quad x_{23} = \frac{x_3x-xx_2+q^2-k^2}{x_3-x_2}$$

are linear in \(x_3\).

★ Substitute \(x_3 = \frac{f}{F}\) into \(x_{13}\) to get

$$x_{13} = \frac{xf+(p^2-k^2-xx_1)F}{f-x_1F}$$

★ On the other hand, \(x_3 = \frac{f}{F} \quad \rightarrow \quad x_{13} = \frac{f_1}{F_1} \).

Thus, \(x_{13} = \frac{f_1}{F_1} = \frac{xf+(p^2-k^2-xx_1)F}{f-x_1F}\).

Hence, \(F_1 = t(f-x_1F)\) and

\(f_1 = t\left(xf+(p^2-k^2-xx_1)F\right)\)
In matrix form

\[
\begin{bmatrix}
F_1 \\
f_1
\end{bmatrix} = t \begin{bmatrix}
1 & -x_1 \\
x & p^2 - k^2 - xx_1
\end{bmatrix} \begin{bmatrix}
F \\
f
\end{bmatrix}
\]

Matches \( \psi_1 = L \psi \) with \( \psi = \begin{bmatrix}
F \\
f
\end{bmatrix} \)

Similarly, from \( x_{23} = \frac{f_2}{F_2} = \frac{xf + (q^2 - k^2 - xx_2)F}{f - x_2F} \)

\[
\psi_2 = \begin{bmatrix}
F_2 \\
f_2
\end{bmatrix} = s \begin{bmatrix}
1 & -x_2 \\
x & q^2 - k^2 - xx_2
\end{bmatrix} \begin{bmatrix}
F \\
f
\end{bmatrix} = M \psi.
\]
Therefore,

\[
L = t \quad L_c = t \begin{bmatrix}
1 & -x_1 \\
x & p^2 - k^2 - xx_1
\end{bmatrix}
\]

\[
M = s \quad M_c = s \begin{bmatrix}
1 & -x_2 \\
x & q^2 - k^2 - xx_2
\end{bmatrix}
\]
• Step 3: Determine \( t \) and \( s \)

★ Substitute \( L = t L_c, M = s M_c \) into \( L_2 M - M_1 L = 0 \)

\[ \rightarrow t_2 s (L_c)_2 M_c - s_1 t (M_c)_1 L_c = 0 \]

★ Solve the equation from the (2-1)-element for

\[ \frac{t_2}{t} \frac{s}{s_1} = f(x, x_1, x_2, p, q, \ldots) \]

Find rational \( t \) and \( s \).

★ Apply determinant to get

\[ \frac{t_2}{t} \frac{s}{s_1} = \sqrt{\frac{\det L_c}{\det (L_c)_2}} \sqrt{\frac{\det (M_c)_1}{\det M_c}} \]

Solution: \( t = \frac{1}{\sqrt{\det L_c}}, \quad s = \frac{1}{\sqrt{\det M_c}} \)

\[ \rightarrow \text{Always works but introduces roots!} \]
The ratio \( \frac{t_2}{t} \frac{s}{s_1} \) is invariant under the change

\[ t \rightarrow \frac{a_1}{a} t, \quad s \rightarrow \frac{a_2}{a} s, \]

where \( a(x) \) is arbitrary.

Proper choice of \( a(x) \) \( \implies \) Rational \( t \) and \( s \).

No roots needed!
Algorithmic Computation of Lax Pairs for Systems of PΔEs

Example 2: Schwarzian-Boussinesq System

\[ y x_1 - z_1 + z = 0 \]
\[ y x_2 - z_2 + z = 0 \]
\[ x y_{12}(y_1 - y_2) - y (p x_1 y_2 - q x_2 y_1) = 0 \]

- System has two single-edge equations and one full-face equation
• Edge equations require augmentation of system with additional shifted, edge equations

\[ y_2 x_{12} - z_{12} + z_2 = 0 \]
\[ y_1 x_{12} - z_{12} + z_1 = 0 \]

• Edge equations will provide additional constraints during homogenization (Step 2).

The way you handle edge equations leads to gauge-equivalent Lax pairs!
• Step 1: Verify the consistency around the cube
  ★ System on the front face:

\[
\begin{align*}
y x_1 - z_1 + z &= 0 \\
y x_2 - z_2 + z &= 0 \\
x y_{12}(y_1 - y_2) - y (p x_1 y_2 - q x_2 y_1) &= 0 \\
y_2 x_{12} - z_{12} + z_2 &= 0 \\
y_1 x_{12} - z_{12} + z_1 &= 0 
\end{align*}
\]

Solve for \( x_{12}, y_{12}, \) and \( z_{12} \):

\[
\begin{align*}
x_{12} &= \frac{z_2 - z_1}{y_1 - y_2} \\
y_{12} &= \frac{y(px_1 y_2 - qx_2 y_1)}{x(y_1 - y_2)} \\
z_{12} &= \frac{y_1 z_2 - y_2 z_1}{y_1 - y_2}
\end{align*}
\]
Compute $x_{123}$, $y_{123}$, and $z_{123}$:

$$x_{123} = \frac{z_{23} - z_{13}}{y_{13} - y_{23}}$$

$$y_{123} = \frac{y_3(px_{13}y_{23} - qx_{23}y_{13})}{x_3(y_{13} - y_{23})}$$

$$z_{123} = \frac{y_{13}z_{23} - y_{23}z_{13}}{y_{13} - y_{23}}$$
System on the bottom face:

\[ y x_1 - z_1 + z = 0 \]
\[ y x_3 - z_3 + z = 0 \]
\[ x y_{13}(y_1 - y_3) - y(px_1 y_3 - k x_3 y_1) = 0 \]
\[ y_3 x_{13} - z_{13} + z_3 = 0 \]
\[ y_1 x_{13} - z_{13} + z_1 = 0 \]

