Invited Lecture 2

Symbolic Computation of Conserved Densities for Systems of Nonlinear Evolution Equations

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Outline Talk

• Purpose
• Motivation
• Other Programs
• Algorithm
• Implementation
• Examples
• Applications
• More Examples
• Scope and Limitations of Code condens.m
• Sample Data File and Output
• Conclusions and Future Work
• **Purpose**

Design and implement an algorithm to compute polynomial-type conservation laws for nonlinear systems of evolution equations

• **Conservation Laws**

Conservation law for a nonlinear PDE

\[ \rho_t + J_x = 0 \]

\( \rho \) is the density, \( J \) is the flux

Consider a single nonlinear evolution equation

\[ u_t = \mathcal{F}(u, u_x, u_{xx}, \ldots, u_{n,x}) \]

If \( \rho \) is a polynomial in \( u \) and its \( x \) derivatives, and does not depend explicitly on \( x \) and \( t \), then \( \rho \) is called a polynomial conserved density

If \( J \) is also polynomial in \( u \) and its \( x \) derivatives then this is called a polynomial conservation law

Consequently

\[ P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant} \]

provided \( J \) vanishes at infinity
Motivation

– Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics.

– or nonlinear PDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability.

– Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of PDEs and their solutions, e.g. Hamiltonian structure(s).


– Conservation laws can be used to test numerical integrators for PDEs.
For KdV equation, $u$ and $u^2$ are conserved quantities. Thus, a numerical scheme should preserves the quantities

$$\sum_j U_{j}^{n-1} = \sum_j U_{j}^{n}$$

and

$$\sum_j [U_{j}^{n-1}]^2 = \sum_j [U_{j}^{n}]^2$$

For two such schemes see Sanz-Serna, J. Comput. Phys. 47, 1982
• Conserved Densities Software

– Conserved densities programs **CONSD** and **SYMCD** by Ito and Kako (Reduce, 1985, 1994 & 1996)

– Conserved densities in **DELiA** by Bocharov (Pascal, 1990)

– Conserved densities and formal symmetries **FS** by Gerdt and Zharkov (Reduce, 1993)


– Conserved densities **condens.m** by Hereman and Göktaş (Mathematica, 1995)

– Conservation laws, based on **CRACK**, by Wolf (Reduce, 1995)

– Conserved densities by Ahner *et al.* (Mathematica, 1995)

Our program is available at ftp site: **mines.edu** in subdirectory

*pub/papers/math_cs_dept/software/condens*
Example

Consider the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities

$$\rho_1 = u, \quad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0$$

$$\rho_2 = u^2, \quad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - u_x^2)_x = 0$$

$$\rho_3 = u^3 - 3u_x^2,$$

$$\left( u^3 - 3u_x^2 \right)_t + \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x} \right)_x = 0$$

$$\vdots$$

$$\rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2$$

$$+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2, \quad \ldots \ldots \text{long} \ldots \ldots$$

$$\vdots$$
Note: KdV equation is invariant under the scaling symmetry

\[(x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u)\]

\(u\) and \(t\) carry the weight of 2, resp. 3 derivatives with respect to \(x\)

\[u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}\]
Key Idea behind Construction of Densities

Compute the building blocks of density with rank 6

(i) Take all the variables, except \( \frac{\partial}{\partial t} \), with positive weight

Here, only \( u \) with \( w(u) = 2 \)

List all possible powers of \( u \), up to rank 6

\[ [u, u^2, u^3] \]

Introduce \( x \) derivatives to ‘complete’ the rank

\( u \) has weight 2, so introduce \( \frac{\partial^4}{\partial x^4} \),

\( u^2 \) has weight 4, so introduce \( \frac{\partial^2}{\partial x^2} \),

\( u^3 \) has weight 6, so no derivative needed

(ii) Apply the derivatives

Remove terms that are total derivatives with respect to \( x \)
or total derivative up to terms kept earlier in the list

\[ [u_{4x}] \rightarrow [\ ] \text{ empty list} \]

\[ [u_x^2, uu_{2x}] \rightarrow [u_x^2] \quad (uu_{2x} = (uu_x)_x - u_x^2) \]

\[ [u^3] \rightarrow [u^3] \]
Combine the ‘building blocks’

\[ \rho = u^3 + c_1 u_x^2 \]

the constant \( c_1 \) must be determined

(iii) Determine the unknown coefficients (\( c_1 \))

1. Compute \( \frac{\partial \rho}{\partial t} = 3u^2 u_t + 2c_1 u_x u_{xt} \),

2. Replace \( u_t \) by \(-(uu_x + u_{3x})\) and \( u_{xt} \) by \(-(uu_x + u_{3x})_x\)

3. Integrate the result with respect to \( x \)

   Carry out all integrations by parts

   \[
   \frac{\partial \rho}{\partial t} = -\left[ \frac{3}{4} u^4 + (c_1 - 3) uu_x^2 + 3u^2 u_{2x} - c_1 u_{2x}^2 + 2c_1 u_x u_{3x} \right]_x
   - (c_1 + 3) u_x^3,
   \]

4. The non-integrable (last) term must vanish. Thus, \( c_1 = -3 \)

Result:

\[ \rho = u^3 - 3u_x^2 \]

Expression [...] yields

\[
J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x}
\]
• Algorithm and Implementation

Consider a system of \( N \) nonlinear evolution equations

\[
  u_{i,t} + \mathcal{F}_i(u_j, u_j^{(1)}, \ldots, u_j^{(n)}) = 0 \quad i, j = 1, 2, \ldots, N
\]

where \( u_{i,t} \triangleq \frac{\partial u_i}{\partial t}, \quad u_i^{(n)} \triangleq \frac{\partial^n u_i}{\partial x^n} \)

All \( u_i \) depend on \( x \) and \( t \)

Algorithm consists of three major steps

1. Determine weights (scaling properties) of variables & parameters

2. Construct the form of the density (building blocks)

3. Determine the unknown numerical coefficients
Procedure to determine the weights (scaling properties)

Define

weight of a variable: the number of partial derivatives with respect to $x$ the variable carries

rank of a term: the total weight of that term in terms of partial derivatives with respect to $x$

For example:

$$u_t \rightarrow r_{1,1} = w(u) + w\left(\frac{\partial}{\partial t}\right)$$

$$uu_x \rightarrow r_{1,2} = 2w(u) + 1$$

$$u_{3x} \rightarrow r_{1,3} = w(u) + 3$$

where $r_{i,k}$ denotes the rank of the $k^{th}$ term in the $i^{th}$ equation

$w$ denotes the weight of its argument

Uniformity in rank requires

$$r_{1,1} = r_{1,2} = r_{1,3}$$

Thus

$$w(u) = 2, \quad w\left(\frac{\partial}{\partial t}\right) = 3$$
Require that all terms in any particular equation have the same rank
(\textit{uniformity in rank})

