Los Alamos Days ’92

SYMOLIC SOFTWARE
FOR
SOLITON THEORY

Willy Hereman

Dept. of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887

Saturday, April 25, 1992
10:00am
Symbolic Software Packages for Soliton Theory

- Painlevé Integrability Test of ODEs and PDEs
  - Painlevé Test for 3rd order equations by Hajee (Reduce, 1982)
  - ODE_Painlevé by Winternitz and Rand (Macsyma, 1986)
  - PDE_Painlevé by Hereman and Van de Bulck (Macsyma, 1987)
  - Painlevé program (parts) by Hlavatý (Reduce, 1986)

- Construction of Explicit Solitary Wave and Soliton Solutions
  - Solitary wave solutions by Hereman (Macsyma, 1990)
  - Solitons via Hirota’s method by Hereman and Zhuang (Macsyma, 1990)
• Calculation of Lie-point and Generalized Symmetries
  – SPDE by Schwarz (Reduce, Scratchpad, 1986)
  – Symmetries via exterior differential forms
    by Kersten and Gragert (Reduce, 1987)
  – Lie-Bäcklund symmetries by Fedorova, Kornyak and
    Fushchich (Reduce, 1987)
  – Lie-point symmetries by Schwarzmeier
    and Rosenau (Macsyma, 1988)
  – Special symmetries by Mikhailov (Pascal, 1988)
  – LIE by Head (muMath, 1990)
  – PDELIE by Vafeades (Macsyma, 1990)
  – DEliA by Bocharov (Pascal, 1990)
  – SYM_DE by Steinberg (Macsyma, 1990)
  – SYMCAL by Reid (Macsyma, 1990)
  – SYMMGRP.MAX by Champagne, Hereman and Win-
    ternitz (1990)
Classes of Bilinear Equations

Major types of bilinear representations with examples:

**Type I:** Korteweg-de Vries equation

\[ P(D_x, D_t)(f \cdot f) \equiv (D_x D_t + D_x^4)(f \cdot f) = 0 \]

**Type II:** modified Korteweg-de Vries equation

\[ P_1(D_x, D_t)(f \cdot g) \equiv (D_t + D_x^3)(f \cdot g) = 0, \]
\[ P_2(D_x, D_t)(f \cdot g) \equiv D_x^2(f \cdot g) = 0 \]

**Type III:** sine-Gordon equation

\[ P_1(D_x, D_t)(g \cdot f) \equiv (D_x D_t - 1)(g \cdot f) = 0, \]
\[ P_2(D_x, D_t)(f \cdot f - g \cdot g) \equiv D_x D_t(f \cdot f - g \cdot g) = 0 \]

**Type IV:** Nonlinear Schrödinger equation

\[ P_1(D_x, D_t)(g \cdot f) \equiv (D_x^2 + iD_t)(g \cdot f) = 0, \]
\[ P_2(D_x, D_t)(f \cdot f) - P_3(D_x, D_t)(g \cdot g^*) \equiv D_x^2(f \cdot f) - gg^* = 0 \]

**Type V:** Benjamin-Ono equation

\[ P(D_x, D_t)(f \cdot f^*) \equiv (D_x^2 + iD_t)(f \cdot f^*) = 0 \]
For equations of Type II, the three-soliton solution follows from

\[ f = 1 + i \exp \theta_1 + i \exp \theta_2 + i \exp \theta_3 \\
- a_{12} \exp(\theta_1 + \theta_2) - a_{13} \exp(\theta_1 + \theta_3) - a_{23} \exp(\theta_2 + \theta_3) \\
- ib_{123} \exp(\theta_1 + \theta_2 + \theta_3), \]

\[ g = 1 - i \exp \theta_1 - i \exp \theta_2 - i \exp \theta_3 \\
- a_{12} \exp(\theta_1 + \theta_2) - a_{13} \exp(\theta_1 + \theta_3) - a_{23} \exp(\theta_2 + \theta_3) \\
+ ib_{123} \exp(\theta_1 + \theta_2 + \theta_3). \]

For equations of Types I, III through V other forms for \( f \) and \( g \) are needed.
Example 1 - Macsyma
Soliton Solutions of Nonlinear PDEs

• Hirota’s Direct Method
  allows to construct exact soliton solutions of
  – nonlinear evolution equations
  – wave equations
  – coupled systems

