Construction of Lax Pairs in Matrix Form and the Drinfel’d-Sokolov Method for Conservation Laws

Jacob Rezac

Colorado School of Mines

06/15/2012
1 Introduction

2 Construction of Lax pairs

3 Conservation Laws

4 Conclusions

Collaborators: Willy Hereman, Sara Clifton, Oscar Aguilar (CSM)
Funded by NSF research award no. CCF-0830783
What is a matrix Lax pair?

For a PDE,

\[ u_t = f(u, u_x, u_{xx}, \ldots) , \]
What is a matrix Lax pair?

For a PDE,

\[ u_t = f(u, u_x, u_{xx}, \ldots), \]

matrices \( X \) and \( T \) which satisfy

\[ \Phi_x = X\Phi \quad \text{and} \quad \Phi_t = T\Phi, \]

for an eigenfunction \( \Phi \) are called a \textit{Lax pair}. 
What is a matrix Lax pair?

For a PDE,

\[ u_t = f(u, u_x, u_{xx}, \ldots), \]

matrices \( X \) and \( T \) which satisfy

\[ \Phi_x = X\Phi \quad \text{and} \quad \Phi_t = T\Phi, \]

for an eigenfunction \( \Phi \) are called a \textit{Lax pair}.

This is equivalent to the equation
What is a matrix Lax pair?

For a PDE,

\[ u_t = f(u, u_x, u_{xx}, \ldots) , \]

matrices \( X \) and \( T \) which satisfy

\[ \Phi_x = X \Phi \quad \text{and} \quad \Phi_t = T \Phi, \]

for an eigenfunction \( \Phi \) are called a \textit{Lax pair}.

This is equivalent to the equation

\[ X_t - T_x + [X, T] = 0 \]

for \([X, T] = XT - TX\).
What is a matrix Lax pair?

For a PDE,

\[ u_t = f(u, u_x, u_{xx}, \ldots), \]

matrices \( X \) and \( T \) which satisfy

\[ \Phi_x = X\Phi \quad \text{and} \quad \Phi_t = T\Phi, \]

for an eigenfunction \( \Phi \) are called a \textit{Lax pair}.

This is equivalent to the equation

\[ X_t - T_x + [X, T] \dot{=} 0 \]

for \([X, T] = XT - TX\).

As before, \( \dot{=} \) means equality on solutions of the PDE.
Example

Consider the Korteweg-de Vries (KdV) equation,

\[ u_t + \alpha uu_x + u_{xxx} = 0. \]
Example

Consider the Korteweg-de Vries (KdV) equation,

\[ u_t + \alpha uu_x + u_{xxx} = 0. \]

A known Lax pairs is

\[ X_{KdV} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6} \alpha u & 0 \end{bmatrix}, \]

\[ T_{KdV} = \begin{bmatrix} \frac{1}{6} \alpha u_x & -4\lambda - \frac{1}{3} \alpha u \\ -4\lambda^2 + \frac{1}{3} \alpha \lambda u + \frac{1}{18} \alpha^2 u^2 + \frac{1}{6} \alpha u_{xx} & -\frac{1}{6} \alpha u_x \end{bmatrix}. \]
Example

Consider the Korteweg-de Vries (KdV) equation,

\[ u_t + \alpha uu_x + u_{xxx} = 0. \]

A known Lax pairs is

\[ X_{\text{KdV}} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6} \alpha u & 0 \end{bmatrix}, \]

\[ T_{\text{KdV}} = \begin{bmatrix} \frac{1}{6} \alpha u_x & -4 \lambda - \frac{1}{3} \alpha u \\ -4 \lambda^2 + \frac{1}{3} \alpha \lambda u + \frac{1}{18} \alpha^2 u^2 + \frac{1}{6} \alpha u_{xx} & -\frac{1}{6} \alpha u_x \end{bmatrix}. \]

The isospectral parameter \( \lambda (\lambda_t = 0) \) is necessary for non-triviality.
Substituting these into the Lax equation,
Example

Substituting these into the Lax equation,

\[(X_{KdV})_t - (T_{KdV})_x + [X_{KdV}, T_{KdV}] = \begin{bmatrix} 0 & 0 \\ -\frac{1}{6} \alpha (u_t + \alpha uu_x + u_{xxx}) & 0 \end{bmatrix} \]

\[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \]
Rescale the variables in the KdV equation to

\[
(\kappa^{-1} x, \kappa^{-3} t, \kappa^2 u) = (\xi, \tau, \nu).
\]

(1)