Different equations in the same system may have different ranks
Introduce the following notations:

- \( w \) returns the weight of its argument
- \( g \) returns the degree of nonlinearity of its argument
- \( d \) returns the number of partial derivatives with respect to its argument
- \( r_{i,k} \) denotes the rank of the \( k^{th} \) term in the \( i^{th} \) equation

Pick

\[
 w\left(\frac{\partial}{\partial x}\right) = 1, \ldots, w\left(\frac{\partial^n}{\partial x^n}\right) = n
\]

All weight are assumed nonnegative and rational

List of ‘variables’ that carry weights

\[
 \left\{ \frac{\partial}{\partial t}, u_1, u_2, \ldots, u_N, p_1, p_2, \ldots, p_P \right\}
\]
Step 1  Take the $i^{th}$ equation with $K_i$ terms

Step 2  For each of its terms compute the rank

$$r_{i,k} = d(x) + d(t) \ w(\frac{\partial}{\partial t}) + \sum_{j=1}^{N} g(u_j) \ w(u_j) + \sum_{j=1}^{P} g(p_j) \ w(p_j)$$

$k = 1, 2, \ldots, K$

If the variable $u_j$ and/or the parameter $p_j$ is missing then $g(u_j) = 0$ or $g(p_j) = 0$, or both

Step 3  Use uniformity in rank in the $i^{th}$ equation

Form the linear system

$$A_i = \{r_{i,1} = r_{i,2} = \cdots = r_{i,K_i}\}$$

Step 4  Repeat steps (1) through (3) for all equations

Step 5  Gather the equations $A_i$

Form the global linear system

$$\mathcal{A} = \bigcup_{i=1}^{N} A_i$$

Step 6  Solve $\mathcal{A}$ for the $N + P + 1$ unknowns $w(u_j)$, $w(p_j)$ and $w(\frac{\partial}{\partial t})$
Example

Consider the Boussinesq equation

\[ u_{tt} - u_{2x} + uu_{2x} + u_x^2 + a u_{4x} = 0 \]

with nonzero parameter \( a \).

Can be written as a system of evolution equations

\[ u_{1,t} + u_2' = 0 \]

\[ u_{2,t} + u_1' - u_1u_1' - a u_1^{(3)} = 0 \]

In the second equation

\[ u_1' \text{ and } a u_1^{(3)} \]

do not allow for uniformity in rank.

Introduce an auxiliary parameter \( b \) with weight and replace the system by

\[ u_{1,t} + u_2' = 0 \]

\[ u_{2,t} + b u_1' - u_1u_1' - a u_1^{(3)} = 0 \]
Determine ranks and weights

\begin{align*}
    r_{1,1} &= 1 \ w\left(\frac{\partial}{\partial t}\right) + 1 \ w(u_1) \\
    r_{1,2} &= 1 + 1 \ w(u_2) \\
    r_{2,1} &= 1 \ w\left(\frac{\partial}{\partial t}\right) + 1 \ w(u_2) \\
    r_{2,2} &= 1 + 1 \ w(u_1) + 1 \ w(b) \\
    r_{2,3} &= 1 + 2w(u_1) \\
    r_{2,4} &= 3 + 1 \ w(u_1)
\end{align*}

Uniformity in rank for each equation requires

\begin{align*}
    A_1 &= \{r_{1,1} = r_{1,2}\} \\
    A_2 &= \{r_{2,1} = r_{2,2} = r_{2,3} = r_{2,4}\} \\
    \text{and} \quad A &= A_1 \cup A_2
\end{align*}

Solve \( A \) for \( w(u_1), w(u_2), w\left(\frac{\partial}{\partial t}\right) \) and \( w(b) \)

\begin{align*}
    w(u_1) &= 2, \quad w(b) = 2, \quad w(u_2) = 3 \quad \text{and} \quad w\left(\frac{\partial}{\partial t}\right) = 2
\end{align*}

or

\begin{align*}
    u_1 &\sim b \sim \frac{\partial^2}{\partial x^2}, \quad u_2 \sim \frac{\partial^3}{\partial x^3}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}
\end{align*}
• **Construct the Form of the Density**

Let \( \mathcal{V} = \{v_1, v_2, \ldots, v_Q\} \) be the sorted list (descending weights) of all variables, including all parameters, but excluding \( \frac{\partial}{\partial t} \).

**Step 1** Form all combinations of variables of rank \( R \) or less

Recursively, form sets consisting of ordered pairs

\[(T_{q,s}; W_{q,s})\]

where \( T_{q,s} \) denotes a combination of different powers of the variables
and \( W_{q,s} \) denotes the total weight of \( T_{q,s} \)

\( q \) refers to the variable \( v_q \)

\( s \) refers to the allowable power of \( v_q \) such that \( W_{q,s} \leq R \)
Set $\mathcal{B}_0 = \{(1; 0)\}$ and proceed as follows:

**For** $q = 1$ through $Q$ **do**

**For** $m = 0$ through $M - 1$ **do**

Form $B_{q,m} = \bigcup_{s=0}^{b_{q,m}} \{(T_{q,s}; W_{q,s})\}$

$M$ is the number of pairs in $\mathcal{B}_{q-1}$

$T_{q,s} = T_{q-1,m} v_q^s$

$W_{q,s} = W_{q-1,m} + s w(v_q)$

$(T_{q-1,m}; W_{q-1,m})$ is the $(m + 1)^{st}$ ordered pair in $\mathcal{B}_{q-1}$

$b_{q,m} = \left\lfloor \frac{R - W_{q-1,m}}{w(v_q)} \right\rfloor$ is the maximum allowable power of $v_q$

Set $\mathcal{B}_q = \bigcup_{m=0}^{M-1} B_{q,m}$

**Step 2** Set $\mathcal{G} = \mathcal{B}_Q$

Note: $\mathcal{G}$ has all possible combinations of powers of variables that produce rank $R$ or less
Step 3  Introduce partial derivatives with respect to $x$

For each pair $(T_{Q,s}; W_{Q,s})$ in $\mathcal{G}$, apply $\frac{\partial^\ell}{\partial x^\ell}$ to the term $T_{Q,s}$ provided $\ell = R - W_{Q,s}$ is integer

This introduces just enough partial derivatives with respect to $x$
so that all the pairs retain weight $R$

Gather in set $\mathcal{H}$ all the terms that result from computing $\frac{\partial^\ell(T_{Q,s})}{\partial x^\ell}$