• Test conditions for existence of soliton solutions

• Examples:
  – Korteweg-de Vries equation (KdV)
    \[ u_t + 6uu_x + u_{3x} = 0 \]
  – Kadomtsev-Petviashvili equation (KP)
    \[ (u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0 \]
  – Sawada-Kotera equation (SK)
    \[ u_t + 45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x} = 0 \]
**Hirota’s Method**

Korteweg-de Vries equation

\[ u_t + 6uu_x + u_{3x} = 0 \]

Substitute

\[ u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} \]

Integrate with respect to \( x \)

\[ ff_{xt} - f_x f_t + ff_{4x} - 4f_x f_{3x} + 3f_{2x}^2 = 0 \]

**Bilinear form**

\[ B(f \cdot f) \overset{\text{def}}{=} (D_x D_t + D_x^4)(f \cdot f) = 0 \]

Introduce the bilinear operator

\[ D_x^m D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x, t) g(x', t') \big|_{x' = x, t' = t} \]

Use the expansion

\[ f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n \]

Substitute \( f \) into the bilinear equation

Collect powers in \( \epsilon \) (book keeping parameter)

\[ O(\epsilon^0) : B(1 \cdot 1) = 0 \]
\[ O(\epsilon^1) : B(1 \cdot f_1 + f_1 \cdot 1) = 0 \]
\( O(\epsilon^2) : B (1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0 \)
\( O(\epsilon^3) : B (1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0 \)
\( O(\epsilon^4) : B (1 \cdot f_4 + f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1 + f_4 \cdot 1) = 0 \)
\( O(\epsilon^n) : B (\sum_{j=0}^{n} f_j \cdot f_{n-j}) = 0 \quad \text{with} \quad f_0 = 1 \)

If the original PDE admits a N-soliton solution then the expansion will truncate at level \( n = N \) provided

\[ f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp (k_i x - \omega_i t + \delta_i) \]

\( k_i, \omega_i \) and \( \delta_i \) are constants

Dispersion law

\[ \omega_i = k_i^3 \quad (i = 1, 2, ..., N) \]

Consider the case \( N=3 \)

Terms generated by \( B(f_1, f_1) \) justify

\[ f_2 = a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \]
\[ = a_{12} \exp [(k_1 + k_2) x - (\omega_1 + \omega_2) t + (\delta_1 + \delta_2)] \]
\[ + a_{13} \exp [(k_1 + k_3) x - (\omega_1 + \omega_3) t + (\delta_1 + \delta_3)] \]
\[ + a_{23} \exp [(k_2 + k_3) x - (\omega_2 + \omega_3) t + (\delta_2 + \delta_3)] \]

Calculate the constants \( a_{12}, a_{13} \) and \( a_{23} \)
\[ a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \quad i, j = 1, 2, 3 \]

\[ B(f_1 \cdot f_2 + f_2 \cdot f_1) \text{ motivates} \]

\[ f_3 = b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \]
\[ = b_{123} \exp [(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t + (\delta_1 + \delta_2 + \delta_3)] \]

with

\[ b_{123} = a_{12} a_{13} a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2} \]

Subsequently, \( f_i = 0 \) for \( i > 3 \)

Set \( \epsilon = 1 \)

\[ f = 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 \]
\[ + a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \]
\[ + b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \]

Return to the original \( u(x, t) \)

\[ u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} \]
Example 2 - Macsyma
Solitary Wave Solutions

- Korteweg-de Vries equation and generalizations

\[
u_t + au^n u_x + u_{xxx} = 0, \quad n \in \mathbb{N}
\]

\[
u(x, t) = \left\{ \frac{c(n + 1)(n + 2)}{2a} \ \text{sech}^2 \left[ \frac{n}{2} \sqrt{c}(x - ct) + \delta \right] \right\}^{\frac{1}{n}}
\]

- Burgers equation

\[
u_t + auu_x - u_{xx} = 0
\]

\[
u(x, t) = \frac{c}{a} \left\{ 1 - \tanh \left[ \frac{c}{2}(x - ct) + \delta \right] \right\}
\]