Call each exponent on \(\kappa\) in (1) the weight of a variable (denoted \(W(\cdot)\)). That is,

\[
W(\partial_x) = 1 \quad W(\partial_t) = 3 \quad W(u) = 2.
\]
Rescale the variables in the KdV equation to

\[ (x, t, u) \rightarrow (\kappa^{-1}x, \kappa^{-3}t, \kappa^2u) = (\xi, \tau, \nu). \]  \quad (1)
Scaling Invariance and Weight

Rescale the variables in the KdV equation to

$$(x, t, u) \rightarrow (\kappa^{-1}x, \kappa^{-3}t, \kappa^2u) = (\xi, \tau, \nu). \quad (1)$$

Then,
Rescale the variables in the KdV equation to
\[(x, t, u) \rightarrow (\kappa^{-1}x, \kappa^{-3}t, \kappa^2u) = (\xi, \tau, \nu). \quad (1)\]

Then,
\[u_t + uu_x + u_{xxx} = 0 \rightarrow \kappa^{-5} (\nu_{\tau} + \nu \nu_{\xi} + \nu \xi \xi \xi).\]
Scaling Invariance and Weight

Rescale the variables in the KdV equation to

$$ (x, t, u) \rightarrow (\kappa^{-1} x, \kappa^{-3} t, \kappa^2 u) = (\xi, \tau, \nu). $$  \hspace{1cm} (1)

Then,

$$ u_t + uu_x + u_{xxx} = 0 \rightarrow \kappa^{-5} (\nu_{\tau} + \nu u_{\xi} + \nu_{\xi\xi\xi}). $$

Call each exponent on $\kappa$ in (1) the \textit{weight} of a variable (denoted $W(\cdot)$). That is,
Scaling Invariance and Weight

Rescale the variables in the KdV equation to

\[(x, t, u) \rightarrow (\kappa^{-1}x, \kappa^{-3}t, \kappa^2u) = (\xi, \tau, \nu).\]  \hspace{1cm} (1)

Then,

\[u_t + uu_x + u_{xxx} = 0 \rightarrow \kappa^{-5} (\nu_{\tau} + \nu\nu_{\xi} + \nu_{\xi\xi\xi}).\]

Call each exponent on \(\kappa\) in (1) the weight of a variable (denoted \(W(\cdot)\)). That is,

\[W \left( \frac{\partial}{\partial x} \right) = 1 \hspace{1cm} W \left( \frac{\partial}{\partial t} \right) = 3 \hspace{1cm} W(u) = 2.\]
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
\begin{array}{cc}
\text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0}
\end{array}
\end{bmatrix}$$

Why is $W(\lambda) = 2$?

Set $\lambda = i \kappa^2$. 
Consider again the KdV equation $X$ matrix,

$$X_{KdV} = \begin{bmatrix}
\text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0}
\end{bmatrix}$$

We have

$$W(\partial_x) = 1, \quad W(\partial_t) = 3, \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{KdV} = \begin{bmatrix} \text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0} \end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
\text{weight 0} & \text{weight 0} \\
\text{0} & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0}
\end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
0 & 0 \\
\lambda - \frac{1}{6} \alpha u & 1 \\
\text{weight 0} & \text{weight 0} \\
\text{weight 0} & \text{weight 0} \\
\text{weight 2} & \text{weight 0}
\end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{KdV} = \begin{bmatrix} 
\text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0} 
\end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
\text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0}
\end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
\begin{array}{cc}
\text{weight 0} & \text{weight 0} \\
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0 \\
\text{weight 2} & \text{weight 0}
\end{array}
\end{bmatrix}$$

We have

$$W\left(\frac{\partial}{\partial x}\right) = 1 \quad W\left(\frac{\partial}{\partial t}\right) = 3 \quad W(u) = 2.$$ 

Why is $W(\lambda) = 2$?
Consider again the KdV equation $X$ matrix,

$$X_{\text{KdV}} = \begin{bmatrix}
\text{weight 0} & \text{weight 0} \\
\text{weight 0} & 0 \\
\lambda - \frac{1}{6} \alpha u & 1 \\
\text{weight 2} & 0 \end{bmatrix} \begin{bmatrix}
\text{weight 0} \\
1 \\
0 \\
\text{weight 0}
\end{bmatrix}$$

