

Solving Nonlinear Wave Equations and Lattices with Mathematica

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OUTLINE

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Purpose & Motivation

- **Develop** and implement various **methods** to find exact (closed form) solutions of nonlinear PDEs and DDEs: Lie symmetry methods, similarity methods, etc.
- Fully **automate** the tanh and sech methods to compute closed form solitary wave solutions of nonlinear (systems) of partial differential equations (PDEs) and differential-difference equations (DDEs or lattices).
- **Class** of nonlinear PDEs and DDEs solvable with the tanh/sech method includes famous evolution and wave equations.
Typical examples: Korteweg-de Vries, Fisher and Boussinesq PDEs, Toda and Volterra lattices (DDEs).
- Solutions of tanh or sech type **model** solitary waves in fluid dynamics, plasmas, electrical circuits, optical fibers, bio-genetics, etc.
- **Benchmark** solutions for numerical PDE solvers.
- **Research aspect**: Design a high-quality application package for the computation of exact solitary wave solutions of large classes of nonlinear evolution and wave equations.
- **Educational aspect**: Software as course ware for courses in nonlinear PDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software. REU Projects.
- **Users**: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.

Typical Examples of Single PDEs and Systems of PDEs

- The Korteweg-de Vries (KdV) equation:

$$u_t + \alpha u u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \Delta],$$

or, equivalently,

$$u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \operatorname{sech}^2 [c_1 x + c_2 t + \Delta].$$

- The modified Korteweg-de Vries (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x, t) = \pm \sqrt{\frac{6}{\alpha}} c_1 \operatorname{sech} [c_1 x - c_1^3 t + \Delta].$$

- Three-dimensional modified Korteweg-de Vries equation:

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

Solitary wave solution:

$$u(x, y, z, t) = \pm \sqrt{c_2 c_3} \operatorname{sech} [c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta].$$

- The combined KdV-mKdV equation:

$$u_t + 6\alpha uu_x + 6\beta u^2 u_x + \gamma u_{3x} = 0.$$

Real solitary wave solution:

$$u(x, t) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\gamma}{\beta}} c_1 \operatorname{sech}(c_1 x + \frac{c_1}{2\beta}(3\alpha^2 - 2\beta\gamma c_1^2)t + \Delta).$$

Complex solutions:

$$u(x, t) = -\frac{\alpha}{2\beta} \pm i \sqrt{\frac{\gamma}{\beta}} c_1 \tanh(c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + 4\beta\gamma c_1^2)t + \Delta),$$

$$u(x, t) = -\frac{\alpha}{2\beta} + \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 (\operatorname{sech}\xi \pm i \tanh\xi),$$

and

$$u(x, t) = -\frac{\alpha}{2\beta} - \frac{1}{2} \sqrt{\frac{\gamma}{\beta}} c_1 (\operatorname{sech}\xi \mp i \tanh\xi)$$

with $\xi = c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + \beta\gamma c_1^2)t + \Delta$.

- The Fisher equation:

$$u_t - u_{xx} - u(1 - u) = 0.$$

Solitary wave solution:

$$u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh\xi + \frac{1}{4} \tanh^2\xi,$$

with

$$\xi = \pm \frac{1}{2\sqrt{6}} x \pm \frac{5}{12} t + \Delta.$$

- The generalized Kuramoto-Sivashinski equation:

$$u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0.$$

Solitary wave solutions

(ignoring symmetry $u \rightarrow -u, x \rightarrow -x, \sigma \rightarrow -\sigma$):

For $\sigma = 4$:

$$u(x, t) = 9 - 2c_2 - 15 \tanh \xi (1 + \tanh \xi - \tanh^2 \xi)$$

with $\xi = \frac{x}{2} + c_2 t + \Delta$.

For $\sigma = \frac{12}{\sqrt{47}}$:

$$u(x, t) = \frac{45 \mp 4418c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh \xi - \frac{45}{47\sqrt{47}} \tanh^2 \xi \pm \frac{15}{47\sqrt{47}} \tanh^3 \xi$$

with $\xi = \pm \frac{1}{2\sqrt{47}} x + c_2 t + \Delta$.

For $\sigma = 16/\sqrt{73}$:

$$u(x, t) = \frac{2(30 \mp 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \tanh \xi - \frac{60}{73\sqrt{73}} \tanh^2 \xi \pm \frac{15}{73\sqrt{73}} \tanh^3 \xi$$

with $\xi = \pm \frac{1}{2\sqrt{73}} x + c_2 t + \Delta$.

