Symbolic Computation of Conserved Densities, Generalized Symmetries, and Recursion Operators for Nonlinear Evolution and Lattice Equations

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• **Purpose**

Design and implement algorithms to compute polynomial conservation laws and generalized symmetries (later recursion operators) for nonlinear systems of PDEs and differential-difference equations (DDEs).

• **Motivation**

– Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.). Compare with constants of motion (linear momentum, energy) in mechanics.

– Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

– Conserved densities can be used to test numerical integrators.

– For PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

– Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).
Part I  PDEs: A Brief Review

• Given: Nonlinear system of evolution equations

\[ u_t = F(u, u_x, u_{2x}, \ldots, u_{mx}) \]

in a (single) space variable \( x \) and time \( t \), and with

\[ u = (u_1, u_2, \ldots, u_n), \quad F = (F_1, F_2, \ldots, F_n). \]

Notation:

\[ u_t = \frac{\partial u}{\partial t}, \quad u_{mx} = u^{(m)} = \frac{\partial u}{\partial x^m}. \]

\( F \) is polynomial in \( u, u_x, \ldots, u_{mx} \).

PDEs of higher order in \( t \) should be recast as a first-order system.

Prototypical Examples

The Korteweg-de Vries (KdV) equation:

\[ u_t + uu_x + u_{3x} = 0. \]

The Boussinesq (wave) equation:

\[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

written as a first-order system (\( v \) auxiliary variable):

\[ u_t + v_x = 0, \]
\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0. \]

• Dilation invariance

The KdV equation, \( u_t + uu_x + u_{3x} = 0 \), has scaling symmetry

\[ (t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u). \]
The Boussinesq system

\[ u_t + v_x = 0, \]

\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \]

is not scaling invariant \((u_x \text{ and } u_{3x} \text{ are conflicting terms})\). If one introduces an auxiliary parameter \(\beta\), then

\[ u_t + v_x = 0, \]

\[ v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0, \]

has scaling symmetry:

\[(x, t, u, v, \beta) \rightarrow (\lambda^{-1} x, \lambda^{-2} t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta).\]

- **Conservation Laws**

\[ D_t \rho + D_x J = 0, \]

with conserved density \(\rho\) and flux \(J\).

Both are polynomial in \(u, u_x, u_{2x}, u_{3x}, \ldots\)

\[ P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant} \]

if \(J\) vanishes at infinity.

**Examples**

For the Korteweg-de Vries (KdV) equation, \(u_t + uu_x + u_{3x} = 0\), the first few conserved densities (and fluxes) are:

\[ \rho_1 = u, \quad D_t(u) + D_x(\frac{u^2}{2} + u_{2x}) = 0. \]

\[ \rho_2 = u^2, \quad D_t(u^2) + D_x(\frac{2u^3}{3} + 2uu_{2x} - u_x^2) = 0. \]
\[ \rho_3 = u^3 - 3u_x^2, \]

\[ D_t \left( u^3 - 3u_x^2 \right) + D_x \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x} \right) = 0. \]

\[ \vdots \]

\[ \rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \]

The Boussinesq system:

\begin{align*}
\rho_1 &= u, & \rho_2 &= v, \\
\rho_3 &= uv, & \rho_4 &= \beta u^2 - u^3 + v^2 + \alpha u_x^2.
\end{align*}

(then set \( \beta = 1 \)).

• **Generalized Symmetries.**

\[ \mathbf{G}(x,t,u,u_x,u_{2x},\ldots) \]

with \( \mathbf{G} = (G_1,G_2,\ldots,G_n) \) is a symmetry iff it leaves the PDE invariant for the replacement \( u \to u + \epsilon \mathbf{G} \) within order \( \epsilon \). i.e.

\[ D_t(u + \epsilon \mathbf{G}) = F(u + \epsilon \mathbf{G}) \]

must hold up to order \( \epsilon \) on the solutions of PDE.

Consequently, \( \mathbf{G} \) must satisfy the linearized equation

\[ D_t \mathbf{G} = F'(u)[\mathbf{G}] = \frac{\partial}{\partial \epsilon} F(u + \epsilon \mathbf{G})|_{\epsilon=0}. \]

where \( F' \) is the Fréchet derivative of \( F \).

Here \( u \) is replaced by \( u + \epsilon \mathbf{G} \), and \( u_{nx} \) by \( u_{nx} + \epsilon D_x^n \mathbf{G} \).
Example

For the KdV equation, \( u_t = 6uu_x + u_{3x} \), the first few generalized symmetries are:

\[
\begin{align*}
G^{(1)} &= u_x, \\
G^{(2)} &= 6uu_x + u_{3x}, \\
G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}, \\
G^{(4)} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} \\
& \quad + 70u_xu_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}.
\end{align*}
\]

• Recursion Operators.

A recursion operator \( \Phi \) connects two consecutive symmetries

\[
G^{(i+1)} = \Phi G^{(i)}.
\]

For \( n \)-component systems, \( \Phi \) is an \( n \times n \) matrix.

Defining equation for \( \Phi \) :

\[
D_t \Phi + [\Phi, F'(u)] = \frac{\partial \Phi}{\partial t} + \Phi'[F] + \Phi \circ F'(u) - F'(u) \circ \Phi = 0,
\]

where \([ , , ]\) means commutator, \( \circ \) stands for composition, and \( \Phi'[F] \) is the variational derivative of \( \Phi \).