We have

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 2.$$

Why is $W(\lambda) = 2$? Set

$$\lambda = i\kappa^2.$$
More interestingly, the modified KdV (mKdV) equation,
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]
More interestingly, the modified KdV (mKdV) equation,

$$u_t + \alpha u^2 u_x + u_{xxx} = 0,$$

with weights

$$W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1$$
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \]

has Lax pair with \( T \)-component

\[
T_{mKdV} = \begin{bmatrix}
-4i\lambda^3 - 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x + 2u^3 - u_{xx} \\
4\lambda^2 u - 2i\lambda u_x + 2u^3 - u_{xx} & 4i\lambda^3 + 2i\lambda u^2
\end{bmatrix}
\]
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \]

has Lax pair with \( T \)-component

\[ T_{mKdV} = \begin{bmatrix} \text{weight 3} & \underbrace{-4i\lambda^3 - 2i\lambda u^2}_{4\lambda^2 u - 2i\lambda u_x + 2u^3 - u_{xx}} \\ \text{weight 3} & \underbrace{4\lambda^2 u + 2i\lambda u_x + 2u^3 - u_{xx}}_{4i\lambda^3 + 2i\lambda u^2} \end{bmatrix} \]
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \]

has Lax pair with \( T \)-component

\[ T_{\text{mKdV}} = \begin{bmatrix}
-4i\lambda^3 - 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x + 2u^3 - u_{xx} \\
4\lambda^2 u - 2i\lambda u_x + 2u^3 - u_{xx} & 4i\lambda^3 + 2i\lambda u^2
\end{bmatrix} \]
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \]

has Lax pair with \( T \)-component

\[
T_{mKdV} = \begin{bmatrix}
\underbrace{-4i\lambda^3 - 2i\lambda u^2}_{\text{weight 3}} \\
\underbrace{4\lambda^2 u - 2i\lambda u_x + 2u^3 - u_{xx}}_{\text{weight 3}} \\
\underbrace{4i\lambda^3 + 2i\lambda u^2}_{\text{weight 3}}
\end{bmatrix}
\]
More interestingly, the modified KdV (mKdV) equation,

\[ u_t + \alpha u^2 u_x + u_{xxx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \]

has Lax pair with \( T \)-component

\[
T_{\text{mKdV}} = \begin{bmatrix}
\begin{array}{c}
\text{weight 3}
\end{array}
\end{bmatrix} \begin{bmatrix}
-4i\lambda^3 - 2i\lambda u^2 \\
4\lambda^2 u - 2i\lambda u_x + 2u^3 - u_{xx}
\end{bmatrix}
\begin{bmatrix}
\text{weight 3}
\end{bmatrix} \begin{bmatrix}
4\lambda^2 u + 2i\lambda u_x + 2u^3 - u_{xx} \\
4i\lambda^3 + 2i\lambda u^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{weight 3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{weight 3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{weight 3}
\end{bmatrix}
\]
Assume Lax matrices have weight invariant elements, e.g.,
Assume Lax matrices have weight invariant elements, e.g.,

\[
W(X) = \begin{bmatrix}
W(X_{11}) & W(X_{12}) \\
W(X_{21}) & W(X_{22})
\end{bmatrix} \quad W(T) = \begin{bmatrix}
W(T_{11}) & W(T_{12}) \\
W(T_{21}) & W(T_{22})
\end{bmatrix}.
\]
Construction

Assume Lax matrices have weight invariant elements, e.g.,

\[
W(X) = \begin{bmatrix}
W(X_{11}) & W(X_{12}) \\
W(X_{21}) & W(X_{22})
\end{bmatrix} \quad W(T) = \begin{bmatrix}
W(T_{11}) & W(T_{12}) \\
W(T_{21}) & W(T_{22})
\end{bmatrix}.
\]

We substitute these weights into the Lax equation and force scale invariance,
Construction

Assume Lax matrices have weight invariant elements, e.g.,

\[ W(X) = \begin{bmatrix} W(X_{11}) & W(X_{12}) \\ W(X_{21}) & W(X_{22}) \end{bmatrix}, \quad W(T) = \begin{bmatrix} W(T_{11}) & W(T_{12}) \\ W(T_{21}) & W(T_{22}) \end{bmatrix}. \]

We substitute these weights into the Lax equation and force scale invariance,

\[ W(X_t) = W(T_x) = W([X, T]), \]

or,

\[ W(X_t) + W(\partial \partial x) = W(T) + W(\partial \partial t). \]
Construction

Assume Lax matrices have weight invariant elements, e.g.,

\[ W(X) = \begin{bmatrix} W(X_{11}) & W(X_{12}) \\ W(X_{21}) & W(X_{22}) \end{bmatrix} \quad \text{and} \quad W(T) = \begin{bmatrix} W(T_{11}) & W(T_{12}) \\ W(T_{21}) & W(T_{22}) \end{bmatrix}. \]