Step 4  Remove the terms from $\mathcal{H}$ that can be written as a derivative with respect to $x$, or as a derivative up to terms kept prior in the set

Call the resulting set $\mathcal{I}$, which consists of the building blocks of the density $\rho$ with desired rank $R$

Step 5  If $\mathcal{I}$ has $I$ elements, then their linear combination will produce the polynomial density of rank $R$

$$\rho = \sum_{i=1}^{I} c_i \mathcal{I}(i)$$

$\mathcal{I}(i)$ denotes the $i^{th}$ element in $\mathcal{I}$

$c_i$ are numerical coefficients, still to be determined
Example

Return to the Boussinesq equation, where

\[ w(u_1) = 2, \ w(b) = 2, \ \text{and} \ w(u_2) = 3 \]

For example, construct the density with rank \( R = 6 \)
\[ \mathcal{V} = \{ u_2, u_1, b \} \]

Hence, \( v_1 = u_2, \ v_2 = u_1, \ v_3 = b \) and \( Q = 3 \)

**Step 1** For \( q = 1, \ m = 0 \):

\[ b_{1,0} = \left[ \frac{6}{3} \right] = 2 \]

Thus, with \( T_{1,s} = u_2^s \), and \( W_{1,s} = 3s \), where \( s = 0, 1, 2 \)
we obtain

\[ B_1 = B_{1,0} = \{ (1; 0), (u_2; 3), (u_2^2; 6) \} \]

For \( q = 2, \ m = 0 \):

\[ b_{2,0} = \left[ \frac{6-0}{2} \right] = 3 \]

So, with \( T_{2,s} = u_1^s \), and \( W_{2,s} = 2s \), with \( s = 0, 1, 2, 3 \)
we obtain

\[ B_{2,0} = \{ (1; 0), (u_1; 2), (u_1^2; 4), (u_1^3; 6) \} \]
For $q = 2$, $m = 1$:

\[ B_{2,1} = \{(u_2; 3), (u_1 u_2; 5)\} \text{ since } b_{2,1} = \left\lceil \frac{6-3}{2} \right\rceil = 1 \]

\[ T_{2,s} = u_2 u_1^s \]

and

\[ W_{2,s} = 3 + 2s, \text{ and } s = 0, 1 \]

For $q = 2$, $m = 2$:

\[ b_{2,2} = \left\lceil \frac{6-6}{2} \right\rceil = 0 \]

Therefore $B_{2,2} = \{(u_2^2; 6)\}$

Hence,

\[ \mathcal{B}_2 = \{(1; 0), (u_1; 2), (u_1^2; 4), (u_1^3; 6), (u_2; 3), (u_1 u_2; 5), (u_2^2; 6)\} \]

For $q = 3$: introduce possible powers of $b$

An analogous procedure leads to

\[ B_{3,0} = \{(1; 0), (b; 2), (b^2; 4), (b^3; 6)\} \quad B_{3,4} = \{(u_2; 3), (b u_2; 5)\} \]

\[ B_{3,1} = \{(u_1; 2), (b u_1; 4), (b^2 u_1; 6)\} \quad B_{3,5} = \{(u_1 u_2; 5)\} \]

\[ B_{3,2} = \{(u_1^2; 4), (b u_1^2; 6)\} \quad B_{3,6} = \{(u_2^2; 6)\} \]

\[ B_{3,3} = \{(u_1^3; 6)\} \]

Thus

\[ \mathcal{B}_3 = \{(1; 0), (b; 2), (b^2; 4), (b^3; 6), (u_1; 2), (b u_1; 4), (b^2 u_1; 6), (u_1^2; 4), (b u_1^2; 6), (u_1^3; 6), (u_2; 3), (b u_2; 5), (u_1 u_2; 5), (u_2^2; 6)\} \]
Step 2  Set $\mathcal{G} = \mathcal{B}_3$

Step 3  Apply derivatives to the first components of the pairs in $\mathcal{G}$

Compute $\ell$ for each pair of $\mathcal{G}$:

$$\ell = 6, 4, 2, 0, 4, 2, 0, 2, 0, 0, 3, 1, 1,$$ and 0

Gather the terms after applying partial derivatives w.r.t. $x$

Hence

$$\mathcal{H} = \{0, b^3, u_1^{(4)}, bu_1^{(2)}, b^2u_1, (u_1')^2, u_1u_1^{(2)},$$

$$bu_1^2, u_1^3, u_2^{(3)}, bu_2', u_1u_2', u_1'u_2, u_2^2\}$$

Step 4  Remove from $\mathcal{H}$ the terms that can be written as a derivative with respect to $x$ or as a derivative up to terms retained earlier in that set

This gives

$$\mathcal{I} = \{b^2u_1, bu_1^2, u_1^3, u_2^2, u_1'u_2, (u_1')^2\}$$

Step 5  Combine these building blocks and form $\rho$ of rank 6

$$\rho = c_1 b^2u_1 + c_2 bu_1^2 + c_3 u_1^3 + c_4 u_2^2 + c_5 u_1'u_2 + c_6 (u_1')^2$$
Calculus of Variations
provides a useful tool to verify if an expression is a derivative

**Theorem**

If

\[ f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)}) \]

then

\[ \mathcal{L}_{\vec{y}}(f) \equiv \vec{0} \]

if and only if

\[ f = \frac{d}{dx} g \]

where

\[ g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)}) \]

Notations:

\[ \vec{y} = [y_1, \ldots, y_N]^T \]

\[ \mathcal{L}_{\vec{y}}(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T \]

\[ \vec{0} = [0, \ldots, 0]^T \]

\((T \text{ for transpose})\)
and **Euler Operator:**

\[ \mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx}(\frac{\partial}{\partial y_i'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial y_i''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial y_i^{(n)}}) \]

Proof: see Olver (1986, pp. 252)

**• Determine the Unknown Coefficients**

**Step 1** Compute \( \frac{\partial \rho}{\partial t} \)

Replace all \((u_{i,t})^{(j)}, \ i, j = 0, 1, \ldots\) from the given system

**Step 2** The resulting expression \( E \) must be the total derivative of some functional \( -J \)

Two options:

- Integrate by parts
  
  Isolate the non-integrable part

  Set it equal to zero

  The latter leads to a linear system for the coefficients \( c_i \) to be solved

- Use the Euler-Lagrange equations

  Apply the Euler operator

\[ \mathcal{L}_{u_i} = \frac{\partial}{\partial u_i} - \frac{d}{dx}(\frac{\partial}{\partial u_i'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial u_i''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial u_i^{(n)}}) \]

  to \( E \)
If $E$ is completely integrable no terms will be left, i.e.