Example

The recursion operator for the KdV equation:

\[
\Phi = D_x^2 + 2u + 2D_x u D_x^{-1} = D_x^2 + 4u + 2u_x D_x^{-1},
\]

where \( D_x^{-1} \) is the integration operator.
For example,
\[
\Phi u_x = (D_x^2 + 2u + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x},
\]
\[
\Phi(6uu_x + u_{3x}) = (D_x^2 + 2u + 2D_x u D_x^{-1})(6uu_x + u_{3x})
= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}.
\]

- **Key observations:**

Conserved densities, generalized symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

The dilation invariance allows one to design algorithms to compute densities, symmetries, and recursion operators.

- **Example: Algorithm for Conserved Densities of PDEs.**

1). Determine weights (scaling properties) of variables and auxiliary parameters.

2). Construct the form of the density (find monomial building blocks).

3). Determine the constant coefficients.

**Example:** Density of rank 6 for the KdV equation.

**Step 1: Compute the weights (dilation symmetry).**

The *weight*, \( w \), of a variable is by definition the number of \( x \)-derivatives the variable corresponds to.

The *rank* of a monomial is its total weight in terms of \( x \)-derivatives. Set \( w(D_x) = 1 \) or \( w(x) = -1 \) and require that all terms in 
\[
u_t + uu_x + u_{3x} = 0.
\]
have the same rank. Hence,

\[ w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3. \]

Solve the linear system: \( w(u) = 2, \ w(D_t) = 3 \), so, \( w(t) = -3 \).

\[ (t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u). \]

**Step 2: Determine the form of the density.**

List all possible powers of \( u \), up to rank 6: \([u, u^2, u^3]\).

Introduce \( x \) derivatives to ‘complete’ the rank.

\( u \) has weight 2, introduce \( D_x^4 \).

\( u^2 \) has weight 4, introduce \( D_x^2 \).

\( u^3 \) has weight 6, no derivative needed.

Apply the \( D_x \) derivatives.

Remove terms of the form \( D_x u_{px} \), or \( D_x \) up to terms kept prior in the list.

\[
[u_{4x}] \rightarrow [\ ] \quad \text{empty list.}
\]

\[
[u_x^2, u u_{2x}] \rightarrow [u_x^2] \quad \text{since } uu_{2x} = (u u_x)_x - u_x^2.
\]

\[
[u^3] \rightarrow [u^3].
\]

Linearly combine the ‘building blocks’:

\[
\rho = c_1 u^3 + c_2 u_x^2.
\]

**Step 3: Determine the coefficients \( c_i \).**

Compute \( D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt} \).

Replace \( u_t \) by \( -(u u_x + u_{3x}) \) and \( u_{xt} \) by \( -(u u_x + u_{3x})_x \).
Integrate the result, \( E \), with respect to \( x \). To avoid integration by parts, apply the Euler operator (variational derivative)

\[
L_u = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}} = \frac{\partial}{\partial u} - D_x \left( \frac{\partial}{\partial u_x} \right) + D_x^2 \left( \frac{\partial}{\partial u_{2x}} \right) + \cdots + (-1)^m D_x^m \left( \frac{\partial}{\partial u_{mx}} \right).
\]

to \( E \) of order \( m \).

If \( L_u(E) = 0 \) immediately, then \( E \) is a total \( x \)-derivative.

If \( L_u(E) \neq 0 \), the remaining expression must vanish identically.

\[
D_t \rho = -D_x \left[ \frac{3}{4} c_1 u^4 - (3c_1 - c_2) uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x} \right] - (3c_1 + c_2) u_x^3.
\]

The non-integrable term must vanish.

So, \( c_1 = -\frac{1}{3} c_2 \). Set \( c_2 = -3 \), hence, \( c_1 = 1 \).

Result:

\[
\rho = u^3 - 3 u_x^2.
\]

Expression \([\ldots]\) yields

\[
J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x}.
\]

Generalized symmetries and recursion operators can be constructed in a similar way!
Recursion Operators

• A *recursion operator* for a PDE system is the linear operator $\Phi$ connecting two symmetries $G$ and $\hat{G}$:

$$\hat{G} = \Phi G.$$

For $n$-component systems, $\Phi$ is an $n \times n$ matrix.

• **Example.**

The recursion operator for the KdV equation (has rank 2)

$$\Phi = D_x^2 + 2u + 2D_x u D_x^{-1} = D_x^2 + 4u + 2u_x D_x^{-1},$$

where $D_x^{-1}$ is the integration operator.

For example

$$\Phi u_x = (D_x^2 + 2u + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x},$$

$$\Phi(6uu_x + u_{3x}) = (D_x^2 + 2u + 2D_x u D_x^{-1})(6uu_x + u_{3x})$$

$$= 30u^2u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}.$$  

• **Key Observations.**

The terms in the recursion operator are monomials in $D_x$, $D_x^{-1}$, $u$, $u_x$, ...

Recursion operators split naturally in $\Phi = \Phi_0 + \Phi_1$.

$\Phi_0$ is a differential operator (no $D_x^{-1}$ terms).