We substitute these weights into the Lax equation and force scale invariance,

\[ W(X_t) = W(T_x) = W([X, T]), \]

or,

\[ W(X) + W\left(\frac{\partial}{\partial x}\right) = W(T) + W\left(\frac{\partial}{\partial t}\right) = W(X) + W(T). \]
Solving the system of equations produced from this yields
Solving the system of equations produced from this yields

\[ W(X) = \begin{bmatrix} W\left(\frac{\partial}{\partial x}\right) & W(X_{12}) \\ 2W\left(\frac{\partial}{\partial x}\right) - W(X_{12}) & W\left(\frac{\partial}{\partial t}\right) \end{bmatrix} \]

and
Solving the system of equations produced from this yields

\[ W(X) = \begin{bmatrix} W \left( \frac{\partial}{\partial x} \right) & W(\mathbf{X}_{12}) \\ 2W \left( \frac{\partial}{\partial x} \right) - W(\mathbf{X}_{12}) & W \left( \frac{\partial}{\partial x} \right) \end{bmatrix} \]

and

\[ W(T) = \begin{bmatrix} W \left( \frac{\partial}{\partial t} \right) & W \left( \frac{\partial}{\partial t} \right) + W(\mathbf{X}_{12}) - W \left( \frac{\partial}{\partial x} \right) \\ W \left( \frac{\partial}{\partial t} \right) - W(\mathbf{X}_{12}) + W \left( \frac{\partial}{\partial x} \right) & W \left( \frac{\partial}{\partial t} \right) \end{bmatrix} . \]
Method of Construction

Step 1: Pick $W(X_{12})$
Method of Construction

Step 1: Pick $W(X_{12})$

Step 2: Based on the weight matrices of $X$ and $T$, generate elements of $X$ and $T$ with undetermined coefficients.
Method of Construction

Step 1: Pick $W(X_{12})$

Step 2: Based on the weight matrices of $X$ and $T$, generate elements of $X$ and $T$ with undetermined coefficients.

Step 3: Substitute matrices into Lax equation.
Method of Construction

Step 1: Pick $W(X_{12})$

Step 2: Based on the weight matrices of $X$ and $T$, generate elements of $X$ and $T$ with undetermined coefficients.

Step 3: Substitute matrices into Lax equation.

Step 4: Solve for undetermined coefficients.
Method of Construction

Step 1: Pick $W(X_{12})$

Step 2: Based on the weight matrices of $X$ and $T$, generate elements of $X$ and $T$ with undetermined coefficients.

Step 3: Substitute matrices into Lax equation.

Step 4: Solve for undetermined coefficients.

Step 5: Substitute back into $X$ and $T$. 
Example

Consider the KdV equation \((\alpha = 6)\) and assume (Step 1) \(W(X_{12}) = 0\). Then, our candidate Lax pair is (Step 2)
Example

Consider the KdV equation ($\alpha = 6$) and assume (Step 1) $W(X_{12}) = 0$. Then, our candidate Lax pair is (Step 2)

$$X = \begin{bmatrix} c_1 k & c_3 \\ c_4 k^2 + c_5 u & -c_1 k \end{bmatrix}$$

and
Example

Consider the KdV equation \((\alpha = 6)\) and assume (Step 1) \(W(X_{12}) = 0\). Then, our candidate Lax pair is (Step 2)

\[
X = \begin{bmatrix}
  c_1 k & c_3 \\
  c_4 k^2 + c_5 u & -c_1 k
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
  T_{11} & T_{12} \\
  T_{21} & -T_{11}
\end{bmatrix},
\]

where
Example

Consider the KdV equation \( (\alpha = 6) \) and assume (Step 1) \( W(X_{12}) = 0 \). Then, our candidate Lax pair is (Step 2)

\[
X = \begin{bmatrix}
c_1 k^2 + c_5 u & -c_1 k \\
c_4 k^2 + c_5 u & c_3 \\
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & -T_{11} \\
\end{bmatrix},
\]

where

\[
T_{11} = c_6 k^3 + c_7 k u + c_8 u_x,
\]

\[
T_{12} = c_9 k^2 + c_{10} u,
\]

\[
T_{21} = c_{11} k^4 + c_{12} k^2 u + c_{13} k u_x + c_{14} u^2 + c_{15} u_{xx}.
\]
We substitute this into the Lax equation (Step 3) and get, for the (1, 1)-element,
We substitute this into the Lax equation (Step 3) and get, for the (1, 1)-element,

\[
(c_{11} c_3 - c_4 c_9) k^4 + (c_{12} c_3 - c_{10} c_4 - c_5 c_9) k^2 u + (c_{14} c_3 - c_{10} c_5) u^2 \\
+ (c_{13} c_3 - c_7) k u_x + (c_{15} c_3 - c_8) u_{xx} = 0.
\]
We substitute this into the Lax equation (Step 3) and get, for the (1, 1)-element,

\[
(c_{11} c_3 - c_4 c_9) k^4 + (c_{12} c_3 - c_{10} c_4 - c_5 c_9) k^2 u + (c_{14} c_3 - c_{10} c_5) u^2 \\
+ (c_{13} c_3 - c_7) ku_x + (c_{15} c_3 - c_8) u_{xx} = 0.
\]