For $\sigma = 0$:

$$u(x, t) = -2\sqrt{\frac{19}{11}}c_2 - \frac{135}{19}\sqrt{\frac{11}{19}}\tanh \xi + \frac{165}{19}\sqrt{\frac{11}{19}}\tanh^3 \xi$$

with $\xi = \frac{1}{2}\sqrt{\frac{11}{19}} x + c_2 t + \Delta$.

- The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0,$$

or written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x, t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta], \\ v(x, t) &= b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta]. \end{aligned}$$

- The Broer-Kaup system:

$$\begin{aligned} u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xxy} &= 0, \\ v_t + 2(uv)_x + v_{xx} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x, t) &= -\frac{c_3}{2c_1} + c_1 \tanh [c_1 x + c_2 y + c_3 t + \Delta], \\ v(x, t) &= c_1 c_2 - c_1 c_2 \tanh^2 [c_1 x + c_2 y + c_3 t + \Delta]. \end{aligned}$$

- System of three nonlinearly coupled equations (Gao & Tian, 2001):

$$u_t - u_x - 2v = 0,$$

$$v_t + 2uw = 0,$$

$$w_t + 2uv = 0.$$

Solutions:

$$u(x, t) = \pm c_2 \tanh \xi,$$

$$v(x, t) = \mp \frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2 \xi,$$

$$w(x, t) = -\frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2 \xi,$$

and

$$u(x, t) = \pm i c_2 \operatorname{sech} \xi,$$

$$v(x, t) = \pm \frac{1}{2} i c_2 (c_1 - c_2) \tanh \xi \operatorname{sech} \xi,$$

$$w(x, t) = \frac{1}{4} c_2 (c_1 - c_2) (1 - 2 \operatorname{sech}^2 \xi),$$

and also

$$u(x, t) = \pm \frac{1}{2} i c_2 (\operatorname{sech} \xi + i \tanh \xi),$$

$$v(x, t) = \pm \frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech} \xi (\operatorname{sech} \xi + i \tanh \xi),$$

$$w(x, t) = -\frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech} \xi (\operatorname{sech} \xi + i \tanh \xi)$$

with $\xi = c_1 x + c_2 t + \Delta$.

- Nonlinear sine-Gordon equation (light cone coordinates):

$$\Phi_{xt} = \sin \Phi.$$

Set $u = \Phi_x$, $v = \cos(\Phi) - 1$,

$$\begin{aligned} u_{xt} - u - uv &= 0, \\ u_t^2 + 2v + v^2 &= 0. \end{aligned}$$

Solitary wave solution (kink):

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{-c}} \operatorname{sech}\left[\frac{1}{\sqrt{-c}}(x - ct) + \Delta\right], \\ v &= 1 - 2 \operatorname{sech}^2\left[\frac{1}{\sqrt{-c}}(x - ct) + \Delta\right]. \end{aligned}$$

Solution:

$$\Phi(x, t) = \int u(x, t) dx = \pm 4 \arctan \left[\exp \left(\frac{1}{\sqrt{-c}}(x - ct) + \Delta \right) \right].$$

- ODEs from quantum field theory:

$$\begin{aligned} u_{xx} &= -u + u^3 + auv^2, \\ v_{xx} &= bv + cv^3 + av(u^2 - 1). \end{aligned}$$

Solitary wave solutions:

$$\begin{aligned} u &= \pm \tanh\left[\sqrt{\frac{a^2 - c}{2(a - c)}}x + \Delta\right], \\ v &= \pm \sqrt{\frac{1 - a}{a - c}} \operatorname{sech}\left[\sqrt{\frac{a^2 - c}{2(a - c)}}x + \Delta\right], \end{aligned}$$

provided $b = \sqrt{\frac{a^2 - c}{2(a - c)}}$.

Typical Examples of DDEs (lattices)

- The Toda lattice:

$$\ddot{u}_n = (1 + \dot{u}_n) (u_{n-1} - 2u_n + u_{n+1}).$$

Solitary wave solution:

$$u_n(t) = a_0 \pm \sinh(c_1) \tanh [c_1 n \pm \sinh(c_1) t + \Delta].$$

- The Volterra lattice:

$$\begin{aligned}\dot{u}_n &= u_n(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n).\end{aligned}$$

Solitary wave solution:

$$\begin{aligned}u_n(t) &= -c_2 \coth(c_1) + c_2 \tanh [c_1 n + c_2 t + \Delta], \\ v_n(t) &= -c_2 \coth(c_1) - c_2 \tanh [c_1 n + c_2 t + \Delta].\end{aligned}$$

- The Relativistic Toda lattice:

$$\begin{aligned}\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).\end{aligned}$$

Solitary wave solution:

$$\begin{aligned}u_n(t) &= -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh [c_1 n + c_2 t + \Delta], \\ v_n(t) &= \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh [c_1 n + c_2 t + \Delta].\end{aligned}$$

Algorithm for Tanh Solutions for system of PDEs

Given: System of nonlinear PDEs of order m

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \dots, \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable \mathbf{u} has M components u_i (or u, v, w, \dots).