$$\mathcal{L}_{u_1}(E) \equiv 0, \ldots, \mathcal{L}_{u_N}(E) \equiv 0$$

otherwise set the remaining terms equal to zero and form the linear system for the coefficients $c_i$

With either option, construct a linear system, denoted by $S$

**Step 3** Two cases may occur, depending on whether or not there are parameters in the system

**Case I:**

If the only unknowns in $S$ are $c_i$’s, just solve $S$ for $c_i$’s

Substitute the nonempty solution into $\rho$ to get its final form

**Case II:**

If in addition to the coefficients $c_i$’s there are also parameters $p_i$ in $S$

Determine the conditions on the parameters so that $\rho$ of the given form exists for at least some $c_i$’s nonzero
These **compatibility conditions** assure that the system has other than trivial solutions

- Set $\mathcal{C} = \{c_1, c_2, \ldots, c_I\}$ and $i = 1$
- **While** $\mathcal{C} \neq \{\}$ **do**:
  - For the building block $\mathcal{I}(i)$ with coefficient $c_i$ to appear in $\rho$, one needs $c_i \neq 0$
  - Therefore, set $c_i = 1$ and eliminate all the other $c_j$ from $\mathcal{S}$
  - This gives compatibility conditions consistent with the presence of the term $c_i \mathcal{I}(i)$ in $\rho$
If compatibility conditions require that some of the parameters are zero

then

\( c_i \) must be zero since parameters are assumed to be nonzero

Hence, set \( C = C \setminus \{c_i\} \), and \( i = i' \)

where \( i' \) is the smallest index of the \( c_j \) that remain in \( C \)

else

Solve the compatibility conditions and for each resulting branch

Solve the system \( S \) for \( c_j \)

Substitute the solution into \( \rho \) to obtain its final form

Collect those \( c_j \) which are zero for \textit{all} of the branches into a set \( Z \)

The \( c_i \) in \( Z \) might not have occurred in any density yet

Give them a chance to occur:

Set \( C = C \cap Z \), and \( i = i' \)

where \( i' \) is the smallest index of the \( c_j \) that are still in \( C \)
• Example

Consider the parameterized coupled KdV equations (Hirota-Satsuma)

\[
\begin{align*}
  u_t - 6\alpha uu_x + 6vv_x - \alpha u_{3x} &= 0 \\
  v_t + 3uv_x + v_{3x} &= 0
\end{align*}
\]

Here, \( w(u) = w(u) = 2 \) and the form of the density of rank 4 is

\[
\rho = c_1 u^2 + c_2 uv + c_3 v^2 = c_1 u_1^2 + c_2 u_1 u_2 + c_3 u_2^2
\]

**Step 1** Compute \( \rho_t \) and replace all \((u_{i,t})^{(j)}\) to get

\[
E = 2c_1u_1 \left( 6\alpha u_1u_1^{(1)} - 6u_2u_2^{(1)} + \alpha u_1^{(3)} \right) \\
+ c_2u_2 \left( 6\alpha u_1u_1^{(1)} - 6u_2u_2^{(1)} + \alpha u_1^{(3)} \right) \\
- c_2u_1 \left( 3u_1u_2^{(1)} + u_2^{(3)} \right) - 2c_3u_2 \left( 3u_1u_2^{(1)} + u_2^{(3)} \right)
\]

**Step 2** Either integrate by parts or apply the Euler operator

Get the linear system for the coefficients \( c_1, c_2 \) and \( c_3 \)

\[
S = \{(1 + \alpha)c_2 = 0, \ 2c_1 + c_3 = 0\}
\]

Obviously, \( C = \{c_1, c_2, c_3\} \) with one parameter \( \alpha \)
Step 3  Search for compatibility conditions

- Set $c_1 = 1$, which gives

$$\{c_1 = 1, \; c_2 = 0, \; c_3 = -2\}$$

without any constraint on the parameter $\alpha$

Since only $c_2 = 0$, $\mathcal{Z} = \{c_2\}$ and $\mathcal{C} = \mathcal{C} \cap \mathcal{Z} = \{c_2\}$, with $i' = 2$

- Set $c_2 = 1$

This leads to the compatibility condition $\alpha = -1$, and

$$\{c_1 = -\frac{1}{2} c_3, \; c_2 = 1\}$$

Since $\mathcal{Z} = \{\}$ the procedure ends

One gets two densities of rank 4, one without any constraint on $\alpha$, one with a constraint

In summary:

$$\rho = u_1^2 - 2u_2^2$$

and

$$\rho = -\frac{1}{2} c_3 u_1^2 + u_1 u_2 + c_3 u_2^2$$

with compatibility condition $\alpha = -1$
Search for densities of rank $R \leq 8$

**Rank 2:** No condition on $\alpha$

One always has the trivial density $\rho = u$

**Rank 4:** At this level, two branches emerge

1. No condition on $\alpha$
   \[
   \rho = u^2 - 2v^2
   \]

2. For $\alpha = -1$
   \[
   \rho = uv + c \left( v^2 - \frac{1}{2}u^2 \right), \quad c \text{ is free}
   \]
Rank 6: No condition on $\alpha$ and

$$\rho = u^3 - \frac{3}{\alpha + 1} uv^2 - \frac{1}{2} u_x^2 + \frac{3}{\alpha + 1} v_x^2, \quad \alpha \neq -1$$

Rank 8: The system has conserved density

$$\rho = u^4 - \frac{12}{5} u^2 v^2 + \frac{12}{5} v^4 - 2 uu_x^2 - \frac{24}{5} uv_x^2 - \frac{8}{5} v^2 u_{2x} + \frac{1}{5} u_{2x}^2 + \frac{8}{5} v_{2x}^2$$

provided that $\alpha = \frac{1}{2}$

Therefore, $\alpha = \frac{1}{2}$ (integrable case!) appears in the computation of density of rank 8

For $\alpha = \frac{1}{2}$, we also computed the density of Rank 10

$$\rho = -\frac{7}{20} u^5 + u^3 v^2 - uv^4 - \frac{7}{4} u^2 u_x^2 + \frac{1}{2} v^2 u_x^2 + u^2 v_x^2$$

$$+ 4 v^2 v_x^2 + uv^2 u_{2x} + v_x^2 u_{2x} - \frac{7}{20} uu_{2x}^2 - 2 uv_{2x}^2 + \frac{1}{40} u_{3x}^2$$

$$+ \frac{2}{5} v_{3x}^2 + \frac{1}{10} v^2 u_{4x}$$
A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters

\[ u \sim \frac{\partial^2}{\partial x^2} \]

Special cases:

- \( \alpha = 30 \) \( \beta = 20 \) \( \gamma = 10 \) Lax
- \( \alpha = 5 \) \( \beta = 5 \) \( \gamma = 5 \) Sawada–Kotera
  or Caudry–Dodd–Gibbon
- \( \alpha = 20 \) \( \beta = 25 \) \( \gamma = 10 \) Kaup–Kupershmidt
- \( \alpha = 2 \) \( \beta = 6 \) \( \gamma = 3 \) Ito

Under what conditions for the parameters \( \alpha, \beta, \) and \( \gamma \) does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)
  \[ \rho = u^2 \]
– **Rank 6:**
Condition:

\[ 10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2 \]

(Lax, SK, and KK cases)

\[ \rho = u^3 + \frac{15}{(-2\beta + \gamma)}u_x^2 \]

– **Rank 8:**

1. \( \beta = 2\gamma \) (Lax and Ito cases)

\[ \rho = u^4 - \frac{6\gamma}{\alpha}uu_x^2 + \frac{6}{\alpha}u_{2x}^2 \]

2. \( \alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45} \) (SK, KK and Ito cases)

\[ \rho = u^4 - \frac{135}{2\beta + \gamma}uu_x^2 + \frac{675}{(2\beta + \gamma)^2}u_{2x}^2 \]

– **Rank 10:**
Condition:

\[ \beta = 2\gamma \]

and

\[ 10\alpha = 3\gamma^2 \]

(Lax case)

\[ \rho = u^5 - \frac{50}{\gamma}u^2u_x^2 + \frac{100}{\gamma^2}uu_{2x}^2 - \frac{500}{7\gamma^3}u_{3x}^2. \]
What are the necessary conditions for the parameters $\alpha, \beta,$ and $\gamma$ so that this equation could admit $\infty$ many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If 
  \[
  \alpha = -\frac{2\beta^2 - 7\beta \gamma + 4\gamma^2}{45}
  \]
  or 
  \[
  \beta = 2\gamma
  \]
  then there is a conserved density of rank 8

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)
Application 2

A Class of Seventh-Order Evolution Equations

\[ u_t + au^3u_x + bu^3_x + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0 \]

where \( a, b, c, d, e, f, g \) are nonzero parameters

\[ u \sim \frac{\partial^2}{\partial x^2} \]

Special cases:

**SK – Ito Case** \( a = 252, \ b = 63, \ c = 378, \ d = 126, \)

\[ e = 63, \quad f = 42, \quad g = 21, \]

**Lax Case** \( a = 140, \ b = 70, \ c = 280, \ d = 70, \)

\[ e = 70, \quad f = 42, \quad g = 14 \]
What are the necessary conditions for the parameters so that this equation could admit \( \infty \) many polynomial conservation laws?

Combine the conditions **Rank 2** through **Rank 8**:

- If \( a = \frac{5}{98} g^3 \), \( b = \frac{5}{14} g^2 \), \( c = \frac{10}{7} g^2 \), \( d = \frac{5}{14} g^2 \), \( e = 5g \), \( f = 3g \) then there is a sequence (without gaps!) of conserved densities (Lax case)

- If \( a = \frac{4}{147} g^3 \), \( b = \frac{1}{7} g^2 \), \( c = \frac{6}{7} g^2 \), \( d = \frac{2}{7} g^2 \), \( e = 3g \), \( f = 2g \) then there is a sequence (with gaps!) of conserved densities (SK-Ito case)

- What if \( a = \frac{4}{147} g^3 \), \( b = \frac{5}{14} g^2 \), \( c = \frac{9}{7} g^2 \), \( d = \frac{2}{7} g^2 \), \( e = 6g \), \( f = \frac{7}{2} g \)?

*This case is not mentioned in the literature!*

With \( g = 42 \) first five densities

\[
\begin{align*}
\rho_1 &= u, \\
\rho_2 &= -8u^3 + u_x^2, \\
\vdots \\
\rho_5 &= -\frac{480}{53}u^7 + \frac{3780}{53}u^4u_x^2 + \frac{861}{106}uu_x^4 \\
&\quad -\frac{644}{53}u^3u_{2x}^2 - \frac{291}{212}u^2u_xu_{2x}^2 - \frac{737}{318}uu_{2x}^3 \\
&\quad +u^2u_{3x}^2 + \frac{133}{636}u_{2x}u_{3x}^2 - \frac{2}{53}uu_{4x}^2 + \frac{1}{1908}u_{5x}^2
\end{align*}
\]

Extension of Kaup-Kupershmidt case? YES, proof by Sanders
• More Examples

• Nonlinear Schrödinger Equation

\[ iq_t - q_{2x} + 2|q|^2q = 0 \]

Program can not handle complex equations
Replace by

\[ u_t - v_{2x} + 2v(u^2 + v^2) = 0 \]
\[ v_t + u_{2x} - 2u(u^2 + v^2) = 0 \]

where \( q = u + iv \)

Scaling properties

\[ u \sim v \sim \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2} \]

First seven conserved densities:

\[ \rho_1 = u^2 + v^2 \]
\[ \rho_2 = vu_x \]
\[ \rho_3 = u^4 + 2u^2v^2 + v^4 + u_x^2 + v_x^2 \]
\[ \rho_4 = u^2vu_x + \frac{1}{3}v^3u_x - \frac{1}{6}vu_{3x} \]
\[ \rho_5 = -\frac{1}{2} u^6 - \frac{3}{2} u^4 v^2 - \frac{3}{2} u^2 v^4 - \frac{1}{2} v^6 - \frac{5}{2} u^2 u_x^2 - \frac{1}{2} v^2 u_x^2 - \frac{3}{2} u^2 v_x^2 - \frac{5}{2} v^2 v_x^2 + u v^2 u_{2x} - \frac{1}{4} u_{2x}^2 - \frac{1}{4} v_{2x}^2 \]

\[ \rho_6 = -\frac{3}{4} u^4 v u_x - \frac{1}{2} u^2 v^3 u_x - \frac{3}{20} v^5 u_x + \frac{1}{4} v u_x^3 - \frac{1}{4} v u_x v_x^2 + u v u_x u_{2x} + \frac{1}{4} u^2 v u_{3x} + \frac{1}{12} v^3 u_{3x} - \frac{1}{40} v u_{5x} \]