We break this up in powers of \(k\) and \(u\) to get the system of equations
We substitute this into the Lax equation (Step 3) and get, for the (1, 1)-element,

\[
(c_{11} c_3 - c_4 c_9) k^4 + (c_{12} c_3 - c_{10} c_4 - c_5 c_9) k^2 u + (c_{14} c_3 - c_{10} c_5) u^2 \\
+ (c_{13} c_3 - c_7) ku_x + (c_{15} c_3 - c_8) u_{xx} = 0.
\]

We break this up in powers of \( k \) and \( u \) to get the system of equations

\[
\begin{align*}
c_{11} c_3 - c_4 c_9 &= 0, \\
c_{12} c_3 - c_{10} c_4 - c_5 c_9 &= 0, \\
c_{14} c_3 - c_{10} c_5 &= 0, \\
c_{13} c_3 - c_7 &= 0, \\
c_{15} c_3 - c_8 &= 0.
\end{align*}
\]
Solving with the equations from the other three matrix elements (Step 4) gives many solutions. Choose (Step 5) a non-trivial one – more on this later:
Solving with the equations from the other three matrix elements (Step 4) gives many solutions. Choose (Step 5) a non-trivial one – more on this later:

\[ X = \begin{bmatrix} 0 & -\frac{c_{10}}{c_{11}} \\ \frac{c_{11}}{2c_{10}} \left( c_{10} k^2 + 2c_{11} u \right) & 0 \end{bmatrix} \]

and

\[ T = \begin{bmatrix} T_{11} \\ T_{12} \\ T_{21} \end{bmatrix}, \quad T_{11} = u_x, \quad T_{12} = 2c_{10}c_{31} \left( -c_{10}k^2 + c_{11}u \right), \quad T_{21} = c_{10}k^4 + c_{11}k^2u - c_{21}c_{10} \left( 2u^2 + u_{xx} \right). \]
Solving with the equations from the other three matrix elements (Step 4) gives many solutions. Choose (Step 5) a non-trivial one – more on this later:

\[ X = \begin{bmatrix}
0 & -\frac{c_{10}}{c_{11}} \\
\frac{c_{11}}{2c_{10}} \left( c_{10} k^2 + 2c_{11} u \right) & 0
\end{bmatrix} \]

and

\[ T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & -T_{11}
\end{bmatrix}, \]

where
Solving with the equations from the other three matrix elements (Step 4) gives many solutions. Choose (Step 5) a non-trivial one – more on this later:

\[
X = \begin{bmatrix}
0 & -\frac{c_{10}}{c_{11}} \\
\frac{c_{11}}{2c_{10}} (c_{10} k^2 + 2c_{11} u) & 0
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & -T_{11}
\end{bmatrix},
\]

where

\[
T_{11} = u_x,
\]

\[
T_{12} = \frac{2c_{10}}{c_{11}^3} (-c_{10} k^2 + c_{11} u),
\]

\[
T_{21} = c_{10} k^4 + c_{11} k^2 u - \frac{c_{11}^2}{c_{10}} (2u^2 + u_{xx}).
\]
Solving with the equations from the other three matrix elements (Step 4) gives many solutions. Choose (Step 5) a non-trivial one – more on this later:

\[
X = \begin{bmatrix}
0 & -\frac{c_{10}}{c_{11}} \\
\frac{c_{11}}{2c_{10}} (c_{10}k^2 + 2c_{11}u) & 0
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & -T_{11}
\end{bmatrix},
\]

where

\[
T_{11} = u_x,
\]

\[
T_{12} = \frac{2c_{10}}{c_{11}^3} (-c_{10}k^2 + c_{11}u),
\]

\[
T_{21} = c_{10}k^4 + c_{11}k^2u - \frac{c_{11}^2}{c_{10}} (2u^2 + u_{xx}).
\]

Setting \( c_{10} = 4, c_{11} = 2i \), and \( \kappa^2 = -i\lambda \) gives the pair from before.
Problems with Method

- Large number of possible values for undetermined coefficients.
Problems with Method

- Large number of possible values for undetermined coefficients.
- Gauge freedom

\[ \hat{X} = G X G^{-1} + G_x G^{-1} \quad \text{and} \quad \hat{T} = G T G^{-1} + G_t G^{-1} \]
Problems with Method

- Large number of possible values for undetermined coefficients.
- Gauge freedom

\[ \hat{X} = G X G^{-1} + G_x G^{-1} \quad \text{and} \quad \hat{T} = G T G^{-1} + G_t G^{-1} \]