Independent variable \mathbf{x} has N components x_j (or x, y, z, \dots, t).

Step T1:

- Seek solution $\mathbf{u}(\mathbf{x}) = \mathbf{U}(T)$, with

$$T = \tanh \xi = \tanh \left[\sum_j^N c_j x_j + \Delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, *all* derivative of T are polynomial in T . For example, $T'' = -2T(1 - T^2)$, etc.
- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dT} \frac{\partial T}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{dT}.$$

Produces a nonlinear system of ODEs

$$\Delta(T, \mathbf{U}(T), \mathbf{U}'(T), \mathbf{U}''(T), \dots, \mathbf{U}^{(m)}(T)) = \mathbf{0}.$$

Note: Compare with the ultraspherical (linear) ODE:

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0$$

with integer $n \geq 0$ and α real. Includes:

- * Legendre equation ($\alpha = \frac{1}{2}$),
- * ODE for Chebyshev polynomials of type I ($\alpha = 0$),
- * ODE for Chebyshev polynomials of type II ($\alpha = 1$).

- Example: For the Boussinesq system

$$\begin{aligned}u_t + v_x &= 0, \\v_t + u_x - 3uu_x - \alpha u_{3x} &= 0,\end{aligned}$$

after cancelling common factors $1 - T^2$,

$$\begin{aligned}c_2U' + c_1V' &= 0, \\c_2V' + c_1U' - 3c_1UU' \\+ \alpha c_1^3 [2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2U'''] &= 0.\end{aligned}$$

Step T2:

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij}T^j.$$

Determine the highest exponents $M_i \geq 1$.

Substitute $U_i(T) = T^{M_i}$ into the LHS of ODE.

Gives polynomial $\mathbf{P}(T)$.

For every P_i consider all possible balances of the highest exponents in T .

Solve the resulting linear system(s) for the unknowns M_i .

- Example: Balance highest exponents for the Boussinesq system

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.$$

So, $M_1 = M_2 = 2$.

Hence,

$$\begin{aligned}U(T) &= a_{10} + a_{11}T + a_{12}T^2, \\V(T) &= a_{20} + a_{21}T + a_{22}T^2.\end{aligned}$$

Step T3:

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in T .
- Example: Algebraic system for Boussinesq case

$$a_{11} c_1 (3a_{12} + 2\alpha c_1^2) = 0,$$

$$a_{12} c_1 (a_{12} + 4\alpha c_1^2) = 0,$$

$$a_{21} c_1 + a_{11} c_2 = 0,$$

$$a_{22} c_1 + a_{12} c_2 = 0,$$

$$a_{11} c_1 - 3a_{10} a_{11} c_1 + 2\alpha a_{11} c_1^3 + a_{21} c_2 = 0,$$

$$-3a_{11}^2 c_1 + 2a_{12} c_1 - 6a_{10} a_{12} c_1 + 16\alpha a_{12} c_1^3 + 2a_{22} c_2 = 0.$$

Step T4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for Boussinesq system

$$a_{10} = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0,$$

$$a_{12} = -4\alpha c_1^2, \quad a_{20} = \text{free},$$

$$a_{21} = 0, \quad a_{22} = 4\alpha c_1 c_2.$$

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Solitary wave solution for Boussinesq system:

$$u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta],$$

$$v(x, t) = a_{20} + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta].$$

Algorithm for Sech Solutions for system of PDEs

Given: System of PDEs of order m

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \dots, \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable \mathbf{u} has M components u_i (or u, v, w, \dots).

Independent variable \mathbf{x} has N components x_j (or x, y, z, \dots, t).

Step S1:

- Seek solution $u_i(\mathbf{x}) = U_i(S)$, with

$$S = \operatorname{sech} \xi = \operatorname{sech} \left[\sum_j^N c_j x_j + \Delta \right].$$

- Observe $(\operatorname{sech} \xi)' = -\tanh \xi \operatorname{sech} \xi$ or $S' = -TS = -\sqrt{1-S^2} S$.
- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dS} \frac{\partial S}{\partial x_j} = -c_j S \sqrt{1-S^2} \frac{d \bullet}{dS}.$$

Leads to coupled system of nonlinear ODEs

$$\mathbf{\Gamma}(S, \mathbf{U}(S), \mathbf{U}'(S), \dots) + \sqrt{1-S^2} \mathbf{\Pi}(S, \mathbf{U}(S), \mathbf{U}'(S), \dots) = \mathbf{0}.$$

All components of $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ are polynomial ODEs.