$\Phi_1$ is an integral operator (with $D_x^{-1}$ terms).

Application of $\Phi$ to a symmetry should not leave any integrals.

For instance, for the KdV equation:

$$D_x^{-1}(6uu_x + u_{3x}) = 3u^2 + u_{2x}$$ is polynomial.
Use the conserved densities: $\rho^{(1)} = u$, $\rho^{(2)} = u^2$, $\rho^{(3)} = u^3 - \frac{1}{2}u_x^2$

$D_t\rho^{(1)} = D_t u = u_t = -D_x J^{(1)}$,  
$D_t\rho^{(2)} = D_t u^2 = 2uu_t = -D_x J^{(2)}$,  and  
$D_t\rho^{(3)} = D_t(u^3 - \frac{1}{2}u_x^2) = \rho^{(3)'}(u)[u_t] = (3u^2 - u_x D_x)u_t = -D_x J^{(3)}$,

for polynomial $J^{(i)}$, $i = 1, 2, 3$.

So, application of $D^{-1}_x$, or $D^{-1}_x u$, or $D^{-1}_x(3u^2 - u_x D_x)$ to $6uu_x + u_3x$ leads to a polynomial result.

• Algorithm for Recursion Operators of PDEs.

**Step 1:** Determine the rank of the recursion operator.

Recall: symmetries for the KdV equation, $u_t = 6uu_x + u_3x$, are  

$G^{(1)} = u_x$, $G^{(2)} = 6uu_x + u_3x$,  
$G^{(3)} = 30u^2u_x + 20u_x u_2x + 10uu_3x + u_5x$.

Hence,  

$R = \text{rank } \Phi = \text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2$.

**Step 2:** Construct the form of the recursion operator.

**(i) Determine the pieces of operator $\Phi_0$**

List all permutations of type $D^j u^k$ of rank $R$, with $j$ and $k$ nonnegative integers.

$\mathcal{L} = \{D^2, u\}$. 


(ii) Determine the pieces of operator $\Phi_1$

Combine the symmetries $G^{(j)}$ with $D^{-1}$ and $\rho^{(k)'}(u)$, so that every term in

$$\Phi_1 = \sum_j \sum_k G^{(j)} D^{-1} \rho^{(k)'}(u)$$

has rank $\Phi_1 = R$.

The indices $j$ and $k$ are taken so that

$$\text{rank } (G^{(j)}) + \text{rank } (\rho^{(k)'}(u)) - 1 = R.$$

List such terms:

$$\mathcal{M} = \{u_x D^{-1}\}.$$

(iii) Build the operator $\Phi$

Linearly combine the term in

$$\mathcal{R} = \mathcal{L} \cup \mathcal{M} = \{D^2, u, u_x D^{-1}\}.$$

to get

$$\Phi = c_1 D^2 + c_2 u + c_3 u_x D^{-1}.$$

Step 3: Determine the unknown coefficients.

Require that

$$\Phi G^{(k)} = G^{(k+1)}, \quad k = 1, 2, 3, \ldots$$

Solve the linear system:

$$\mathcal{S} = \{c_1 - 1 = 0, 18c_1 + c_3 - 20 = 0, 6c_1 + c_2 - 10 = 0, 2c_2 + c_3 - 10 = 0\},$$

Solution: $c_1 = 1$, $c_2 = 4$, and $c_3 = 2$. So,

$$\Phi = D^2 + 4u + 2u_x D^{-1}.$$
Examples.

The SK equation:

\[ u_t = 5u^2u_x + 5u_xu_{2x} + 5uu_{3x} + u_{5x}. \]

Recursion operator:

\[
\Phi = D^6 + 3uD^4 - 3DuD^3 + 11D^2uD^2 - 10D^3uD + 5D^4u
+ 12u^2D^2 - 19uDuD + 8uD^2u + 8DuDu + 4u^3
+ u_xD^{-1}(u^2 - 2u_xD) + G^{(2)}D^{-1},
\]

with \( G^{(2)} = 5u^2u_x + 5u_xu_{2x} + 5uu_{3x} + u_{5x}. \)

For the vector nonlinear Schrödinger system:

\[
\begin{align*}
  u_t + [u(u^2 + v^2) + \beta u + \gamma v - v_x]_x &= 0, \\
  v_t + [v(u^2 + v^2) + \theta u + \delta v + u_x]_x &= 0.
\end{align*}
\]

Recursion operator:

\[
\Phi = \begin{pmatrix}
  \beta - \delta + 2u^2 + 2u_xD^{-1}u & \theta + 2uv - D + 2u_xD^{-1}v \\
  \theta + 2uv + D + 2v_xD^{-1}u & 2v^2 + 2v_xD^{-1}v
\end{pmatrix}.
\]
Part II  Differential-difference Equations (DDEs)

• **Given:** Nonlinear system of DDEs:

(continuous in time, discretized in space)

\[
\dot{u}_n = F(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)
\]

where \(u_n\) and \(F\) are vector dynamical variables.

\(F\) is polynomial with constant coefficients.

No restrictions on the level of the shifts or the degree of nonlinearity.

• **Conservation Law:**

\[
\dot{\rho}_n = J_n - J_{n+1}
\]

with density \(\rho_n\) and flux \(J_n\).