Solve for \( x_{13}, y_{13}, \) and \( z_{13} \):

\[ x_{13} = \frac{z_3 - z_1}{y_1 - y_3} \]
\[ y_{13} = \frac{y(px_1 y_3 - k x_3 y_1)}{x(y_1 - y_3)} \]
\[ z_{13} = \frac{y_1 z_3 - y_3 z_1}{y_1 - y_3} \]
Compute $x_{123}$, $y_{123}$, and $z_{123}$:

\[
x_{123} = \frac{z_{23} - z_{12}}{y_{12} - y_{23}}
\]

\[
y_{123} = \frac{y_2(p x_{12} y_{23} - k x_{23} y_{12})}{x_2(y_{12} - y_{23})}
\]

\[
z_{123} = \frac{y_{12} z_{23} - y_{23} z_{12}}{y_{12} - y_{23}}
\]
System on the left face:

\[ y x_3 - z_3 + z = 0 \]
\[ y x_2 - z_2 + z = 0 \]
\[ x y_{23}(y_3 - y_2) - y (px_3 y_2 - qx_2 y_3) = 0 \]
\[ y_2 x_{23} - z_2 + z_2 = 0 \]
\[ y_1 x_{23} - z_2 + z_1 = 0 \]

Solve for \( x_{23}, y_{23}, \) and \( z_{23} \):

\[ x_{23} = \frac{z_3 - z_2}{y_2 - y_3} \]
\[ y_{23} = \frac{y(qx_2 y_3 - kx_3 y_2)}{x(y_2 - y_3)} \]
\[ z_{23} = \frac{y_2 z_3 - y_3 z_2}{y_2 - y_3} \]
Compute $x_{123}$, $y_{123}$, and $z_{123}$:

$$x_{123} = \frac{z_{13} - z_{12}}{y_{12} - y_{13}}$$

$$y_{123} = \frac{y_1(q x_{12} y_{13} - k x_{13} y_{12})}{x_1(y_{12} - y_{13})}$$

$$z_{123} = \frac{y_{12} z_{13} - y_{13} z_{12}}{y_{12} - y_{13}}$$

Substitute $x_{12}$, $y_{12}$, $y_{12}$, $x_{13}$, $y_{13}$, $z_{13}$, $x_{23}$, $y_{23}$, $z_{23}$ into the above to get
\[ x_{123} = \frac{x(x_1 - x_2)(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2))}{(z_1 - z_2)(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))} \]

\[ y_{123} = \frac{q(z_2 - z_1)(kx_3y_1 - px_1y_3) + k(z_3 - z_1)(px_1y_2 - qx_2y_1)}{x_1(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))} \]

\[ z_{123} = \frac{px_1(y_3z_2 - y_2z_3) + qx_2(y_1z_3 - y_3z_1) + kx_3(y_2z_1 - y_1z_2)}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)} \]

Answer is unique and independent of \( x \) and \( y \).

Consistency around the cube is satisfied!
• Step 2: Homogenization

★ Observed that $x_3$, $y_3$ and $z_3$ appear linearly in numerators and denominators of

\[
\begin{align*}
x_{13} &= \frac{z_3 - z_1}{y_1 - y_3} \\
y_{13} &= \frac{y(px_1y_3 - kx_3y_1)}{x(y_1 - y_3)} \\
z_{13} &= \frac{y_1z_3 - y_3z_1}{y_1 - y_3}
\end{align*}
\]
Substitute
\[ x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{h}{H}. \]

Use constraints (from left face edges)
\[
y x_1 - z_1 + z = 0, \quad y x_2 - z_2 + z = 0
\Rightarrow y x_3 - z_3 + z = 0
\]

Solve for \[ x_3 = \frac{z_3 - z}{y} \]

Thus, \[ x_3 = \frac{f}{F} = \frac{h - zH}{yH}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{h}{H}. \]
★ Substitute $x_3, y_3, z_3$ into $x_{13}, y_{13}, z_{13}$:

\[
\begin{align*}
x_{13} & = \frac{G(h - z_1 H)}{H(y_1 G - g)} \\
y_{13} & = \frac{y(px_1 g F - ky_1 f G)}{Fx(y_1 G - g)} \\
z_{13} & = \frac{y_1 h G - z_1 g H}{H(y_1 G - g)}
\end{align*}
\]

Require that numerators and denominators are linear in $f, g, h, F, G$, and $H$. That forces $H = G = F$.