First case: $\mathbf{\Gamma} = \mathbf{0}$ or $\mathbf{\Pi} = \mathbf{0}$.

$$\mathbf{\Delta}(S, \mathbf{U}(S), \mathbf{U}'(S), \dots) = \mathbf{0}.$$

$\mathbf{\Delta}$ stands for either $\mathbf{\Gamma}$ or $\mathbf{\Pi}$.

Note: All terms in the given system of PDE must be of even or odd order.

- Example: For the 3D mKdV equation

$$u_t + 6u^2u_x + u_{xyz} = 0,$$

after cancelling a common factor $-\sqrt{1-S^2}S$,

$$c_4U' + 6c_1U^2U' + c_1c_2c_3[(1-6S^2)U' + 3S(1-2S^2)U'' + S^2(1-S^2)U'''] = 0.$$

Step S2:

- Seek polynomial solutions

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij}S^j.$$

Substitute $U_i(S) = S^{M_i}$ and balance the highest power terms in S to determine M_i .

- Example: Balance of exponents for the 3D mKdV case

$$3M_1 - 1 = M_1 + 1.$$

So, $M_1 = 1$. Hence,

$$U(S) = a_{10} + a_{11}S.$$

Step S3:

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in S .
- Example: Algebraic system for 3D mKdV case

$$a_{11}c_1(a_{11}^2 - c_2c_3) = 0,$$

$$a_{11}(6a_{10}^2c_1 + c_1c_2c_3 + c_4) = 0,$$

$$a_{10}a_{11}^2c_1 = 0.$$

Step S4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for 3D mKdV case

$$\begin{aligned}a_{10} &= 0, \\a_{11} &= \pm\sqrt{c_1 c_3}, \\c_4 &= -c_1 c_2 c_3.\end{aligned}$$

Step S5:

- Return to the original variables. Test the final solution(s). Reject trivial solutions.
- Example: Solitary wave solution for the 3D mKdV equation

$$u(x, y, z, t) = \pm\sqrt{c_2 c_3} \operatorname{sech}(c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t).$$

Second case: $\Gamma \neq \mathbf{0}$ and $\Pi \neq \mathbf{0}$.

$$\Gamma(S, \mathbf{U}(S), \mathbf{U}'(S), \dots) + \sqrt{1 - S^2} \Pi(S, \mathbf{U}(S), \mathbf{U}'(S), \dots) = \mathbf{0}.$$

Most general solution

$$U_i(S) = \sum_{j=0}^{\tilde{M}_i} \sum_{k=0}^{\tilde{N}_i} \tilde{a}_{i,jk} S^j T^k.$$

Double series is not necessary! Solution can be rearranged as

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j + T \sum_{j=0}^{N_i} b_{ij} S^j.$$

Algorithm for Mixed Tanh/Sech Solutions for PDEs

Step ST1:

- Seek solution in $u_i(\mathbf{x}) = U_i(S)$, with

$$S = \operatorname{sech}\xi = \operatorname{sech} \left[\sum_j^N c_j x_j + \Delta \right].$$

Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dS} \frac{\partial S}{\partial x_j} = -c_j S \sqrt{1 - S^2} \frac{d \bullet}{dS}.$$

- Example: Coupled system due to Gao and Tian (2001)

$$u_t - u_x - 2v = 0,$$

$$v_t + 2uw = 0,$$

$$w_t + 2uv = 0,$$

transforms into

$$(c_1 - c_2)S\sqrt{1 - S^2}U' - 2V = 0,$$

$$c_2S\sqrt{1 - S^2}V' - 2UW = 0,$$

$$c_2S\sqrt{1 - S^2}W' - 2UV = 0.$$

Step ST2:

- Seek solution

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j + \sqrt{1 - S^2} \sum_{j=0}^{N_i} b_{ij} S^j.$$

First, determine the leading exponents M_i, N_i . Substitute

$$U_i(S) = a_{i0} + a_{iM_i} S^{M_i} + \sqrt{1 - S^2} (b_{i0} + b_{iN_i} S^{N_i})$$

to get

$$\mathbf{P}(S) + \sqrt{1 - S^2} \mathbf{Q}(S).$$

\mathbf{P} and \mathbf{Q} are polynomials.