- Fisher equation and generalizations

\[
u_t - u_{xx} - u(1 - u^n) = 0, \quad n \in \mathbb{N}
\]

\[
u(x, t) = \left\{ \frac{1}{2} \left[ 1 - \tanh \left[ \frac{n}{2\sqrt{2n + 4}}(x - \frac{(n + 4)}{\sqrt{2n + 4}}t) + \delta \right] \right] \right\}^{\frac{2}{n}}
\]
• Fitzhugh-Nagumo equation

\[ u_t - u_{xx} + u(1 - u)(a - u) = 0 \]

\[ u(x, t) = \frac{a}{2} \left\{ 1 + \tanh \left[ \frac{a}{2\sqrt{2}} (x - \frac{(2 - a)}{\sqrt{2}} t) + \delta \right] \right\} \]

• Kuramoto-Sivashinski equation

\[ u_t + uu_x + au_{xx} + bu_{xxxx} = 0 \]

\[ u(x, t) = c + \frac{165ak}{19} \left\{ \tanh^3 \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]
\[ - \frac{135ak}{19} \left\{ \tanh \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

with \( k = \sqrt{\frac{11a}{19b}} \)

\[ u(x, t) = c - \frac{15ak}{19} \left\{ \tanh^3 \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]
\[ + \frac{45ak}{19} \left\{ \tanh \left[ \frac{k(x - ct)}{2} + \delta \right] \right\} \]

with \( k = \sqrt{-\frac{a}{19b}} \)
\begin{itemize}
  \item Harry Dym equation

  \[ u_t + (1 - u)^3 u_{xxx} = 0 \]

  \[ u(x, t) = \text{sech}^2 \left[ \frac{1}{2} \sqrt{c} \left[ x - ct + \delta(x, t) \right] \right] \]

  \[ \delta(x, t) = \frac{2}{\sqrt{c}} \tanh \left[ \frac{\sqrt{c}}{2} \left[ x - ct + \delta(x, t) \right] \right] \]

  \item sine-Gordon equation

  \[ u_{tt} - u_{xx} - \sin u = 0 \]

  \[ u(x, t) = 4 \arctan \left\{ \exp \left[ \frac{1}{\sqrt{-c}} (x - ct) + \delta \right] \right\} \]
\end{itemize}
• Coupled Korteweg-de Vries equations

\[ u_t - a(6uu_x + u_{xxx}) - 2b \, vv_x = 0, \]
\[ v_t + 3uv_x + v_{xxx} = 0 \]

\[ u(x, t) = 2 \, c \, \text{sech}^2 \left[ \sqrt{c}(x - ct) + \delta \right], \]
\[ v(x, t) = \pm c \sqrt{\frac{-2(4a + 1)}{b}} \, \text{sech} \left[ \sqrt{c}(x - ct) + \delta \right], \]

\[ u(x, t) = c \, \text{sech}^2 \left[ \frac{1}{2} \sqrt{c}(x - ct) + \delta \right], \]
\[ v(x, t) = \frac{3}{\sqrt{6|b|}} \, u(x, t) = \frac{3 \, c}{\sqrt{6|b|}} \, \text{sech}^2 \left[ \frac{1}{2} \sqrt{c}(x - ct) + \delta \right] \]
A class of generalized KdV equations

\[ u_t + (a + bu^m)u^m u_x + u_{3x} = 0 \]

with \( a, b \in \mathbb{R}; m \in \mathbb{Q} \)

- **CASE 1:** \( a \neq 0, b = 0 \):
  \[
  u(x, t) = \left\{ \frac{c(m + 2)(m + 1)}{2a} \text{sech}^2\left[ \frac{m}{2}\sqrt{c(x - ct)} + \frac{\Delta}{2} \right] \right\}^{\frac{1}{m}}
  \]
  with arbitrary velocity \( c \)

- **CASE 2:** \( b \neq 0 \):
  \[
  u(x, t) = \left\{ \frac{-a(2m + 1)}{2b(m + 2)} (1 - \tanh\left[ \frac{m}{2}\sqrt{c(x - ct)} + \frac{\Delta}{2} \right]) \right\}^{\frac{1}{m}}
  \]
  with \( c = -\frac{a^2(2m + 1)}{b(m + 1)(m + 2)^2} \)
Example 4 - Macsyma
Lie-point Symmetries

• System of \( m \) differential equations of order \( k \)

\[
\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m
\]

with \( p \) independent and \( q \) dependent variables

\[
x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p
\]
\[
u = (u^1, u^2, ..., u^q) \in \mathbb{R}^q
\]

• The group transformations have the form

\[
\tilde{x} = \Lambda_{group}(x, u), \quad \tilde{u} = \Omega_{group}(x, u)
\]

where the functions \( \Lambda_{group} \) and \( \Omega_{group} \) are to be determined