Hence, $x_3 = \frac{h - z F}{y F}, \ y_3 = \frac{g}{F}, \ \text{and} \ z_3 = \frac{h}{F}$. 
Compute

\[
x_3 = \frac{h - zF}{yF} \quad \rightarrow \quad x_{13} = \frac{h_1 - z_1F_1}{y_1F_1}
\]

\[
y_3 = \frac{g}{F} \quad \rightarrow \quad y_{13} = \frac{g_1}{F_1}
\]

\[
z_3 = \frac{h}{F} \quad \rightarrow \quad z_{13} = \frac{h_1}{F_1}
\]

Hence,

\[
x_{13} = \frac{h - z_1F}{y_1F - g} = \frac{h_1 - z_1F_1}{y_1F_1}
\]

\[
y_{13} = \frac{y_p x_1 g - ky_1 h + kzy_1 F}{x(y_1 F - g)} = \frac{g_1}{F_1}
\]

\[
z_{13} = \frac{y_1 h - z_1 g}{y_1 F - g} = \frac{h_1}{F_1}
\]
Note that

\[ x_{13} = \frac{h - z_1 F}{y_1 F - g} = \frac{h_1 - z_1 F_1}{y_1 F_1} \]

is automatically satisfied as a result of the relation

\[ x_3 = \frac{z_3 - z}{y}. \]

\[ \star \] Write in matrix form:

\[ \psi_1 = \begin{bmatrix} F_1 \\ g_1 \\ h_1 \end{bmatrix} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = L \psi \]

\[ \star \] Repeat the same steps for \( x_{23}, y_{23}, z_{23} \) to obtain
\[ \psi_2 = \begin{bmatrix} F_2 \\ g_2 \\ h_2 \end{bmatrix} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kz y_2}{x} & \frac{qy x_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = M \psi \]

\[
\star \text{ Therefore, } \\
L = t L_{\text{core}} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kz y_1}{x} & \frac{p y x_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix} \\
M = s M_{\text{core}} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kz y_2}{x} & \frac{qy x_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix} \]
• **Step 3: Determine** $t$ and $s$

  ✰ Substitute $L = t L_{\text{core}}, M = s M_{\text{core}}$ into

  $L_2 M - M_1 L = 0$

  $\rightarrow t_2 s (L_{\text{core}})_2 M_{\text{core}} - s_1 t (M_{\text{core}})_1 L_{\text{core}} = 0$

  ✰ Solve the equation from the (2-1)-element:

  \[
  \frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}.
  \]

  Thus, $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

  or (from determinant method) $t = \frac{3\sqrt{x}}{y_1 y x_1}$ and

  $s = \frac{3\sqrt{x}}{y_2 y x_2}$.  

Summary: Lax Pair for Schwarzian-BSQ System

• Option a: Solving the edge equation for $x_3 = \frac{z_3 - z}{y}$ yields

$$x_3 = \frac{h - zF}{yF}, \quad y_3 = \frac{g}{F}, \quad \text{and} \quad z_3 = \frac{h}{F}, \quad \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

• Corresponding Lax matrix:

$$L_a = \frac{1}{y} \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$
• Option b: Solving the edge equation for
\[ z_3 = x_3 y + z \text{ yields} \]
\[ x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \text{ and } z_3 = \frac{zF + yf}{F}, \quad \psi_b = \begin{bmatrix} F \\ f \\ g \end{bmatrix} \]

• Corresponding Lax matrix:
\[ L_b = \frac{1}{y} \begin{bmatrix} y_1 & 0 & -1 \\ z - z_1 & y & 0 \\ 0 & -\frac{kyy_1}{x_1} & \frac{pyz_1}{x} \end{bmatrix} \]
• Gauge Equivalences between these Lax Matrices

\[ L_b = G_1 L_a G^{-1}, \quad \psi_b = G \psi_a \]

with

\[
G = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{z}{y} & 0 & \frac{1}{y} \\
0 & 1 & 0
\end{bmatrix}
\]
Software Demonstration
Conclusions and Future Work

• **Mathematica code** works for **scalar** $P\Delta E_s$ in 2D defined on quad-graphs (quadrilateral faces).
• **Mathematica code** has been extended to systems of $P\Delta E_s$ in 2D defined on quad-graphs.
• Code can be used to **test** (i) consistency around the cube and compute or test (ii) Lax pairs.
• Consistency around cube $\iff P\Delta E$ has Lax pair.
• $P\Delta E$ has Lax pair $\not\iff$ consistency around cube. Indeed, there are $P\Delta E$s with a Lax pair that are not consistent around the cube.
  **Example:** discrete sine-Gordon equation.
• Avoid the determinant method to avoid square roots! Factorization plays an essential role!

• Future Work: Extension to more complicated systems of $P\Delta Es$. 
Thank You
Additional Examples

• **Example:** Discrete pKdV Equation

\[
(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0
\]

• **Lax pair:**

\[
L = tL_{\text{core}} = t \begin{bmatrix}
 x & p^2 - k^2 - xx_1 \\
 1 & -x_1
\end{bmatrix}
\]

\[
M = sM_{\text{core}} = s \begin{bmatrix}
 x & q^2 - k^2 - xx_2 \\
 1 & -x_2
\end{bmatrix}
\]

with \( t = s = 1 \) or \( t = \frac{1}{\sqrt{\det L_{\text{core}}}} = \frac{1}{\sqrt{k^2 - p^2}} \)

and \( s = \frac{1}{\sqrt{\det M_{\text{core}}}} = \frac{1}{\sqrt{k^2 - q^2}} \).