Consider possible balances of the highest exponents in P_i and Q_i .

Get a linear system of $2M$ (or less) equations for the $2M$ unknown M_i and N_i .

No longer assume $M_i \geq 1, N_i \geq 1$ (some M_i or N_i may be zero).

Trouble. Strongly underdetermined problem. Set *all* $M_i = 2$ and $N_i = 1$.

- Example: Quadratic solutions in S and T only.

Substitute

$$\begin{aligned} U(S) &= a_{10} + a_{11}S + a_{12}S^2 + \sqrt{1 - S^2} (b_{10} + b_{11}S), \\ V(S) &= a_{20} + a_{21}S + a_{22}S^2 + \sqrt{1 - S^2} (b_{20} + b_{21}S), \\ W(S) &= a_{30} + a_{31}S + a_{32}S^2 + \sqrt{1 - S^2} (b_{30} + b_{31}S). \end{aligned}$$

leads to

$$\mathbf{P}(S) + \sqrt{1 - S^2} \mathbf{Q}(S) = \mathbf{0},$$

\mathbf{P} and \mathbf{Q} are polynomials.

Step ST3:

- Derive the algebraic system for the coefficients a_{ij}, b_{ij} by setting to zero the coefficients of power terms in S in $\mathbf{P} = \mathbf{0}$ and $\mathbf{Q} = \mathbf{0}$ separately.
- Example: Algebraic system has 25 equations (not shown).

Step ST4:

- Solve the nonlinear algebraic system with parameters.
- Example: 11 solutions in total: 3 are trivial ($U_i = \text{constant}$), 8 are nontrivial.

Step ST5:

- Return to the original variables. Test the final solution(s). Reject trivial (constant) solutions.
- Example: Solitary wave solutions:

$$\begin{aligned}u(x, t) &= \pm c_2 \tanh \xi, \\v(x, t) &= \mp \frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2 \xi, \\w(x, t) &= -\frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2 \xi,\end{aligned}$$

(could have been obtained with tanh-method), and

$$\begin{aligned}u(x, t) &= \pm i c_2 \operatorname{sech} \xi, \\v(x, t) &= \pm \frac{1}{2} i c_2 (c_1 - c_2) \tanh \xi \operatorname{sech} \xi, \\w(x, t) &= \frac{1}{4} c_2 (c_1 - c_2) (1 - 2 \operatorname{sech}^2 \xi),\end{aligned}$$

and also

$$\begin{aligned}u(x, t) &= \pm \frac{1}{2} i c_2 (\operatorname{sech} \xi + i \tanh \xi), \\v(x, t) &= \pm \frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech} \xi (\operatorname{sech} \xi + i \tanh \xi), \\w(x, t) &= -\frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech} \xi (\operatorname{sech} \xi + i \tanh \xi).\end{aligned}$$

plus the c.c. solutions.

In all solutions $\xi = c_1 x + c_2 t + \Delta$.

Algorithm for Tanh Solutions for system of DDEs

Given: System of nonlinear differential-difference equations (DDEs) of order m

$$\Delta(\dots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \dots, \dot{\mathbf{u}}_n, \dots, \mathbf{u}_n^{(m)}) = 0.$$

Dependent variable \mathbf{u}_n has M components $u_{i,n}$ (or u_n, v_n, w_n, \dots)

Independent variable \mathbf{x} has 2 components x_i (or n, t).

No derivatives on shifted variables!

Step D1:

- Seek solution $\mathbf{u}_n(t) = \mathbf{U}_n(T)$, with

$$T = T_n(t) = \tanh [c_1 n + c_2 t + \Delta].$$

- Note: The argument of T depends on n .
- Repeatedly apply the operator rule

$$\frac{d\bullet}{dt} = \frac{d\bullet}{dT} \frac{dT}{dt} = c_2(1 - T^2) \frac{d\bullet}{dT}.$$

Produces a nonlinear system of type

$$\Delta(T, \dots, \mathbf{U}_{n-1}, \mathbf{U}_n, \mathbf{U}_{n+1}, \dots, \mathbf{U}'_n, \mathbf{U}''_n, \dots, \mathbf{U}_n^{(m)}) = \mathbf{0}.$$

- Example: Toda lattice

$$\ddot{u}_n = (1 + \dot{u}_n) (u_{n-1} - 2u_n + u_{n+1})$$

transforms into

$$c_2^2(1-T^2) [2TU'_n - (1-T^2)U''_n] + [1 + c_2(1-T^2)U'_n] [U_{n-1} - 2U_n + U_{n+1}] = 0.$$