Both are polynomials in \(u_n\) and its shifts.

\[
\frac{d}{dt}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})
\]

if \(J_n\) is bounded for all \(n\).

Subject to suitable boundary or periodicity conditions

\[
\sum_n \rho_n = \text{constant}.
\]

• **Example.**

Consider the one-dimensional Toda lattice

\[
\ddot{y}_n = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1})
\]

\(y_n\) is the displacement from equilibrium of the \(n\)th particle with unit mass under an exponential decaying interaction force between nearest neighbors.
Change of variables:

\[ u_n = \dot{y}_n, \quad v_n = \exp(y_n - y_{n+1}) \]

yields

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_nv_{n-1}. \]

**Key concept: Dilation invariance.**

The conservation laws and symmetries are invariant under the dilation symmetry of the Toda lattice:

\[ (t, u_n, v_n) \to (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n). \]

Thus, \( u_n \) corresponds to one \( t \)-derivative: \( u_n \sim \frac{d}{dt} \).

Similarly, \( v_n \sim \frac{d^2}{dt^2} \).

**Weight, \( w \), of variables are defined in terms of \( t \)-derivatives.**

With \( w(\frac{d}{dt}) = 1 \), we get \( w(u_n) = 1, \ w(v_n) = 2 \).

Weights of dependent variables are nonnegative, rational, and independent of \( n \).

The **rank** of a monomial is its total weight in terms of \( t \)-derivatives.
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<th>Semi-discrete Case (DDEs)</th>
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<tr>
<td><strong>System</strong></td>
<td></td>
</tr>
<tr>
<td>( u_t = F(u, u_x, u_{2x}, ...) )</td>
<td>( \dot{u}<em>n = F(..., u</em>{n-1}, u_n, u_{n+1}, ...) )</td>
</tr>
<tr>
<td><strong>Conservation Law</strong></td>
<td></td>
</tr>
<tr>
<td>( D_t \rho + D_x J = 0 )</td>
<td>( \dot{\rho}<em>n + J</em>{n+1} - J_n = 0 )</td>
</tr>
<tr>
<td><strong>Symmetry</strong></td>
<td></td>
</tr>
<tr>
<td>( D_t G = F'(u)[G] )</td>
<td>( D_t G = F'(u_n)[G] )</td>
</tr>
<tr>
<td>( = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)</td>
<td>_{\epsilon=0} )</td>
</tr>
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Table 1: Conservation Laws and Symmetries

<table>
<thead>
<tr>
<th>KdV Equation</th>
<th>Volterra Lattice</th>
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<tbody>
<tr>
<td><strong>Equation</strong></td>
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<tr>
<td>( u_t = 6uu_x + u_{3x} )</td>
<td>( \dot{u}<em>n = u_n (u</em>{n+1} - u_{n-1}) )</td>
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<tr>
<td><strong>Densities</strong></td>
<td></td>
</tr>
<tr>
<td>( \rho = u, \rho = u^2 )</td>
<td>( \rho_n = u_n, \rho_n = u_n (\frac{1}{2}u_n + u_{n+1}) )</td>
</tr>
<tr>
<td>( \rho = u^3 - \frac{1}{2}u_x^2 )</td>
<td>( \rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1} (u_n + u_{n+1} + u_{n+2}) )</td>
</tr>
<tr>
<td><strong>Symmetries</strong></td>
<td></td>
</tr>
<tr>
<td>( G = u_x, G = 6uu_x + u_{3x} )</td>
<td>( G = u_n u_{n+1} (u_n + u_{n+1} + u_{n+2}) )</td>
</tr>
<tr>
<td>( G = 30u^2u_x + 20u_x u_{2x} )</td>
<td>( -u_{n-1} u_n (u_{n-2} + u_{n-1} + u_n) )</td>
</tr>
<tr>
<td>( +10uu_{3x} + u_{5x} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Prototypical Examples
Analogy PDEs and DDEs

- Conservation laws for PDEs

\[ D_t \rho = -D_x J \]

density \( \rho \), flux \( J \).

Compute \( E = D_t \rho \).

Use PDE to replace all \( t \)-derivatives: \( u_t, u_{tx}, u_{txx}, \) etc.

To avoid integration by parts, apply the Euler operator (variational derivative)

\[
L_u = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}}
= \frac{\partial}{\partial u} - D_x \left( \frac{\partial}{\partial u_x} \right) + D_x^2 \left( \frac{\partial}{\partial u_{2x}} \right) + \cdots + (-1)^m D_x^m \left( \frac{\partial}{\partial u_{mx}} \right).
\]

to \( E \) of order \( m \).

If \( L_u(E) = 0 \), then \( E \) is a total \( x \)-derivative.

If \( L_u(E) \neq 0 \), the nonzero terms must vanish identically.
- Conservation laws for DDEs

\[ \dot{\rho}_n = J_n - J_{n+1} \]

density \( \rho_n \), flux \( J_n \).

Compute \( E = \dot{\rho}_n \).

Use the DDE to remove all \( t \)–derivatives, \( \dot{u}_n, \dot{u}_{n\pm 1}, \dot{u}_{n\pm 2}, \) etc.