Here, \( x_3 = \frac{f}{F}, \, \psi = [f \ F]^T \), and \( \frac{t_2}{t} \frac{s}{s_1} = 1 \).
• Example: (H1) Equation (ABS classification)

\[(x - x_{12})(x_1 - x_2) + q - p = 0\]

• Lax pair:

\[
L = t \begin{bmatrix} x & p - k - xx_1 \\ 1 & -x_1 \end{bmatrix}
\]

\[
M = s \begin{bmatrix} x & q - k - xx_2 \\ 1 & -x_2 \end{bmatrix}
\]

with \( t = s = 1 \) or \( t = \frac{1}{\sqrt{k-p}} \) and \( s = \frac{1}{\sqrt{k-q}} \)

Here, \( x_3 = \frac{f}{F} \), \( \psi = \begin{bmatrix} f \\ F \end{bmatrix} \), and \( \frac{t_2}{t} \frac{s}{s_1} = 1 \).
• **Example:** (H2) Equation (ABS 2003)

\[(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12})+q^2-p^2=0\]

• **Lax pair:**

\[
L = t \begin{bmatrix} p - k + x & p^2 - k^2 + (p - k)(x + x_1) - xx_1 \\ 1 & -(p - k + x_1) \end{bmatrix}
\]

\[
M = s \begin{bmatrix} q - k + x & q^2 - k^2 + (q - k)(x + x_2) - xx_2 \\ 1 & -(q - k + x_2) \end{bmatrix}
\]

with \( t = \frac{1}{\sqrt{2(k-p)(p+x+x_1)}} \) and \( s = \frac{1}{\sqrt{2(k-q)(q+x+x_2)}} \)

Here, \( x_3 = \frac{f}{F} \), \( \psi = \begin{bmatrix} f \\ F \end{bmatrix} \), and \( \frac{t_2}{t} \frac{s}{s_1} = \frac{p+x+x_1}{q+x+x_2} \).
• Example: (H3) Equation (ABS 2003)

\[ p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0 \]

• Lax pair:

\[
L = t \begin{bmatrix} kx & -(\delta(p^2 - k^2) + pxx_1) \\ \delta p & -kx_1 \end{bmatrix}
\]

\[
M = s \begin{bmatrix} kx & -(\delta(q^2 - k^2) + qxx_2) \\ q & -kx_2 \end{bmatrix}
\]

with \( t = \frac{1}{\sqrt{(p^2-k^2)(\delta p + xx_1)}} \) and \( s = \frac{1}{\sqrt{(q^2-k^2)(\delta q + xx_2)}} \)

Here, \( x_3 = \frac{f}{F}, \psi = \begin{bmatrix} f \\ F \end{bmatrix}, \) and \( \frac{t_2}{t} \frac{s}{s_1} = \frac{\delta p + xx_1}{\delta q + xx_2}. \)
• Example: (H3) Equation \( (\delta = 0) \) (ABS 2003)

\[
p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0
\]

• Lax pair:

\[
L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}
\]

\[
M = s \begin{bmatrix} kx & -qx x_2 \\ q & -kx_2 \end{bmatrix}
\]

with \( t = s = \frac{1}{x} \) or \( t = \frac{1}{x_1} \) and \( s = \frac{1}{x_2} \).

Here, \( x_3 = \frac{f}{F} \), \( \psi = \begin{bmatrix} f \\ F \end{bmatrix} \), and \( \frac{t_2}{t} \frac{s}{s_1} = \frac{x_1 x}{xx_2} = \frac{x_1}{x_2} \).

\[ p(x+x_2)(x_1+x_{12}) - q(x+x_1)(x_2+x_{12}) - \delta^2 pq(p-q) = 0 \]

(Q1) if \( x_1 \rightarrow -x_1 \) and \( x_2 \rightarrow -x_2 \)

Lax pair:

\[
L = t \begin{bmatrix}
(k - p)x_1 + kx & -p \left( \delta^2 k(k - p) + xx_1 \right) \\
p & -((k - p)x + kx_1)
\end{bmatrix}
\]

\[
M = s \begin{bmatrix}
(k - q)x_2 + kx & -q \left( \delta^2 k(k - q) + xx_2 \right) \\
q & -((k - q)x + kx_2)
\end{bmatrix}
\]
with \( t = \frac{1}{\sqrt{k(k-p)((\delta p+x+x_1)(\delta p-x-x_1))}} \) and
\[
s = \frac{1}{\sqrt{k(k-q)((\delta q+x+x_2)(\delta q-x-x_2))}}
\]

Here \( x_3 = \frac{f}{F}, \psi = \begin{bmatrix} f \\ F \end{bmatrix} \), and
\[
\frac{t_2}{t} \frac{s}{s_1} = \frac{q(\delta p+(x+x_1))(\delta p-(x+x_1))}{p(\delta q+(x+x_2))(\delta q-(x+x_2))}.
\]

**Question:** Rational choice for \( t \) and \( s \)?
• Example: (A2) Equation (ABS 2003)

\[(q^2 - p^2)(xx_1x_2x_{12} + 1) + q(p^2 - 1)(xx_2 + x_1x_{12}) - p(q^2 - 1)(xx_1 + x_2x_{12}) = 0\]

(Q3) with \(\delta = 0\) via Möbius transformation:

\[x \rightarrow x, x_1 \rightarrow \frac{1}{x_1}, x_2 \rightarrow \frac{1}{x_2}, x_{12} \rightarrow x_{12}, p \rightarrow p, q \rightarrow q\]