• Look for the Lie algebra \( \mathcal{L} \) realized by the vector field

\[
\alpha = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u) \frac{\partial}{\partial u^l}
\]
Procedure for finding the coefficients

• Construct the $k^{\text{th}}$ prolongation $\text{pr}^{(k)} \alpha$ of the vector field $\alpha$

• Apply it to the system of equations

• Request that the resulting expression vanishes on the solution set of the given system

\[ \text{pr}^{(k)} \alpha \Delta^i |_{\Delta j = 0} \quad i, j = 1, \ldots, m \]

• This results in a system of linear homogeneous PDEs for $\eta^i$ and $\varphi_l$, with independent variables $x$ and $u$ (determining equations)

• Procedure thus consists of two major steps:

  * deriving the determining equations
  * solving the determining equations
Procedure for Computing the Determining Equations

- Use multi-index notation $J = (j_1, j_2, \ldots, j_p) \in \mathbb{N}^p$, to denote partial derivatives of $u^l$

\[ u^l_J \equiv \frac{\partial^{\left| J \right|} u^l}{\partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_p^{j_p}}, \]

where $\left| J \right| = j_1 + j_2 + \ldots + j_p$

- $u^{(k)}$ denotes a vector whose components are all the partial derivatives of order 0 up to $k$ of all the $u^l$

- Steps:
  
 1. Construct the $k^{th}$ prolongation of the vector field

\[ \text{pr}^{(k)} \alpha = \alpha + \sum_{l=1}^{q} \sum_J \psi_J^l(x, u^{(k)}) \frac{\partial}{\partial u^l_J}, \quad 1 \leq \left| J \right| \leq k \]

The coefficients $\psi_J^l$ of the first prolongation are:

\[ \psi_J^l = D_i \varphi_l(x, u) - \sum_{j=1}^{p} u^l_J D_i \eta^j(x, u), \]

where $J_i$ is a $p$–tuple with 1 on the $i^{th}$ position and zeros elsewhere
\( D_i \) is the total derivative operator

\[
D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} u_{J+J_i} \frac{\partial}{\partial u_{J}} \quad 0 \leq |J| \leq k
\]

Higher order prolongations are defined recursively:

\[
\psi^{J+J_i} = D_i \psi^J - \sum_{j=1}^{p} u_{J+J_j} D_i \eta^j(x, u), \quad |J| \geq 1
\]

(2) Apply the prolonged operator \( pr^{(k)} \alpha \) to each equation \( \Delta^i(x, u^{(k)}) = 0 \)

 Require that \( pr^{(k)} \alpha \) vanishes on the solution set of the system

\[
pr^{(k)} \alpha \Delta^i |_{\Delta^j=0} = 0 \quad i, j = 1, \ldots, m
\]

(3) Choose \( m \) components of the vector \( u^{(k)} \), say \( v^1, \ldots, v^m \), such that:

(a) Each \( v^i \) is equal to a derivative of a \( u^l \) \( (l = 1, \ldots, q) \) with respect to at least one variable \( x_i \) \( (i = 1, \ldots, p) \).

(b) None of the \( v^i \) is the derivative of another one in the set.

(c) The system can be solved algebraically for the \( v^i \) in terms of the remaining components of \( u^{(k)} \), which we de-
noted by $w$:

$$v^i = S^i(x, w), \quad i = 1, \ldots, m.$$  

(d) The derivatives of $v^i$,

$$v^i_J = D_J S^i(x, w),$$

where $D_J \equiv D^i_1 D^j_2 \cdots D^p_p$, can all be expressed in terms of the components of $w$ and their derivatives, without ever reintroducing the $v^i$ or their derivatives.

For instance, for a system of evolution equations

$$u^i_t(x_1, \ldots, x_{p-1}, t) = F^i(x_1, \ldots, x_{p-1}, t, u^{(k)}), \quad i = 1, \ldots, m,$$

where $u^{(k)}$ involves derivatives with respect to the variables $x_i$ but not $t$, choose $v^i = u^i_t$.

(4) Eliminate all $v^i$ and their derivatives from the expression prolonged vector field, so that all the remaining variables are independent.

(5) Obtain the determining equations for $\eta^i(x, u)$ and $\varphi^i_l(x, u)$ by equating to zero the coefficients of the remaining independent derivatives $u^i_J$. 