To avoid pattern matching, apply the Euler operator (variational derivative)

\[
L_u = \sum_{k=-q}^{p} (D)^k \frac{\partial}{\partial u_{n+k}}
\]

\[
= \frac{\partial}{\partial u_n} + D(\frac{\partial}{\partial u_{n+1}}) + D^2(\frac{\partial}{\partial u_{n+2}}) + \cdots + D^p(\frac{\partial}{\partial u_{n+p}})
\]

\[
+ U(\frac{\partial}{\partial u_{n-1}}) + U^2(\frac{\partial}{\partial u_{n-2}}) + \cdots + U^q(\frac{\partial}{\partial u_{n-q}}).
\]

to \( E \) with shifts \( n - q, n + p \). Note \( U = D^{-1} \).

If \( L_u(E) = 0 \), then \( E \) can be written as \( J_n - J_{n+1} \).

If \( L_u(E) \neq 0 \), the nonzero terms must vanish identically.
• Equivalence Criterion.

Define $D$ shift-down operator, and $U$ shift-up operator, on the set of all monomials in $u_n$ and their shifts.

For a monomial $m$:

$$Dm = m|_{n\rightarrow n-1}, \quad \text{and} \quad Um = m|_{n\rightarrow n+1}.$$ 

For example

$$Du_{n+2}v_n = u_{n+1}v_{n-1}, \quad Uu_{n-2}v_{n-1} = u_{n-1}v_n.$$ 

Compositions of $D$ and $U$ define an equivalence relation. All shifted monomials are equivalent.

For example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$ 

Equivalence criterion:

Two monomials $m_1$ and $m_2$ are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$.

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}].$$

Main representative of an equivalence class is the monomial with label $n$ on $u$ (or $v$).

For example, $u_nu_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_nv_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.
• **Algorithm for Conserved Densities of DDEs.**

Three-step algorithm to find conserved densities:

1. Determine the weights.
2. Construct the form of density.
3. Determine the coefficients.

**Example:** Density of rank 3 of the Toda lattice,

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

**Step 1: Compute the weights.**

Require uniformity in rank for each equation to compute the weights:

\[
\begin{align*}
   w(u_n) + w(\frac{d}{dt}) &= w(v_{n-1}) = w(v_n), \\
   w(v_n) + w(\frac{d}{dt}) &= w(v_n) + w(u_n) = w(v_n) + w(u_{n+1})
\end{align*}
\]

Weights are shift invariant. Set \( w(\frac{d}{dt}) = 1 \) and solve the linear system: \( w(u_n) = w(u_{n+1}) = 1 \) and \( w(v_n) = w(v_{n-1}) = 2. \)

**Step 2: Construct the form of the density.**

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[ G = \{ u_n^3, u_n^2, u_nv_n, u_n, v_n \}. \]

For each monomial in \( G \), introduce enough \( t \)-derivatives to obtain weight 3. Use the DDE to remove \( \dot{u}_n \) and \( \dot{v}_n \):

\[
\begin{align*}
   \frac{d^0}{dt^0}(u_n^3) &= u_n^3, \\
   \frac{d^0}{dt^0}(u_nv_n) &= u_nv_n, \\
   \frac{d}{dt}(u_n^2) &= 2u_nv_{n-1} - 2u_nv_n,
\end{align*}
\]
\[
\frac{d}{dt}(v_n) = u_n v_n - u_{n+1} v_n,
\]
\[
\frac{d^2}{dt^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.
\]

Gather the resulting terms in a set
\[
\mathcal{H} = \{u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}.
\]

Replace members in the same equivalence class by their main representatives.
For example, \(u_n v_{n-1} \equiv u_{n+1} v_n\) are replaced by \(u_n v_{n-1}\).

Linearly combine the monomials in
\[
\mathcal{I} = \{u_n^3, u_n v_{n-1}, u_n v_n\}
\]
to obtain
\[
\rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n.
\]

**Step 3: Determine the coefficients.**

Require that \(\dot{\rho}_n = J_n - J_{n+1}\), holds.

Compute \(\dot{\rho}_n\) and use the DDE to remove \(\dot{u}_n\) and \(\dot{v}_n\).

Group the terms
\[
\dot{\rho}_n = (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1} v_n
\]
\[
+ c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2.
\]

Use the equivalence criterion to modify \(\dot{\rho}_n\).

Replace \(u_{n-1} u_n v_{n-1}\) by
\[
u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n].
\]
Introduce the main representatives. Thus
\[ \rho_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n 
+ (c_3 - c_2)v_nv_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_nv_{n+1}] 
+ c_2u_nu_{n+1}v_n + [c_2u_{n-1}u_nv_{n-1} - c_2u_nu_{n+1}v_n] 
+ c_2v_n^2 + [c_2v_{n-1}^2 - c_2v_n^2] - c_3u_nu_{n+1}v_n - c_3v_n^2. \]

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom.

Rearrange the terms to match the pattern \([J_n - J_{n+1}]\).