• Lax pair:

\[L = t \begin{bmatrix} k(p^2 - 1)x & -(p^2 - k^2 + p(k^2 - 1)xx_1) \\ p(k^2 - 1) + (p^2 - k^2)xx_1 & -k(p^2 - 1)x_1 \end{bmatrix}\]

\[M = s \begin{bmatrix} k(q^2 - 1)x & -(q^2 - k^2 + q(k^2 - 1)xx_2) \\ q(k^2 - 1) + (q^2 - k^2)xx_2 & -k(q^2 - 1)x_2 \end{bmatrix}\]
with \( t = \frac{1}{\sqrt{(k^2-1)(k^2-p^2)(p-xx_1)(pxx_1-1)}} \)

and \( s = \frac{1}{\sqrt{(k^2-1)(k^2-q^2)(q-xx_2)(qxx_2-1)}} \)

Here, \( x_3 = \frac{f}{F} \), \( \psi = \begin{bmatrix} f \\ F \end{bmatrix} \), and

\[
\frac{t_2}{t} \frac{s}{s_1} = \frac{(q^2-1)(p-xx_1)(pxx_1-1)}{(p^2-1)(q-xx_2)(qxx_2-1)}.
\]

**Question:** Rational choice for \( t \) and \( s \)?
• Example: (Q1) Equation (ABS 2003)

\[ p(x-x_2)(x_1-x_{12}) - q(x-x_1)(x_2-x_{12}) + \delta^2 pq(p-q) = 0 \]

• Lax pair:

\[ L = t \begin{bmatrix} (p-k)x_1 + kx & -p(\delta^2 k(p-k) + xx_1) \\ p & -((p-k)x + kx_1) \end{bmatrix} \]

\[ M = s \begin{bmatrix} (q-k)x_2 + kx & -q(\delta^2 k(q-k) + xx_2) \\ q & -((q-k)x + kx_2) \end{bmatrix} \]

with \( t = \frac{1}{\delta p \pm (x-x_1)} \) and \( s = \frac{1}{\delta q \pm (x-x_2)} \),

or \( t = \frac{1}{\sqrt{k(p-k)((\delta p+x-x_1)(\delta p-x+x_1))}} \) and

\( s = \frac{1}{\sqrt{k(q-k)((\delta q+x-x_2)(\delta q-x+x_2))}} \)
Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

$$\frac{t_2}{t} \frac{s}{s_1} = \frac{q(\delta p+(x-x_1))(\delta p-(x-x_1))}{p(\delta q+(x-x_2))(\delta q-(x-x_2))}.$$
• Example: (Q1) Equation \((\delta = 0)\) (ABS 2003)

\[
p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) = 0
\]

which is the cross-ratio equation

\[
\frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)} = \frac{p}{q}
\]

• Lax pair:

\[
L = t \begin{bmatrix}
(p - k)x_1 + kx & -pxx_1 \\
\quad p & -((p - k)x + kx_1)
\end{bmatrix}
\]

\[
M = s \begin{bmatrix}
(q - k)x_2 + kx & -qx x_2 \\
\quad q & -((q - k)x + kx_2)
\end{bmatrix}
\]
\[ t = \frac{1}{x-x_1} \quad \text{and} \quad s = \frac{1}{x-x_2} \]

or \[ t = \frac{1}{\sqrt{k(k-p)(x-x_1)}} \quad \text{and} \quad s = \frac{1}{\sqrt{k(k-q)(x-x_2)}}. \]

Here, \( x_3 = \frac{f}{F}, \quad \psi = \begin{bmatrix} f \\ F \end{bmatrix}, \quad \text{and} \quad \frac{t_2}{t} \frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}. \]
• Example: (Q2) Equation (ABS 2003)

\[ p(x-x_2)(x_1-x_{12}) - q(x-x_1)(x_2-x_{12}) + pq(p-q) \]
\[ (x+x_1+x_2+x_{12}) - pq(p-q)(p^2-pq+q^2) = 0 \]

• Lax pair:

\[
L = \begin{bmatrix}
(k-p)(kp-x_1) + kx \\
-p \left( k(k-p)(k^2-kp+p^2-x-x_1) + xx_1 \right) \\
p & -((k-p)(kp-x) + kx_1)
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
(k-q)(kq-x_2) + kx \\
-q \left( k(k-q)(k^2-kq+q^2-x-x_2) + xx_2 \right) \\
q & -((k-q)(kq-x) + kx_2)
\end{bmatrix}
\]
with

\[ t = \frac{1}{\sqrt{k(k-p)((x-x_1)^2-2p^2(x+x_1)+p^4)}} \]

and

\[ s = \frac{1}{\sqrt{k(k-q)((x-x_2)^2-2q^2(x+x_2)+q^4)}} \]

Here, \( x_3 = \frac{f}{F} \), \( \psi = [f \ F]^T \), and

\[
\frac{t_2}{t} \quad \frac{s}{s_1} = \frac{q ((x - x_1)^2 - 2p^2(x + x_1) + p^4)}{p ((x - x_2)^2 - 2q^2(x + x_2) + q^4)}
\]

\[
= \frac{p ((X + X_1)^2 - p^2) ((X - X_1)^2 - p^2)}{q ((X + X_2)^2 - q^2) ((X - X_2)^2 - q^2)}
\]

with \( x = X^2 \), and, consequently, \( x_1 = X_1^2 \), \( x_2 = X_2^2 \).
• Example: (Q3) Equation (ABS 2003)

\[(q^2 - p^2)(xx_{12} + x_1 x_2) + q(p^2 - 1)(xx_1 + x_2 x_{12}) \]
\[-p(q^2 - 1)(xx_2 + x_1 x_{12}) - \frac{\delta^2}{4pq}(p^2 - q^2)(p^2 - 1)(q^2 - 1) = 0\]