- Use is limited to scaling invariant PDEs
Problems with Method

- Large number of possible values for undetermined coefficients.
- Gauge freedom

\[ \hat{X} = GXG^{-1} + G_x G^{-1} \quad \text{and} \quad \hat{T} = GTG^{-1} + G_t G^{-1} \]

- Use is limited to scaling invariant PDEs
- Computational complexity
Problems with Method

- Large number of possible values for undetermined coefficients.
- Gauge freedom

\[ \hat{X} = G X G^{-1} + G_x G^{-1} \quad \text{and} \quad \hat{T} = G T G^{-1} + G_t G^{-1} \]

- Use is limited to scaling invariant PDEs
- Computational complexity

How do we choose our undetermined coefficients?
Problems with Method

- Large number of possible values for undetermined coefficients
- Gauge freedom
  \[ \hat{X} = G X G^{-1} + G \dot{x} G^{-1} \quad \text{and} \quad \hat{T} = G T G^{-1} + G_t G^{-1} \]
- Use is limited to scaling invariant PDEs
- Computational complexity

How do we choose our undetermined coefficients?
Definition

A **Conservation Law** in \((1+1)\)-dimensions with density \(\rho\) and flux \(J\) is an equation of the form

\[
\rho_t + J_x \overset{\cdot}{=} 0,
\]

when evaluated on a PDE.
Definition

A Conservation Law in (1+1)-dimensions with density $\rho$ and flux $J$ is an equation of the form

$$\rho_t + J_x = 0,$$

when evaluated on a PDE.

The first few conservation laws of the KdV equation are easy to compute:
A **Conservation Law** in (1+1)-dimensions with density $\rho$ and flux $J$ is an equation of the form

$$\rho_t + J_x = 0,$$

when evaluated on a PDE.

The first few conservation laws of the KdV equation are easy to compute:

$$(u)_t + \left(\frac{1}{2}u^2 + u_{xx}\right)_x = 0,$$

$$(u^2)_t + \left(\frac{2}{3}u^3 - u_x^2 + 2uu_{xx}\right)_x = 0,$$
Definition

A Conservation Law in (1+1)-dimensions with density $\rho$ and flux $J$ is an equation of the form

$$\rho_t + J_x = 0,$$

when evaluated on a PDE.

The first few conservation laws of the KdV equation are easy to compute:

$$(u)_t + \left(\frac{1}{2}u^2 + u_{xx}\right)_x = 0,$$

$$(u^2)_t + \left(\frac{2}{3}u^3 - u_x^2 + 2uu_{xx}\right)_x = 0,$$

and have been known for some time.
Continuing this pattern, we might try $\rho = u^3$. 
Continuing this pattern, we might try $\rho = u^3$.

\[
(u^3)_t = 3u^2u_t = -3(u^3u_x + u^2u_{xxx}) = J_x.
\]
Continuing this pattern, we might try $\rho = u^3$.

\[
(u^3)_t = 3u^2 u_t \\
= -3(u^3 u_x + u^2 u_{xxx}) \\
= J_x.
\]

Thus,

\[
J = -3 \int (u^3 u_x + u^2 u_{xxx}) dx.
\]

However, this integral can’t be written as a polynomial in $\{u, u_x, u_{xx}, \ldots\}$.
Some experimentation yields $\rho = u^3 - 3u_x^2$ and

\[
(u^3 - 3u_x^2)_t + \left(\frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{xx} + 3u_x^2 - 6u_xu_{xxx}\right)_x = 0.
\]

These are well-known and have been for some time (e.g., M. Kruskal’s “speech” during A. Newell’s talk).
Some experimentation yields \( \rho = u^3 - 3u_x^2 \) and

\[
\left( u^3 - 3u_x^2 \right)_t + \left( \frac{3}{4} u^4 - 6u u_x^2 + 3u^2 u_{xx} + 3u_x^2 - 6u_x u_{xxx} \right)_x \equiv 0.
\]

- These are well-known and have been for some time (e.g., M. Kruskal’s “speach” during A. Newell’s talk).
- It has been conjectured that any integrable PDE will have an infinite number of Lax pairs.
Comparison of Lax Pairs

Compare the zero-curvature equation and defining conservation law equation.

\[
X_t - T_x + [X, T] = 0 \quad \text{and} \quad \rho_t + J_x \dot{\rho} = 0.
\]

These are the same if we can make the commutator vanish from the zero-curvature equation,

\[
X_t - T_x = 0.
\]

How do we make the commutator vanish?
Relation to Lax Pairs

Compare the zero-curvature equation and defining conservation law equation

\[ X_t - T_x + [X, T] = 0 \quad \text{and} \quad \rho_t + J_x = 0. \]
Relation to Lax Pairs

Compare the zero-curvature equation and defining conservation law equation

\[ X_t - T_x + [X, T] \dot{=} 0 \quad \text{and} \quad \rho_t + J_x \dot{=} 0. \]

These are the same if we can make the commutator vanish from the zero-curvature equation,

\[ X_t - T_x = 0. \]
Relation to Lax Pairs

Compare the zero-curvature equation and defining conservation law equation

\[ X_t - T_x + [X, T] = 0 \quad \text{and} \quad \rho_t + J_x = 0. \]

These are the same if we can make the commutator vanish from the zero-curvature equation,

\[ X_t - T_x = 0. \]

How do we make the commutator vanish?
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.