Hence
\[ \rho_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n 
+ (c_3 - c_2)v_nv_{n+1} + (c_2 - c_3)v_n^2 
+ \{(c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2\} 
- \{(c_3 - c_2)v_nv_{n+1} + c_2u_nu_{n+1}v_n + c_2v_n^2\}. \]

The terms inside the square brackets determine:
\[ J_n = (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2. \]

The terms outside the square brackets must vanish, thus
\[ S = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}. \]

The solution is \(3c_1 = c_2 = c_3\), so choose \(c_1 = \frac{1}{3}\), and \(c_2 = c_3 = 1\):
\[ \rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2. \]

Analogously, conserved densities of rank \(\leq 5\):
\[ \rho_n^{(1)} = u_n, \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n \]
\[ \rho_n^{(3)} = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) \]

\[ \rho_n^{(4)} = \frac{1}{4} u_n^4 + u_n^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \frac{1}{2} v_n^2 + v_n v_{n+1} \]

\[ \rho_n^{(5)} = \frac{1}{5} u_n^5 + u_n^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\
+ u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1}). \]

- **Generalized Symmetries**

A vector function \( G(..., u_{n-1}, u_n, u_{n+1}, ...) \) is a **symmetry** if the infinitesimal transformation \( u_n \rightarrow u_n + \epsilon G(..., u_{n-1}, u_n, u_{n+1}, ...) \) leaves the DDE system invariant within order \( \epsilon \).

Consequently, \( G \) must satisfy the linearized equation

\[ D_t G = F'(u_n)[G] = \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G) \big|_{\epsilon=0}. \]

where \( F' \) is the Fréchet derivative of \( F \).

Here, \( u_n \rightarrow u_n + \epsilon G(..., u_{n-1}, u_n, u_{n+1}, ...) \) means that \( u_{n+k} \) is replaced by \( u_{n+k} + \epsilon G|_{n\rightarrow n+k} \).

- **Example**

Consider the Toda lattice

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

Higher-order symmetry of rank \((3, 4)\):

\[ G_1 = v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n), \]

\[ G_2 = v_n(u_{n+1}^2 - u_n^2) + v_n(v_{n+1} - v_{n-1}). \]
Algorithm for Generalized Symmetries of DDEs.

Consider the Toda system with \( w(u_n) = 1 \) and \( w(v_n) = 2 \). Compute the form of the symmetry of ranks \((3, 4)\), i.e. the first component of the symmetry has rank 3, the second rank 4.

Step 1: Construct the form of the symmetry.

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[ \mathcal{L}_1 = \{u_n^3, u_n^2, u_nv_n, u_n, v_n\}, \]

and of rank 4 or less:

\[ \mathcal{L}_2 = \{u_n^4, u_n^3, u_n^2v_n, u_n^2, u_nv_n, u_n, v_n^2, v_n\}. \]

For each monomial in \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), introduce enough \( t \)-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the DDEs, for the monomials in \( \mathcal{L}_1 \):

\[ \frac{d^0}{dt^0}(u_n^3) = u_n^3, \quad \frac{d^0}{dt^0}(u_nv_n) = u_nv_n, \]

\[ \frac{d}{dt}(u_n^2) = 2u_n\dot{u}_n = 2u_nv_{n-1} - 2u_nv_n, \]

\[ \frac{d}{dt}(v_n) = \dot{v}_n = u_nv_n - u_{n+1}v_n, \]

\[ \frac{d^2}{dt^2}(u_n) = \frac{d}{dt}(\dot{u}_n) = \frac{d}{dt}(v_{n-1} - v_n) = u_{n-1}v_{n-1} - u_nv_{n-1} - u_nv_n + u_{n+1}v_n. \]

Gather the resulting terms:

\[ \mathcal{R}_1 = \{u_n^3, u_{n-1}v_{n-1}, u_nv_{n-1}, u_nv_n, u_{n+1}v_n\}. \]
\[ \mathcal{R}_2 = \{ u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1} u_n v_{n-1}, u_n^2 v_{n-1}, v_{n-2} v_{n-1}, v_{n-1}^2, u_n^2 v_n, u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_n v_{n+1}, v_n^2, v_{n-1} v_n, v_{n-1}^2, v_n^2, v_{n+1}^2 \}. \]

Linearly combine the monomials in \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \)

\[
\begin{align*}
G_1 &= c_1 u_n^3 + c_2 u_{n-1} v_{n-1} + c_3 u_n v_{n-1} + c_4 u_n v_n + c_5 u_{n+1} v_n, \\
G_2 &= c_6 u_n^4 + c_7 u_{n-1}^2 v_{n-1} + c_8 u_{n-1} u_n v_{n-1} + c_9 u_n^2 v_{n-1} + c_{10} v_{n-2} v_{n-1} + c_{11} v_{n-1}^2 + c_{12} u_n^2 v_n + c_{13} u_n u_{n+1} v_n + c_{14} u_{n+1}^2 v_n + c_{15} v_{n-1} v_n + c_{16} v_n^2 + c_{17} v_n v_{n+1}.
\end{align*}
\]

**Step 2: Determine the unknown coefficients.**

Require that the symmetry condition holds.