Step D2:

- Seek polynomial solutions

$$U_{i,n} = \sum_{j=0}^{M_i} a_{ij} T_n^j.$$

Use

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

to deal with the shift:

$$U_{i,n\pm p} = \sum_{j=0}^{M_i} a_{i,j} T_{n\pm p}^j = \sum_{j=0}^{M_i} a_{i,j} \left[\frac{T_n \pm \tanh(pc_1)}{1 \pm T_n \tanh(pc_1)} \right]^j.$$

Substitute $U_{i,n} = T_n^{M_i}$, and

$$U_{i,n\pm p} = T_{n\pm p}^{M_i} = \left[\frac{T_n \pm \tanh(pc_1)}{1 \pm T_n \tanh(pc_1)} \right]^{M_i},$$

and balance the potential highest exponents in T_n to determine M_i .

Note: $U_{i,n\pm p}$ is homogeneous of degree zero in T .

- Example: Balance of exponents for Toda lattice

$$2M_1 - 1 = M_1 + 1.$$

So, $M_1 = 1$.

Hence,

$$U_n = a_{10} + a_{11} T_n,$$
$$U_{n\pm 1} = a_{10} + a_{11} T_{n\pm 1} = a_{10} + a_{11} \frac{T_n \pm \tanh(c_1)}{1 \pm T_n \tanh(c_1)}.$$

Step D3:

- Determine the algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the powers in T_n .
- Example: Algebraic system for Toda lattice

$$\begin{aligned}c_2^2 - \tanh^2(c_1) - a_{11}c_2 \tanh^2(c_1) &= 0, \\c_2 - a_{11} &= 0.\end{aligned}$$

Step D4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution of algebraic system for Toda lattice

$$\begin{aligned}a_{10} &= \text{free}, \\a_{11} &= \pm \sinh(c_1), \\c_2 &= \pm \sinh(c_1).\end{aligned}$$

Step D5:

- Return to the original variables. Test solution(s) of DDE. Reject trivial ones.
- Example: Solitary wave solution for Toda lattice:

$$u_n(t) = a_0 \pm \sinh(c_1) \tanh [c_1 n \pm \sinh(c_1) t + \Delta].$$

Example: System of DDEs: Relativistic Toda lattice

$$\begin{aligned}\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).\end{aligned}$$

Change of variables

$$u_n(t) = U_n(T_n), \quad v_n(t) = V_n(T_n),$$

with

$$T_n(t) = \tanh [c_1 n + c_2 t + \Delta].$$

gives

$$\begin{aligned}c_2(1 - T^2)U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) &= 0, \\ c_2(1 - T^2)V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) &= 0.\end{aligned}$$

Seek polynomial solutions

$$U_n = \sum_{j=0}^{M_1} a_{1j} T_n^j, \quad V_n = \sum_{j=0}^{M_2} a_{2j} T_n^j.$$

Balance the highest exponents in T_n to determine M_1 , and M_2 :

$$M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2.$$

So, $M_1 = M_2 = 1$. Hence,

$$U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n.$$

Algebraic system for a_{ij} :

$$\begin{aligned}-a_{11} c_2 + a_{21} \tanh(c_1) + \alpha a_{10} a_{21} \tanh(c_1) &= 0, \\ a_{11} \tanh(c_1) (\alpha a_{21} + c_2) &= 0, \\ -a_{21} c_2 + a_{11} a_{20} \tanh(c_1) + 2\alpha a_{20} a_{21} \tanh(c_1) &= 0, \\ \tanh(c_1) (a_{11} a_{21} + 2\alpha a_{21}^2 - a_{11} a_{20} \tanh(c_1)) &= 0, \\ a_{21} \tanh^2(c_1) (c_2 - a_{11}) &= 0.\end{aligned}$$

Solution of the algebraic system

$$\begin{aligned}a_{10} &= -c_2 \coth(c_1) - \frac{1}{\alpha}, \\a_{11} &= c_2, \\a_{20} &= \frac{c_2 \coth(c_1)}{\alpha}, \\a_{21} &= -\frac{c_2}{\alpha}.\end{aligned}$$

Solitary wave solution in original variables:

$$\begin{aligned}u_n(t) &= -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh [c_1 n + c_2 t + \Delta], \\v_n(t) &= \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh [c_1 n + c_2 t + \Delta].\end{aligned}$$

Solving/Analyzing Systems of Algebraic Systems with Parameters

Class of fifth-order evolution equations with parameters:

$$u_t + \alpha\gamma^2 u^2 u_x + \beta\gamma u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0.$$

Well-Known Special cases

Lax case: $\alpha = \frac{3}{10}, \beta = 2, \gamma = 10$. **Two** solutions:

$$u(x, t) = 4c_1^2 - 6c_1^2 \tanh^2 [c_1 x - 56c_1^5 t + \Delta],$$

and

$$u(x, t) = a_0 - 2c_1^2 \tanh^2 [c_1 x - 2(15a_0^2 c_1 - 40a_0 c_1^3 + 28c_1^5)t + \Delta],$$

where a_0 is arbitrary.