• Lax pair:

\[
L = t \begin{bmatrix}
-4kp (p(k^2 - 1)x + (p^2 - k^2)x_1) \\
(p^2 - 1)(\delta^2 k^2 - \delta^2 k^4 - \delta^2 p^2 + \delta^2 k^2 p^2 - 4k^2 pxx_1) \\
-4k^2 p(p^2 - 1) & 4kp (p(k^2 - 1)x_1 + (p^2 - k^2)x)
\end{bmatrix}
\]

\[
M = s \begin{bmatrix}
-4kq (q(k^2 - 1)x + (q^2 - k^2)x_2) \\
(q^2 - 1)(\delta^2 k^2 - \delta^2 k^4 - \delta^2 q^2 + \delta^2 k^2 q^2 - 4k^2 qxx_2) \\
-4k^2 q(q^2 - 1) & 4kq (q(k^2 - 1)x_2 + (q^2 - k^2)x)
\end{bmatrix}
\]
with

\[ t = \frac{1}{2k \sqrt{p(k^2 - 1)(k^2 - p^2)} \left( 4p^2(x^2 + x_1^2) - 4p(1 + p^2)x_1 + \delta^2 (1-p^2)^2 \right)} \]

and

\[ s = \frac{1}{2k \sqrt{q(k^2 - 1)(k^2 - q^2)} \left( 4q^2(x^2 + x_2^2) - 4q(1 + q^2)x_2 + \delta^2 (1-q^2)^2 \right)} \].
Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

\[
\begin{align*}
\frac{t_2}{t} & \quad s \\
\frac{s}{s_1} & = \frac{q(q^2 - 1) \left(4p^2(x^2 + x_1^2) - 4p(1+p^2)xx_1 + \delta^2(1-p^2)^2\right)}{p(p^2 - 1) \left(4q^2(x^2 + x_2^2) - 4q(1+q^2)xx_2 + \delta^2(1-q^2)^2\right)} \\
& = \frac{q(q^2 - 1) \left(4p^2(x - x_1)^2 - 4p(p-1)^2xx_1 + \delta^2(1-p^2)^2\right)}{p(p^2 - 1) \left(4q^2(x - x_2)^2 - 4q(q-1)^2xx_2 + \delta^2(1-q^2)^2\right)} \\
& = \frac{q(q^2 - 1) \left(4p^2(x + x_1)^2 - 4p(p+1)^2xx_1 + \delta^2(1-p^2)^2\right)}{p(p^2 - 1) \left(4q^2(x + x_2)^2 - 4q(q+1)^2xx_2 + \delta^2(1-q^2)^2\right)}
\end{align*}
\]
where

$$4p^2(x^2 + x_1^2) - 4p(1 + p^2)xx_1 + \delta^2(1 - p^2)^2$$

$$= \delta^2(p - e^{X + X_1})(p - e^{-(X + X_1)})(p - e^{X - X_1})(p - e^{-(X - X_1)})$$

$$= \delta^2(p - \cosh(X + X_1) + \sinh(X + X_1))$$

$$\quad (p - \cosh(X + X_1) - \sinh(X + X_1))$$

$$\quad (p - \cosh(X - X_1) + \sinh(X - X_1))$$

$$\quad (p - \cosh(X - X_1) - \sinh(X - X_1))$$

with $x = \delta \cosh(X)$, and, consequently,

$x_1 = \delta \cosh(X_1)$, $x_2 = \delta \cosh(X_2)$. 
• **Example:** (Q3) Equation \((\delta = 0)\) (ABS 2003)

\[
(q^2 - p^2)(x x_{12} + x_1 x_2) + q(p^2 - 1)(x x_1 + x_2 x_{12}) \\
- p(q^2 - 1)(x x_2 + x_1 x_{12}) = 0
\]

• **Lax pair:**

\[
L = t \begin{bmatrix}
(p^2 - k^2)x_1 + p(k^2 - 1)x & -k(p^2 - 1)x x_1 \\
(p^2 - 1)k & -((p^2 - k^2)x + p(k^2 - 1)x_1)
\end{bmatrix}
\]

\[
M = s \begin{bmatrix}
(q^2 - k^2)x_2 + q(k^2 - 1)x & -k(q^2 - 1)x x_2 \\
(q^2 - 1)k & -((q^2 - k^2)x + q(k^2 - 1)x_2)
\end{bmatrix}
\]
\textbullet \ with \ t = \frac{1}{px-x_1} \ and \ s = \frac{1}{qx-x_2} \\

or \ t = \frac{1}{px_1-x} \ and \ s = \frac{1}{qx_2-x} \\

or \ t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}} \\

and \ s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}. \\

Here, \ x_3 = \frac{f}{F}, \ \psi = \begin{bmatrix} f \\ F \end{bmatrix}, \ and \\

\frac{t_2}{t} \frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}. 
• Example: \((\alpha, \beta)\)-equation (Quispel 1983)

\[
\begin{align*}
((p-\alpha)x-(p+\beta)x_1) & \cdot ((p-\beta)x_2-(p+\alpha)x_{12}) \\
-((q-\alpha)x-(q+\beta)x_2) & \cdot ((q-\beta)x_1-(q+\alpha)x_{12}) = 0
\end{align*}
\]