\[ \rho_7 = \frac{5}{4} u^8 + 5 u^6 v^2 + \frac{15}{2} u^4 v^4 + 5 u^2 v^6 + \frac{5}{4} v^8 + \frac{35}{2} u^4 u_x^2 - 5 u^2 v^2 u_x^2 + \frac{5}{2} v^4 u_x^2 - \frac{7}{4} u_x^4 + \frac{15}{2} u^4 v_x^2 + 25 u^2 v^2 v_x^2 + \frac{35}{2} v^4 v_x^2 - \frac{5}{2} u_x^2 v_x^2 - \frac{7}{4} v_x^4 - 10 u^3 v^2 u_{2x} - 5 u v^4 u_{2x} - 5 u v v_x^2 u_{2x} + \frac{7}{2} u^2 u_{2x}^2 + \frac{1}{2} v^2 u_{2x}^2 + \frac{5}{2} u^2 v_{2x}^2 + \frac{7}{2} v^2 v_{2x}^2 \]
• The Ito system

\[
\begin{align*}
  u_t - u_{3x} - 6uu_x - 2vv_x &= 0 \\
  v_t - 2u_xv - 2uv_x &= 0
\end{align*}
\]

\[
\begin{align*}
  u &\sim \frac{\partial^2}{\partial x^2} \\
  v &\sim \frac{\partial^2}{\partial x^2}
\end{align*}
\]

\[
\begin{align*}
  \rho_1 &= c_1u + c_2v \\
  \rho_2 &= u^2 + v^2 \\
  \rho_3 &= 2u^3 + 2uv^2 - u_x^2 \\
  \rho_4 &= 5u^4 + 6u^2v^2 + v^4 - 10uu_x^2 + 2v^2u_{2x} + u_{2x}^2 \\
  \rho_5 &= 14u^5 + 20u^3v^2 + 6uv^4 - 70uu_x^2 + 10v^2u_x^2 \\
  &\quad - 4v^2v_x^2 + 20uv^2u_{2x} + 14uu_{2x}^2 - u_{3x}^2 + 2v^2u_{4x}
\end{align*}
\]

and more conservation laws
• The dispersiveless long-wave system

\[
\begin{align*}
    u_t + vu_x + uv_x &= 0 \\
    v_t + u_x + vv_x &= 0
\end{align*}
\]

\(u\) free, \(v\) free, but \(u \sim 2v\)

choose \(u \sim \frac{\partial}{\partial x}\) and \(2v \sim \frac{\partial}{\partial x}\)

\[
\begin{align*}
    \rho_1 &= v \\
    \rho_2 &= u \\
    \rho_3 &= uv \\
    \rho_4 &= u^2 + uv^2 \\
    \rho_5 &= 3u^2v + uv^3 \\
    \rho_6 &= \frac{1}{3}u^3 + u^2v^2 + \frac{1}{6}uv^4 \\
    \rho_7 &= u^3v + u^2v^3 + \frac{1}{10}uv^5 \\
    \rho_8 &= \frac{1}{3}u^4 + 2u^3v^2 + u^2v^4 + \frac{1}{15}uv^6
\end{align*}
\]

and more

Always the same set irrespective the choice of weights
A generalized Schamel equation

\[ n^2 u_t + (n + 1)(n + 2)u^n u_x + u_{3x} = 0 \]

where \( n \) is a positive integer

\[ \rho_1 = u, \quad \rho_2 = u^2 \]

\[ \rho_3 = \frac{1}{2} u_x^2 - \frac{n^2}{2} u^{2+\frac{2}{n}} \]

For \( n \neq 1, 2 \) no further conservation laws
**Three-Component Korteweg-de Vries Equation**

\[
\begin{align*}
    u_t - 6uu_x + 2vv_x + 2ww_x - u_{3x} &= 0 \\
    v_t - 2vu_x - 2uv_x &= 0 \\
    w_t - 2wu_x - 2uw_x &= 0
\end{align*}
\]

Scaling properties

\[
    u \sim v \sim w \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}
\]

First five densities:

\[
\begin{align*}
    \rho_1 &= c_1u + c_2v + c_3w \\
    \rho_2 &= u^2 - v^2 - w^2 \\
    \rho_3 &= -2u^3 + 2uv^2 + 2uw^2 + u_x^2 \\
    \rho_4 &= -\frac{5}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 + 3u^2w^2 - v^2w^2 - \frac{1}{2}w^4 \\
    &\quad + 5uu_x^2 + v^2u_{2x} + w^2u_{2x} - \frac{1}{2}u_{2x}^2 \\
    \rho_5 &= -\frac{7}{10}u^5 + u^3v^2 - \frac{3}{10}uv^4 + u^3w^2 - \frac{3}{5}uv^2w^2 - \frac{3}{10}uw^4 \\
    &\quad + \frac{7}{2}u^2u_x^2 + \frac{1}{2}v^2u_x^2 + \frac{1}{2}w^2u_x^2 + \frac{1}{5}v^2v_x^2 \\
    &\quad - \frac{1}{5}w^2v_x^2 + \frac{1}{5}w^2w_x^2 + uv^2u_{2x} + uw^2u_{2x} - \frac{7}{10}uu_{2x}^2 \\
    &\quad - \frac{1}{5}vw^2v_{2x} + \frac{1}{20}u_{3x}^2 + \frac{1}{10}v^2u_{4x} + \frac{1}{10}w^2u_{4x}
\end{align*}
\]
• A modified vector derivative NLS equation

$$\frac{\partial B_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 B_\perp) + \alpha B_\perp B_\perp^0 \cdot \frac{\partial B_\perp}{\partial x} + e_x \times \frac{\partial^2 B_\perp}{\partial x^2} = 0$$

Replace the vector equation by

$$u_t + (u(u^2 + v^2) + \beta u - v_x)_x = 0$$
$$v_t + (v(u^2 + v^2) + u)_x = 0$$

\(u\) and \(v\) denote the components of \(B_\perp\) parallel and perpendicular to \(B_\perp^0\) and \(\beta = \alpha B_\perp^2\)

$$u^2 \sim \frac{\partial}{\partial x}, \quad v^2 \sim \frac{\partial}{\partial x}, \quad \beta \sim \frac{\partial}{\partial x}$$

First 6 conserved densities

$$\rho_1 = c_1 u + c_2 v$$
$$\rho_2 = u^2 + v^2$$
$$\rho_3 = \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2$$
$$\rho_4 = \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$
\[\rho_5 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{2x} - u_{2x} v_x) + \frac{4}{5}(u u_x + v v_x)^2 + \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(u v_x - u_x v) + \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4\]

\[\rho_6 = \frac{7}{16}(u^2 + v^2)^5 + \frac{1}{2}(u_{2x}^2 + v_{2x}^2) - \frac{5}{2}(u^2 + v^2)(u_x v_{2x} - u_{2x} v_x) + 5(u^2 + v^2)(u u_x + v v_x)^2 + \frac{15}{4}(u^2 + v^2)^2(u_x^2 + v_x^2) - \frac{35}{16}(u^2 + v^2)^3(u v_x - u_x v) + \frac{\beta}{8}(5u^8 + 10u^6 v^2 - 10u^2 v^6 - 5v^8 + 20u^2 u_x^2 - 12u^5 v_x + 60u v^4 v_x - 20v^2 v_x^2) + \frac{\beta^2}{4}(u^6 + v^6)\]
• **Scope and Limitations**