Sawada-Kotera case: $\alpha = \frac{1}{5}, \beta = 1, \gamma = 5$. **Two** solutions:

$$u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 [c_1 x - 16c_1^5 t + \Delta],$$

and

$$u(x, t) = a_0 - 6c_1^2 \tanh^2 [c_1 x - (5a_0^2 c_1 - 40a_0 c_1^3 + 76c_1^5)t + \Delta],$$

where a_0 is arbitrary.

Kaup-Kupershmidt case: $\alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10$. **Two** solutions:

$$u(x, t) = c_1^2 - \frac{3}{2}c_1^2 \tanh^2 [c_1 x - c_1^5 t + \Delta]$$

and

$$u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 [c_1 x - 176c_1^5 t + \Delta].$$

No free constants!

Ito case: $\alpha = \frac{2}{9}, \beta = 2, \gamma = 3$. **One** solution:

$$u(x, t) = 20c_1^2 - 30c_1^2 \tanh^2 [c_1 x - 96c_1^5 t + \Delta].$$

What about the General case?

Q1: Can we retrieve the special solutions?

Q2: What are the condition(s) on the parameters α, β, γ for solutions of tanh-type to **exist**?

Tanh solutions:

$$u(x, t) = a_0 + a_1 \tanh [c_1 x + c_2 t + \Delta] + a_2 \tanh^2 [c_1 x + c_2 t + \Delta].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha\gamma^2 a_2^2 + 6\gamma a_2 c_1^2 + 2\beta\gamma a_2 c_1^2 + 24c_1^4) = 0,$$

$$a_1(\alpha\gamma^2 a_1^2 + 6\alpha\gamma^2 a_0 a_2 + 6\gamma a_0 c_1^2 - 18\gamma a_2 c_1^2 - 12\beta\gamma a_2 c_1^2 - 120c_1^4) = 0,$$

$$\alpha\gamma^2 a_2^2 + 12\gamma a_2 c_1^2 + 6\beta\gamma a_2 c_1^2 + 360c_1^4 = 0,$$

$$2\alpha\gamma^2 a_1^2 a_2 + 2\alpha\gamma^2 a_0 a_2^2 + 3\gamma a_1^2 c_1^2 + \beta\gamma a_1^2 c_1^2 + 12\gamma a_0 a_2 c_1^2 - 8\gamma a_2^2 c_1^2 - 8\beta\gamma a_2^2 c_1^2 - 480a_2 c_1^4 = 0,$$

$$a_1(\alpha\gamma^2 a_0^2 c_1 - 2\gamma a_0 c_1^3 + 2\beta\gamma a_2 c_1^3 + 16c_1^5 + c_2) = 0,$$

$$\alpha\gamma^2 a_0 a_1^2 c_1 + \alpha\gamma^2 a_0^2 a_2 c_1 - \gamma a_1^2 c_1^3 - \beta\gamma a_1^2 c_1^3 - 8\gamma a_0 a_2 c_1^3 + 2\beta\gamma a_2^2 c_1^3 + 136a_2 c_1^5 + a_2 c_2 = 0.$$

Unknowns: a_0, a_1, a_2 .

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

Solve and **Reduce** cannot be used on the whole system!

Strategy to Solve/Reduce Nonlinear Systems

Assumptions:

- All $c_i \neq 0$
- Parameters ($\alpha, \beta, \gamma, \dots$) are nonzero. Otherwise the maximal exponents M_i may change.
- All $M_i \geq 1$ in tanh- and sech-methods.
- All $a_i M_i \neq 0$ in tanh- and sech-methods. Highest power terms in U_i must be present, except in mixed sech-tanh-method.
- Solve for a_{ij} , then c_i , then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in a_{ij} first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in c_i if they are free of a_{ij} .
- Solve linear equations in parameters if they free of a_{ij}, c_i .
- Solve quasi-linear equations for a_{ij}, c_i , parameters.
- Solve quadratic equations for a_{ij}, c_i , parameters.
- Eliminate cubic terms for a_{ij}, c_i , parameters, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner Basis Techniques.
- Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$.