• Lax pair:

\[
L = t \begin{bmatrix}
(p-\alpha)(p-\beta)x + (k^2-p^2)x_1 & -(k-\alpha)(k-\beta)xx_1 \\
(k+\alpha)(k+\beta) & -(p+\alpha)(p+\beta)x_1 + (k^2-p^2)x
\end{bmatrix}
\]

\[
M = s \begin{bmatrix}
(q-\alpha)(q-\beta)x + (k^2-q^2)x_2 & -(k-\alpha)(k-\beta)xx_2 \\
(k+\alpha)(k+\beta) & -(q+\alpha)(q+\beta)x_2 + (k^2-q^2)x
\end{bmatrix}
\]
• with \( t = \frac{1}{(\alpha-p)x+(\beta+p)x_1} \) and \( s = \frac{1}{(\alpha-q)x+(\beta+q)x_2} \)

or \( t = \frac{1}{(\beta-p)x+(\alpha+p)x_1} \) and \( s = \frac{1}{(\beta-q)x+(\alpha+q)x_2} \)

or \( t = \frac{1}{\sqrt{(p^2-k^2)(\beta-p)x+(\alpha+p)x_1)((\alpha-p)x+(\beta+p)x_1)} \)

and \( s = \frac{1}{\sqrt{(q^2-k^2)(\beta-q)x+(\alpha+q)x_2)((\alpha-q)x+(\beta+q)x_2)} \)

Here, \( x_3 = \frac{f}{F}, \quad y = \begin{bmatrix} f \\ F \end{bmatrix}, \) and

\[
\frac{t_2}{t} \frac{s}{s_1} = \frac{((\beta-p)x+(\alpha+p)x_1)((\alpha-p)x+(\beta+p)x_1)}{((\beta-q)x+(\alpha+q)x_2)((\alpha-q)x+(\beta+q)x_2)}.
\]
Example: Discrete sine-Gordon Equation

\[
xx_1x_2x_{12} - pq(xx_{12} - x_1x_2) - 1 = 0
\]

(H3) with \( \delta = 0 \) via extended Möbius transformation:

\[
x \rightarrow x, \quad x_1 \rightarrow x_1, \quad x_2 \rightarrow \frac{1}{x_2}, \quad x_{12} \rightarrow -\frac{1}{x_{12}}, \quad p \rightarrow \frac{1}{p}, \quad q \rightarrow q
\]

Discrete sine-Gordon equation is **NOT** consistent around the cube, but has a Lax pair!

Lax pair:

\[
L = \begin{bmatrix}
p & -kx_1 \\
-k & px_1 \\
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
\frac{qx_2}{x} & -\frac{1}{kx} \\
-x_2 & q \\
\end{bmatrix}
\]
Example: Lattice due to Hietarinta (2011)

\[ zx_1 - y_1 - x = 0 \]
\[ zx_2 - y_2 - x = 0 \]
\[ z_{12} - \frac{y}{x} - \frac{1}{x} \left( \frac{px_1 - qx_2}{z_1 - z_2} \right) = 0 \]

System has two single-edge equations and one full-face equation.

Lax pair:

\[
L = t \begin{bmatrix}
\frac{yz}{x} & \frac{k}{x} & \frac{kx - px_1 z - yzz_1}{x} \\
-x_1 z & z_1 & xz_1 \\
z & 0 & -z z_1
\end{bmatrix}
\]
and

\[
M = s \begin{bmatrix}
\frac{yz}{x} & k & \frac{kx-qx_2z-yzz_2}{x} \\
-x_2z & z_2 & xz_2 \\
z & 0 & -zz_2
\end{bmatrix},
\]

where \( t = s = \frac{1}{z} \), or \( t = \frac{1}{z_1}, s = \frac{1}{z_2} \),

or \( t = \sqrt[3]{\frac{x}{x_1z^2z_1}}, s = \sqrt[3]{\frac{x}{x_2z^2z_2}} \).

Here, \( x_3 = \frac{h}{G}, y_3 = \frac{g}{G}, z_3 = \frac{f}{G}, \psi = \begin{bmatrix} f \\ g \\ G \end{bmatrix} \), and

\( \frac{t_2}{t} \frac{s}{s_1} = \frac{z_1}{z_2} \).
• **Example:** Discrete Boussinesq System  
  (Tongas and Nijhoff 2005)

\[
\begin{align*}
  z_1 - xx_1 + y &= 0 \\
  z_2 - xx_2 + y &= 0 \\
  (x_2 - x_1)(z - xx_{12} + y_{12}) - p + q &= 0
\end{align*}
\]

• **Lax pair:**

\[
L = t
\begin{bmatrix}
  -x_1 & 1 & 0 \\
  -y_1 & 0 & 1 \\
  p - k - xy_1 + x_1z & -z & x
\end{bmatrix}
\]
\[
M = s \begin{bmatrix}
-x_2 & 1 & 0 \\
-y_2 & 0 & 1 \\
q - k - xy_2 + x_2z & -z & x
\end{bmatrix}
\]

with \( t = s = 1 \), or \( t = \frac{1}{\sqrt[3]{p-k}} \) and \( s = \frac{1}{\sqrt[3]{q-k}} \).