– Systems must be polynomial in dependent variables

– Only two independent variables ($x$ and $t$) are allowed

– No terms should *explicitly* depend on $x$ and $t$

– Program only computes polynomial-type conserved densities
  only polynomials in the dependent variables and their derivatives
  no explicit dependencies on $x$ and $t$

– No limit on the number of evolution equations
  In practice: time and memory constraints

– Input systems may have (nonzero) parameters
  Program computes the conditions for parameters such that densities (of a given rank) might exist

– Systems can also have parameters with (unknown) weight
  Allows one to test systems with non-uniform rank

– For systems where one or more of the weights are free
  Program prompts the user to enter values for the free weights

– Negative weights are not allowed

– Fractional weights are permitted

– Form of $\rho$ can be given (testing purposes)
• Sample Data and Output

Data file for the Hirota-Satsuma system

\[ u_t - 6 \alpha uu_x + 6vv_x - \alpha u_{3x} = 0 \]
\[ v_t + 3uv_x + v_{3x} = 0 \]

(* start of data file d_phrsat.m *)

dep = False;

(* Hirota-Satsuma System *)

eq[1][x,t] = D[u[1][x,t],t]-aa*D[u[1][x,t],{x,3}]-
6*aa*u[1][x,t]*D[u[1][x,t],x]+6*u[2][x,t]*D[u[2][x,t],x];

eq[2][x,t] = D[u[2][x,t],t]+D[u[2][x,t],{x,3}]+3*u[1][x,t]*D[u[2][x,t],x];

noeqs = 2;
name = "Hirota-Satsuma System (parameterized)";
parameters = {aa};
weightpars = {};
formrho[x,t] = {};

(* end of data file d_phrsat.m *)
Explanation of the lines in the data file

debug = False;
No trace of output. Set it True to see a detailed trace

eq[k][x,t] = ...;
Give the $k^{th}$ equation of the system in Mathematica notation
Note that there is no == 0 at the end

noeqs = 2;
Specifies the number of equations in the system

name = "Hirota-Satsuma System (parameterized)";
Pick a short name for the system. The quotes are essential

parameters = {aa};
Give a list of the parameters in the system
If there are none, set parameters = {};

weightpars = {};
Give a list of those parameters that are assumed to have weights
Weighted parameters are listed in weightpars, not in parameters
The latter is only a list of weightless parameters

formrho[x,t] = {};
Unless the form of $\rho$ is given, program will automatically compute it
This option allows to test forms of $\rho$ (from the literature)
Anything within (\* and \*) are comments (ignored by Mathematica)

For testing purposes, the form of the density can be given

For example:

\[
\text{formrho}[x,t]=\{c[1]*u[1][x,t]^3+c[2]*D[u[1][x,t],x]^2\};
\]

Density $\rho$ must be given in expanded form and with coefficients $c[i]$

The braces are essential

If $\rho$ is given, the program skips both the computation of scaling properties and the construction of $\rho$

Program continues with given $\rho$, and computes the $c[i]$

For search for densities of specific rank, set \text{formrho}[x,t] = \{ \};
**Menu Interface and Sample Output**

**Example:** Compute $\rho$ of rank 4 for Drinfel’d-Sokolov system

\[ u_t + 3vv_x = 0 \]
\[ v_t + 2v_{3x} + 2uv_x + u_xv = 0 \]

Start *Mathematica*

Type

```math
In[1]:= <<condens.m
```

Menu interface: program prompts automatically for answers

```plaintext
*** MENU INTERFACE *** (page: 3)
```

21) Kaup-Broer System (d_broer.m)
22) Drinfel’d-Sokolov System (d_soko.m)
23) Dispersiveless Long Wave System (d_disper.m)
24) 3-Component KdV System (d_3ckdv.m)
25) 2-Component Nonlinear Schrodinger Eq. (d_2cnls.m)
26) Boussinesq System (d_bous.m)