Actual Solution: Two major cases:

CASE 1: $a_1 = 0$, two subcases

Subcase 1-a:

$$a_2 = -\frac{3}{2}a_0,$$

$$c_2 = c_1^3(24c_1^2 - \beta\gamma a_0),$$

where a_0 is one of the two roots of the quadratic equation:

$$\alpha\gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta\gamma a_0 c_1^2 + 160c_1^4 = 0.$$

Subcase 1-b: If $\beta = 10\alpha - 1$, then

$$a_2 = -\frac{6}{\alpha\gamma}c_1^2,$$

$$c_2 = -\frac{1}{\alpha}(\alpha^2\gamma^2 a_0^2 c_1 - 8\alpha\gamma a_0 c_1^3 + 12c_1^5 + 16\alpha c_1^5),$$

where a_0 is arbitrary.

CASE 2: $a_1 \neq 0$, then

$$\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)$$

and

$$a_2 = -\frac{168}{\gamma(3 + 2\beta)}c_1^2,$$

provided β is root of

$$(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0.$$

Subcase 2-a: If $\beta^2 = -\frac{1}{104}(886\beta + 1487)$, then

$$\alpha = -\frac{2\beta + 5}{26},$$

$$a_0 = -\frac{49c_1^2(9983 + 4378\beta)}{26\gamma(8 + 3\beta)(3 + 2\beta)^2},$$

$$a_1 = \pm \frac{336c_1^2}{\gamma(3 + 2\beta)},$$

$$a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},$$

$$c_2 = -\frac{364 c_1^5 (3851 + 1634\beta)}{6715 + 2946\beta}.$$

Subcase 2-b: If $\beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2)$, then

$$\alpha = \frac{39 + 38\beta + 8\beta^2}{392},$$

$$a_0 = \frac{28 c_1^2 (6483 + 5529\beta + 1066\beta^2)}{(3 + 2\beta)(23 + 6\beta)(81 + 26\beta)\gamma},$$

$$a_1^2 = \frac{28224 c_1^4 (4\beta - 1)(26\beta - 17)}{(3 + 2\beta)^2(23 + 6\beta)(81 + 26\beta)\gamma^2},$$

$$a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},$$

$$c_2 = -\frac{8 c_1^5 (1792261977 + 1161063881\beta + 188900114\beta^2)}{959833473 + 632954969\beta + 105176786\beta^2}.$$

Implementation Issues – Software Demo – Future Work

- Demonstration of Mathematica package for tanh/sech methods.
- Long term goal: Develop PDESolve for closed form solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry methods, etc.
- Look at other types of explicit solutions involving
 - hyperbolic functions sinh, cosh, tanh, ...
 - complex exponentials combined with sech or tanh.
- Seek solutions $u(x, t) = U(F(\xi))$, for special functions F , where $F'(\xi)$ is polynomial or irrational expression in F .

Examples:

- If $F = \exp \xi$

$$F'(\xi) = F(\xi).$$

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = c_j F \frac{d \bullet}{dF}.$$

- If $F = \tanh \xi$

$$F'(\xi) = 1 - F^2(\xi).$$

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = c_j (1 - F^2) \frac{d \bullet}{dF}.$$

- If $F = \operatorname{sech} \xi$

$$F'(\xi) = -F(\xi) \sqrt{1 - F^2(\xi)}.$$

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = -c_j F \sqrt{1 - F^2} \frac{d \bullet}{dF}.$$

– If $F = \operatorname{cn} \xi$

$$\operatorname{cn}' \xi = -\operatorname{sn} \xi \operatorname{dn} \xi,$$

$$F'(\xi) = -\sqrt{1 - F^2} \sqrt{1 - k^2 + k^2 F^2}.$$

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{1 - F^2} \sqrt{1 - k^2 + k^2 F^2} \frac{d \bullet}{dF}.$$

The modulus (k) of the elliptic functions is added to the list of c_i .

- Add the constraining differential equations to the system of PDEs directly.
- Why are \tanh and sech solutions so prevalent?
- Other applications (of the nonlinear algebraic solver):
Computation of conservation laws, symmetries, first integrals, etc. leading to **linear** parameterized systems for unknowns coefficients (see *InvariantsSymmetries* by Göktaş and Hereman).
- Preprint:
D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J. Miller, *Symbolic computation of tanh and sech solutions of nonlinear partial differential and differential-difference equations*, *Journal of Symbolic Computation* (2001), to be submitted.