Here, \( x_3 = \frac{f}{F} \), \( y_3 = \frac{g}{F} \), \( \psi = \begin{bmatrix} f \\ F \\ g \end{bmatrix} \), and \( \frac{t_2}{t} \frac{s}{s_1} = 1 \).
• Example: System of pKdV Lattices

(Xenitidis and Mikhailov 2009)

\[(x - x_{12})(y_1 - y_2) - p^2 + q^2 = 0\]
\[(y - y_{12})(x_1 - x_2) - p^2 + q^2 = 0\]

• Lax pair:

\[
L = \begin{bmatrix}
0 & 0 & tx & t(p^2 - k^2 - xy_1) \\
0 & 0 & t & -ty_1 \\
Ty & T(p^2 - k^2 - x_1y) & 0 & 0 \\
T & -Tx_1 & 0 & 0
\end{bmatrix}
\]
\[
M = \begin{bmatrix}
0 & 0 & sx & s(q^2 - k^2 - xy_2) \\
0 & 0 & s & -sy_2 \\
Sy & S(q^2 - k^2 - x_2y) & 0 & 0 \\
S & -Sx_2 & 0 & 0 \\
\end{bmatrix}
\]

with \( t = s = T = S = 1 \),

or \( tT = \frac{1}{\sqrt{\text{Det}L_c}} = \frac{1}{p-k} \) and \( sS = \frac{1}{\sqrt{\text{Det}M_c}} = \frac{1}{q-k} \).

Here, \( x_3 = \frac{f}{F} \), \( y_3 = \frac{g}{G} \), \( \psi = [f \ F \ g \ G]^T \), and

\[
\frac{T_2}{T} \frac{S}{s_1} = 1 \quad \text{and} \quad \frac{T_2}{t} \frac{s}{S_1} = 1,
\]

or \( \frac{T_2}{\mathcal{T}} \frac{S}{S_1} = 1 \), with \( \mathcal{T} = tT, \ S = sS. \)
• Example: Discrete NLS System (Xenitidis and Mikhailov 2009)

\[ y_1 - y_2 - y \left( (x_1 - x_2)y + p - q \right) = 0 \]
\[ x_1 - x_2 + x_{12} \left( (x_1 - x_2)y + p - q \right) = 0 \]

• Lax pair:

\[
L = t \begin{bmatrix}
-1 & x_1 \\
y & k - p - xy_1
\end{bmatrix}
\]
\[
M = s \begin{bmatrix}
-1 & x_2 \\
y & k - q - xy_2
\end{bmatrix}
\]
with $t = s = 1$, or $t = \frac{1}{\sqrt{\text{Det} L_c}} = \frac{1}{\sqrt{\alpha-k}}$ and $s = \frac{1}{\sqrt{\beta-k}}$.

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$. 
• Example: Schwarzian-Boussinesq Lattice
  (Nijhoff 1999)

\[ y z_1 - x_1 + x = 0 \]
\[ y z_2 - x_2 + x = 0 \]
\[ z y_{12} (y_1 - y_2) - y (p y_2 z_1 - q y_1 z_2) = 0 \]

• Lax pair:

\[
L = t \begin{bmatrix}
y & 0 & -yz_1 \\
-kyy_1/z & pyz_1/z & 0 \\
0 & -1 & y_1
\end{bmatrix}
\]
\[
M = s \begin{bmatrix}
  y & 0 & -yz_2 \\
  -\frac{kyy_2}{z} & \frac{pyz_2}{z} & 0 \\
  0 & -1 & y_2 \\
\end{bmatrix}
\]

with \( t = s = \frac{1}{y} \), or \( t = \frac{1}{y_1} \) and \( s = \frac{1}{y_2} \),
or \( t = \sqrt[3]{\frac{z}{y^2y_1z_1}} \) and \( s = \sqrt[3]{\frac{z}{y^2y_2z_2}} \).

Here, \( x_3 = \frac{fy+Fx}{F} \), \( y_3 = \frac{g}{F} \), \( z_3 = \frac{f}{F} \), \( \psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix} \),
and \( \frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2} \).
**Example: Toda modified Boussinesq System**

(Nijhoff 1992)

\[
y_{12}(p - q + x_2 - x_1) - (p - 1)y_2 + (q - 1)y_1 = 0
\]

\[
y_1 y_2(p - q - z_2 + z_1) - (p - 1)yy_2 + (q - 1)yy_1 = 0
\]

\[
y(p + q - z - x_{12})(p - q + x_2 - x_1) - (p^2 + p + 1)y_1
\]

\[
+ (q^2 + q + 1)y_2 = 0
\]

**Lax pair:**

\[
L = t \begin{bmatrix}
k + p - z & \frac{1+k+k^2}{y} & -k^2y - y_1 - p^2(y_1 - yy_1 - kyy_1 + yzx_1) - kyy_1 + yzx_1 \\
0 & p - 1 & (1 - k)y_1 \\
1 & 0 & p - k - x_1
\end{bmatrix}
\]
\[
M = s \begin{bmatrix}
k + q - z & \frac{1+k+k^2}{y} & -k^2 y - y_2 - q^2(y_2 - y) - ky(x_2 - z) + yzx_2 \\
0 & q - 1 & 0 \\
1 & 0 & (1 - k)y_2 \\
\end{bmatrix}
\]

with \( t = s = 1 \), or \( t = 3\sqrt{\frac{y_1}{y}} \) and \( s = 3\sqrt{\frac{y_2}{y}} \).

Here, \( x_3 = \frac{f}{F} \), \( y_3 = \frac{g}{F} \), \( \psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix} \),

and \( \frac{t_2}{t} \frac{s}{s_1} = 1 \).