Solution:

\[
\begin{align*}
c_1 &= c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\
-c_2 &= -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}.
\end{align*}
\]

Therefore, with \( c_{17} = 1 \), the symmetry of rank \((3, 4)\) is:

\[
\begin{align*}
G_1 &= u_n v_n - u_{n-1} v_{n-1} + u_{n+1} v_n - u_n v_{n-1}, \\
G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1} v_n.
\end{align*}
\]

Analogously, the symmetry of rank \((4, 5)\) reads

\[
\begin{align*}
G_1 &= u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n^2 + v_n v_{n+1} - u_{n-1}^2 v_{n-1} \\
&\quad - u_{n-1} u_n v_{n-1} - u_n^2 v_{n-1} - v_{n-2} v_{n-1} - v_{n-1}^2, \\
G_2 &= u_{n+1} v_n^2 + 2u_{n+1} v_n v_{n+1} + u_{n+2} v_{n+1} v_{n+1} - u_n^3 v_n + u_{n+1}^3 v_n \\
&\quad - u_{n-1} v_{n-1} v_n - 2u_n v_{n-1} v_n - u_n v_n^2.
\end{align*}
\]
• Example: The Ablowitz-Ladik DDE.

Consider the Ablowitz and Ladik discretization,

\[ i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1}), \]

of the NLS equation,

\[ iu_t + u_{xx} + \kappa u^2 u^* = 0 \]

\( u_n^* \) is the complex conjugate of \( u_n \). Treat \( u_n \) and \( v_n = u_n^* \) as independent variables and add the complex conjugate equation. Set \( \kappa = 1 \) (scaling) and absorb \( i \) in the scale on \( t \):

\[
\begin{align*}
\dot{u}_n &= \ u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\
\dot{v}_n &= \ - (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).
\end{align*}
\]

Since \( v_n = u_n^* \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Introduce an auxiliary parameter \( \alpha \) with weight.

\[
\begin{align*}
\dot{u}_n &= \ \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\
\dot{v}_n &= \ - \alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).
\end{align*}
\]

Uniformity in rank leads to

\[
\begin{align*}
w(u_n) + 1 &= \ w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \\
w(v_n) + 1 &= \ w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n).
\end{align*}
\]

which yields

\[
w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1.
\]
Conserved densities (for $\alpha = 1$, in original variables):

$\rho_n^{(1)} = u_n u_{n-1}^*$

$\rho_n^{(2)} = u_n u_{n+1}^*$

$\rho_n^{(3)} = \frac{1}{2} u_n^2 u_{n-1}^2 + u_n u_{n+1} u_{n-1}^* v_n + u_n u_{n-2}^*$

$\rho_n^{(4)} = \frac{1}{2} u_n^2 u_{n+1}^2 + u_n u_{n+1} u_{n+2}^* + u_n u_{n+2}^*$

$\rho_n^{(5)} = \frac{1}{3} u_n^3 u_{n-1}^3 + u_n u_{n+1} u_{n-1}^* u_n^* (u_n u_{n-1}^* + u_{n+1}^* + u_{n+2}^* u_{n+1}^*)$

$+ u_n u_{n-1}^* (u_n u_{n-2}^* + u_{n+1} u_{n-1}^*) + u_n u_{n}^* (u_{n+1} u_{n-2}^* + u_{n+2} u_{n-1}^*) + u_n u_{n-3}^*$

$\rho_n^{(6)} = \frac{1}{3} u_n^3 u_{n+1}^3 + u_n u_{n+1} u_{n+1}^* u_{n+2}^* (u_n u_{n+1}^* + u_{n+1}^* + u_{n+2}^* u_{n+3}^*)$

$+ u_n u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^*) + u_n u_{n+3}^* (u_{n+1} u_{n+1}^* + u_{n+2} u_{n+2}^*) + u_n u_{n+3}^*$

The Ablowitz-Ladik lattice has infinitely many conserved densities.

Although a constant of motion, we cannot find the Hamiltonian:

$$H = -i \sum [u_n^* (u_{n-1} + u_{n+1}) - 2 \ln(1 + u_n u_n^*)],$$

since it has a logarithmic term.
• Application: Discretization of the combined KdV-mKdV equation.

Consider the integrable discretization
\[
\dot{u}_n = -(1 + \alpha h^2 u_n + \beta h^2 u_n^2) \left\{ \frac{1}{h^3} \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) \right. \\
+ \frac{\alpha}{2h} [u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \\
+ \frac{\beta}{2h} [u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n)] \left\} \right.
\]
of the combined KdV-mKdV equation
\[
u_t + 6\alpha uu_x + 6\beta u^2 u_x + u_{xxx} = 0.
\]
Discretizations the KdV and mKdV equations are special cases.
Set \( h = 1 \) (scaling). No uniformity in rank!
Introduce auxiliary parameters \( \gamma \) and \( \delta \) with weights.
\[
\dot{u}_n = -(\gamma + \alpha u_n + \beta u_n^2) \left\{ \delta \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) \right. \\
+ \frac{\alpha}{2} \left[ u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2} \right] \\
+ \frac{\beta}{2} \left[ u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n) \right] \left\} \right.
\]
Uniformity in rank requires
\[
w(\gamma) = w(\delta) = 2w(u_n), \quad w(\alpha) = w(u_n), \quad w(\beta) = 0.
\]
Then,
\[
w(u_n) + 1 = 5w(u_n),
\]
Hence,
\[
w(u_n) = w(\alpha) = \frac{1}{4}, \quad w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\beta) = 0,
\]
Conserved densities: Special cases