nn) Next Page
tt) Your System
qq) Exit the Program
```
ENTER YOUR CHOICE: 22

Enter the rank of rho: 4

Use Variational Derivative instead of Integration by Parts? (y/n): y

Enter the name of the output file: d_soko4.o
This is the density: \( u[2][x, t] \)

This is the flux:

\[
2 u[1][x, t] u[2][x, t] - 2 (u[2])^{(1,0)} [x, t] + (2,0) > 4 u[2][x, t] (u[2])^{(2,0)} [x, t]
\]

Result of explicit verification \( (\rho_t + J_x) = 0 \)
In[2] :=

At the end of computation, density and flux are available

To see these, type

In[2] := rho[x,t]

Out[2] = u[2][x, t]

In[3] := j[x,t]

Out[3] = 2 u[1][x, t] u[2][x, t] - (1,0) 2 (u[2]) [x, t] + (2,0) > 4 u[2][x, t] (u[2]) [x, t]
• Conclusions & Further Research

– Comparison with other programs

* Parameter analysis
* Not restricted to uniform rank
* Not restricted to evolution equations provided that one can write the equation(s) as a system of evolution equations

– Usefulness

* Testing models for integrability
* Study of classes of nonlinear PDEs
* Study of generalized symmetries

– Future work

* Generalization towards broader classes of equations
* Generalization towards non-local conservation laws
* Conservation laws with variable coefficients
* Interface issues between Mathematica, Maple and Reduce programs
<table>
<thead>
<tr>
<th>Density</th>
<th>Sawada-Kotera equation</th>
<th>Lax equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>$\frac{1}{2}u^2$</td>
<td></td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$</td>
<td>$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>$\frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{3}{4}u_x^2$</td>
<td>$\frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{1}{20}u_x^2$</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>$\frac{1}{5}u^5 - \frac{2}{5}u^2u_x^2 + \frac{1}{5}uu_x^2 - \frac{1}{50}u_x^2$</td>
<td></td>
</tr>
<tr>
<td>$\rho_6$</td>
<td>$\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_x^2$ + $2u_x^2$</td>
<td>$\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_x^2$ + $2u_x^2$</td>
</tr>
<tr>
<td>$\rho_7$</td>
<td>$\frac{1}{7}u^7 - \frac{9}{7}u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u^3u_x^2$ + $\frac{648}{35}u_x^2u_x^2$ + $\frac{489}{35}uu_x^2$ + $\frac{261}{35}u^2u_x^2$ + $\frac{288}{35}u_x^2u_x^2$ + $\frac{81}{35}uu_x^2$ + $\frac{9}{35}u^2$</td>
<td>$\frac{1}{7}u^7 - \frac{9}{7}u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u^3u_x^2$ + $\frac{648}{35}u_x^2u_x^2$ + $\frac{489}{35}uu_x^2$ + $\frac{261}{35}u^2u_x^2$ + $\frac{288}{35}u_x^2u_x^2$ + $\frac{81}{35}uu_x^2$ + $\frac{9}{35}u^2$</td>
</tr>
<tr>
<td>$\rho_8$</td>
<td>$\frac{1}{8}u^8 - \frac{7}{4}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_x^2$ + $\frac{7}{4}uu_x^2u_x^2$ + $\frac{5}{3}u^2u_x^3$ + $\frac{7}{21}u_x^4$ + $\frac{1}{7}u^3u_x^2$</td>
<td>$\frac{1}{8}u^8 - \frac{7}{4}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_x^2$ + $\frac{7}{4}uu_x^2u_x^2$ + $\frac{5}{3}u^2u_x^3$ + $\frac{7}{21}u_x^4$ + $\frac{1}{7}u^3u_x^2$</td>
</tr>
</tbody>
</table>

**Table 1: Conserved Densities for the Sawada-Kotera and Lax 5th-order equations**
<table>
<thead>
<tr>
<th>Density</th>
<th>Kaup-Kuperschmidt equation</th>
<th>Ito equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>$\frac{1}{8}u_x^2$</td>
<td>$\frac{u^2}{2}$</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>$\frac{u^4}{4} - \frac{9}{16}u u_x^2 + \frac{3}{64}u_2x^2$</td>
<td>$\frac{u^4}{4} - \frac{9}{4}u u_x^2 + \frac{3}{4}u_2x^2$</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>$\frac{u^4}{4} - \frac{9}{16}u u_x^2 + \frac{3}{64}u_2x^2$</td>
<td>$\frac{u^4}{4} - \frac{9}{4}u u_x^2 + \frac{3}{4}u_2x^2$</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>$\frac{u^6}{6} - \frac{35}{16}u^3 u_x^2 - \frac{31}{256}u^4 + \frac{51}{64}u^2 u_2x^2$</td>
<td>$\frac{u^6}{6} - \frac{35}{16}u^3 u_x^2 - \frac{31}{256}u^4 + \frac{51}{64}u^2 u_2x^2$</td>
</tr>
<tr>
<td>$\rho_6$</td>
<td>$\frac{u^6}{6} - \frac{35}{16}u^3 u_x^2 - \frac{31}{256}u^4 + \frac{51}{64}u^2 u_2x^2$</td>
<td>$\frac{u^6}{6} - \frac{35}{16}u^3 u_x^2 - \frac{31}{256}u^4 + \frac{51}{64}u^2 u_2x^2$</td>
</tr>
<tr>
<td>$\rho_7$</td>
<td>$\frac{u^7}{7} - \frac{27}{8}u^4 u_x^2 - \frac{369}{320}u u_x^4 + \frac{69}{60}u^3 u_2x^2$</td>
<td>$\frac{u^7}{7} - \frac{27}{8}u^4 u_x^2 - \frac{369}{320}u u_x^4 + \frac{69}{60}u^3 u_2x^2$</td>
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<td>$\frac{u^7}{7} - \frac{27}{8}u^4 u_x^2 - \frac{369}{320}u u_x^4 + \frac{69}{60}u^3 u_2x^2$</td>
<td>$\frac{u^7}{7} - \frac{27}{8}u^4 u_x^2 - \frac{369}{320}u u_x^4 + \frac{69}{60}u^3 u_2x^2$</td>
</tr>
<tr>
<td>Density</td>
<td>Sawada-Kotera-Ito equation</td>
<td>Lax equation</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------</td>
<td>--------------</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>$-u^2$</td>
<td>$u^2$</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>$-u^3 + u_x^2$</td>
<td>$-2u^3 + u_x^2$</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>$3u^4 - 9uu_x^2 + u_x^2$</td>
<td>$5u^4 - 10uu_x^2 + u_x^2$</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>$-\frac{12}{7}u^6 + \frac{150}{7}u^3u_x^2 + \frac{17}{7}u_x^4 - \frac{48}{7}u^2u_{2x}^2$</td>
<td>$-\frac{7}{3}u^6 + \frac{70}{3}u^3u_x^2 + \frac{35}{18}u_x^4 - \frac{7}{9}u^2u_{2x}^2$</td>
</tr>
<tr>
<td>$\rho_6$</td>
<td>$-\frac{16}{21}u_x^3 + uu_{3x}^2 - \frac{1}{21}u_{4x}^2$</td>
<td>$-\frac{10}{9}u_{2x}^3 + uu_{3x}^2 - \frac{1}{18}u_{4x}^2$</td>
</tr>
<tr>
<td>$\rho_7$</td>
<td>$5u^7 - 105u^4u_x^2 - 42uu_x^4 + \frac{133}{3}u^3u_x^2$</td>
<td>$\frac{7}{3}u^7 + \frac{35}{3}u^4u_x^2 + \frac{35}{9}uu_x^4 - \frac{14}{3}u^3u_{2x}^2$</td>
</tr>
<tr>
<td>$\rho_8$</td>
<td>$+24u_x^2u_{2x}^2 + \frac{163}{9}uu_{2x}^2 - \frac{29}{3}u^2u_{3x}^2$</td>
<td>$-\frac{7}{3}u_x^2u_{2x}^2 - \frac{20}{9}uu_{2x}^3 + u^2u_{3x}^2$</td>
</tr>
<tr>
<td>$\rho_8$</td>
<td>$-\frac{32}{9}u_x^2u_{3x}^2 + uu_{4x}^2 - \frac{1}{27}u_{5x}^2$</td>
<td>$\frac{5}{9}u_{2x}u_{3x}^2 - \frac{1}{9}uu_{4x}^2 + \frac{1}{198}u_{5x}^2$</td>
</tr>
<tr>
<td>$\rho_8$</td>
<td>$\frac{3}{2}u^8 - 42u^5u_x^2 - 35u^2u_x^4 + 21u^4u_{2x}^2$</td>
<td>$\frac{7}{11}u_{2x}u_{4x}^2 - \frac{1}{11}uu_{5x}^2 + \frac{1}{280}u_{6x}^2$</td>
</tr>
</tbody>
</table>