For the KdV case ($\beta = 0$):

\[
\dot{u}_n = - (\gamma + \alpha h^2 u_n) \left\{ \frac{\delta}{h^3} \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) \right. \\
+ \frac{\alpha}{2h} \left[ u_{n+1}^2 - u_{n-1}^2 + u_n (u_{n+1} - u_{n-1}) + u_{n+1} u_{n+2} - u_{n-1} u_{n-2} \right] \right\}
\]

with $\gamma = \delta = 1$ is a completely integrable discretization of the KdV equation

\[ u_t + 6\alpha uu_x + u_{xxx} = 0. \]

Now,

\[ w(\gamma) = w(\delta) = w(u_n), \quad w(\alpha) = 0. \]

Then,

\[ w(u_n) + 1 = 3w(u_n). \]

So,

\[ w(u_n) = w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\alpha) = 0. \]

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

\[ \rho_{n}^{(1)} = u_n, \]

\[ \rho_{n}^{(2)} = u_n \left( \frac{1}{2} u_n + u_{n+1} \right), \]

\[ \rho_{n}^{(3)} = u_n \left( \frac{1}{3} u_n^2 + u_n u_{n+1} + u_{n+1}^2 + \frac{1}{\alpha} u_{n+2} + u_{n+1} u_{n+2} \right) \]

\[ \rho_{n}^{(4)} = u_n \left( \frac{1}{4} u_n^3 + u_n^2 u_{n+1} + \frac{3}{2} u_n u_{n+1}^2 + u_{n+1}^3 + \cdots + u_{n+1} u_{n+2} u_{n+3} \right) \]

\[ \rho_{n}^{(5)} = u_n \left( \frac{1}{5} \alpha u_n^4 - \frac{1}{2} u_n^3 - 2u_n^2 u_{n+1} + \cdots + \alpha u_{n+1} u_{n+2} u_{n+3} u_{n+4} \right) \]
For the mKdV case ($\alpha = 0$):

$$\dot{u}_n = -(\gamma + \beta h^2 u_n^2) \left\{ \frac{\delta}{h^3} \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) + \frac{\beta}{2h} \left[ u_{n+1}^2 (u_{n+2} + u_n) - u_{n-1}^2 (u_{n-2} + u_n) \right] \right\}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the modified KdV equation

$$u_t + 6\beta u^2 u_x + u_{xxx} = 0.$$

Now,

$$w(\gamma) = w(\delta) = 2w(u_n), \quad w(\beta) = 0.$$

Then,

$$w(u_n) + 1 = 5w(u_n).$$

So,

$$w(u_n) = \frac{1}{4}, \quad w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\beta) = 0.$$

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$\rho_n^{(1)} = u_n u_{n+1},$$

$$\rho_n^{(2)} = u_n \left( \frac{1}{2} u_n u_{n+1}^2 + \frac{1}{\beta} u_{n+2} + u_{n+1}^2 u_{n+2} \right)$$

$$\rho_n^{(3)} = u_n \left( \frac{1}{3} u_n^2 u_{n+1}^3 + \frac{1}{\beta} u_n u_{n+1} u_{n+2} + \cdots + u_{n+1}^2 u_{n+2} u_{n+3} \right)$$

$$\rho_n^{(4)} = u_n \left( \frac{1}{4} \beta u_n^3 u_{n+1}^4 + u_n^2 u_{n+1} u_{n+2} + \cdots + \beta u_{n+1}^2 u_{n+2} u_{n+3} u_{n+4} \right)$$
Part III  Software, Future Work, Publications

- **Scope and Limitations of Algorithms.**
  - Systems of PDEs and DDEs must be polynomial in dependent variables. No *explicitly* dependencies on the independent variables. (Transcendental nonlinearities in progress).
  - Only one space variable (continuous or discretized) is allowed.
  - Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).
  - Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.
  - No limit on the number of equations in the system.
    In practice: time and memory constraints.
  - Input systems may have (nonzero) parameters.
    Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
  - Systems can also have parameters with (unknown) weight.
    This allows one to test evolution and lattice equations of non-uniform rank.
  - For systems where one or more of the weights is free, the program prompts the user for info.
  - Fractional weights and ranks are permitted.
  - Complex dependent variables are allowed.
  - PDEs and lattice equations must be of first-order in $t$. 
• Conclusions and Future Research

– Implement the recursion operator algorithm for PDEs.

– Design an algorithm for recursion operators of DDEs.

– Generalization to (3+1)-dimensional case
  (e.g. Kadomtsev-Petviashvili equation).

– Improve software, compare with other packages.

– Add tools for parameter analysis (Gröbner basis).

– Generalization towards broader classes of equations (e.g. $u_{xt}$).

– Exploit other symmetries in the hope to find conserved densities
  of non-polynomial form

– Application: test model DDEs for integrability.
  (study the integrable discretization of KdV-mKdV equation).

– Compute constants of motion for dynamical systems
  (e.g. Lorenz and Hénon-Heiles systems)
• **Implementation in Mathematica – Software**


  – Software: available via FTP, ftp site *mines.edu* in

    pub/papers/math_cs_dept/software/condens
    pub/papers/math_cs_dept/software/diffdens

  or via the Internet

    URL: http://www.mines.edu/fs_home/whereman/
• Publications