Let $S$ be a power series in $\lambda$,

$$S = I + \sum_{i=1}^{\infty} \Gamma_i \lambda^{-i},$$

for unknown (off-diagonal) matrices $\Gamma_i$.

Define $\tilde{X} = SXS^{-1}$ as

$$\tilde{X} = D_x - X_1 \lambda + \sum_{i=0}^{\infty} P_i \lambda^{-i},$$

for undetermined (diagonal) matrices, $P_i$ and $D_x = \frac{\partial}{\partial x} I$.

Finally, set $\tilde{T} = S T S^{-1}$.
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.

Let $S$ be a power series in $\lambda$,

$$S = I + \sum_{i=1}^{\infty} \Gamma_i \lambda^{-i},$$

for unknown (off-diagonal) matrices $\Gamma_i$.

Define $\tilde{X} = SXS^{-1}$ as

$$\tilde{X} = D_x - X_1 \lambda + \sum_{i=0}^{\infty} P_i \lambda^{-i},$$

for undetermined (diagonal) matrices, $P_i$ and $D_x = \frac{\partial}{\partial x} I$.

Finally, set $\tilde{T} = S T S^{-1}$. 

Jacob Rezac

Computation of Lax Pairs
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.

Let $S$ be a power series in $\lambda$,

$$S = I + \sum_{i=1}^{\infty} \Gamma_i \lambda^{-i},$$

for unknown (off-diagonal) matrices $\Gamma_i$. Define $\tilde{X} = SXS^{-1}$ as
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.

Let $S$ be a power series in $\lambda$,

$$S = I + \sum_{i=1}^{\infty} \Gamma_i \lambda^{-i},$$

for unknown (off-diagonal) matrices $\Gamma_i$. Define $\tilde{X} = SXS^{-1}$ as

$$\tilde{X} = D_x - X_1 \lambda + \sum_{i=0}^{\infty} P_i \lambda^{-i},$$

for undetermined (diagonal) matrices, $P_i$ and $D_x = \frac{\partial}{\partial x} I$. 
Drinfel’d-Sokolov Method

If we know a Lax pair, expand $X$ as

$$X = X_0 - X_1 \lambda,$$

for $X_0$ off-diagonal and $X_1$ diagonal.

Let $S$ be a power series in $\lambda$,

$$S = I + \sum_{i=1}^{\infty} \Gamma_i \lambda^{-i},$$

for unknown (off-diagonal) matrices $\Gamma_i$. Define $\tilde{X} = SXS^{-1}$ as

$$\tilde{X} = D_x - X_1 \lambda + \sum_{i=0}^{\infty} P_i \lambda^{-i},$$

for undetermined (diagonal) matrices, $P_i$ and $D_x = \frac{\partial}{\partial x} I$.

Finally, set $\tilde{T} = STS^{-1}$.
This set-up will do two things:
This set-up will do two things:

- $[\tilde{X}, \tilde{T}] = 0$, so $\tilde{X}$ and $\tilde{T}$ are the (matrix) density and flux of a conservation law.
This set-up will do two things:

- \([\tilde{X}, \tilde{T}] = 0\), so \(\tilde{X}\) and \(\tilde{T}\) are the (matrix) density and flux of a conservation law.

- Equating powers of \(\lambda\) in the equation \(\tilde{X} = SXS^{-1}\) gives a recurrence relation between \(P_n\) and \(\Gamma_{n+1}\):
This set-up will do two things:

- $[\tilde{X}, \tilde{T}] = 0$, so $\tilde{X}$ and $\tilde{T}$ are the (matrix) density and flux of a conservation law.
- Equating powers of $\lambda$ in the equation $\tilde{X} = SXS^{-1}$ gives a recurrence relation between $P_n$ and $\Gamma_{n+1}$:

$$P_n + [\Gamma_{n+1}, X_1] = \Gamma_n X_0 - \Gamma'_n - \sum_{i=0}^{n-1} P_i \Gamma_{n-i},$$
This set-up will do two things:

- $[\tilde{X}, \tilde{T}] = 0$, so $\tilde{X}$ and $\tilde{T}$ are the (matrix) density and flux of a conservation law.
- Equating powers of $\lambda$ in the equation $\tilde{X} = SXS^{-1}$ gives a recurrence relation between $P_n$ and $\Gamma_{n+1}$:

$$P_n + [\Gamma_{n+1}, X_1] = \Gamma_n X_0 - \Gamma'_n - \sum_{i=0}^{n-1} P_i \Gamma_{n-i},$$

which will allow us to algorithmically solve for each $P_n$. 
Example

Consider again the KdV equation, with Lax pair $X = X_0 - X_1 \lambda$, 

\[ X_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X_1 = \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix}. \]
Consider again the KdV equation, with Lax pair \( X = X_0 - X_1 \lambda \),

\[
X_0 = \begin{bmatrix} 0 & u \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.
\]
Example

Consider again the KdV equation, with Lax pair \( X = X_0 - X_1 \lambda \),

\[
X_0 = \begin{bmatrix} 0 & u \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.
\]

Set \( P_0 \) as

\[
P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2} i [0 u_1], \quad P_1 = \frac{1}{2} i [u_0 0 u_0] \]

Similarly,

\[ \Gamma_2 = -\frac{1}{8} i [0 u_2 - u_{xx} u_0], \quad P_2 = -\frac{1}{4} [u_x 0 0 u] \]

So, the first few densities are

\[ \rho_1 = u, \quad \rho_2 = u_x, \quad \rho_3 = u_2, \quad \rho_5 = u_3 - 3 u_2 x, \ldots \]
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2} i \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{2} i \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}. \]
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2}i \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{2}i \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}. \]

Similarly,

\[ \Gamma_2 = -\frac{1}{8}i \begin{bmatrix} 0 & u^2 - u_{xx} \\ u & 0 \end{bmatrix} \quad \text{and} \quad P_2 = -\frac{1}{4} \begin{bmatrix} u_x & 0 \\ 0 & 0 \end{bmatrix}. \]
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2}i \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{2}i \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}. \]

Similarly,

\[ \Gamma_2 = -\frac{1}{8}i \begin{bmatrix} 0 & u^2 - u_{xx} \\ u & 0 \end{bmatrix} \quad \text{and} \quad P_2 = -\frac{1}{4} \begin{bmatrix} u_x & 0 \\ 0 & 0 \end{bmatrix}. \]

So, the first few densities are
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2} i \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{2} i \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}. \]

Similarly,

\[ \Gamma_2 = -\frac{1}{8} i \begin{bmatrix} 0 & u^2 - u_{xx} \\ u & 0 \end{bmatrix} \quad \text{and} \quad P_2 = -\frac{1}{4} \begin{bmatrix} u_x & 0 \\ 0 & 0 \end{bmatrix}. \]

So, the first few densities are

\[ \rho_1 = u, \quad \rho_2 = u_x. \]
Using the recurrence relationship, we find

\[ \Gamma_1 = \frac{1}{2} i \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{2} i \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}. \]

Similarly,

\[ \Gamma_2 = -\frac{1}{8} i \begin{bmatrix} 0 & u^2 - u_{xx} \\ u & 0 \end{bmatrix} \quad \text{and} \quad P_2 = -\frac{1}{4} \begin{bmatrix} u_x & 0 \\ 0 & 0 \end{bmatrix}. \]

So, the first few densities are

\[ \rho_1 = u, \quad \rho_2 = u_x. \]

Continuing, we find \( \rho_3 = u^2, \rho_5 = u^3 - 3u_x^2, \ldots \).
Finding the Flux

We could do the same process to find $J$ - but it’s tricky. We use simple integration.
Finding the Flux

We could do the same process to find $J$ - but it’s tricky. We use simple integration. Integrating our conservation law, we have

\[ \int \rho_t \, dx + \int J_x \, dx = 0. \]
We could do the same process to find $J$ - but it’s tricky. We use simple integration. Integrating our conservation law, we have

$$\int \rho_t \, dx + \int J_x \, dx = 0.$$
Finding the Flux

We could do the same process to find $J$ - but it’s tricky. We use simple integration. Integrating our conservation law, we have

$$\int \rho_t \, dx + \int J_x \, dx = 0.$$ 

Rearranging,

$$J = -\int \rho_t \, dx.$$
Example

Continuing with the KdV equation, consider

\[ \rho_1 = u. \]

Then,

\[ J_1 = -\int u_t \, dx = \int -\alpha uu_x + u_{xxx} \, dx. \]

So,

\[ J_1 = -\frac{1}{2} \alpha u^2 + u_{xx}. \]
Example

Continuing with the KdV equation, consider

\[ \rho_1 = u. \]
Example

Continuing with the KdV equation, consider

$$\rho_1 = u.$$
Example

Continuing with the KdV equation, consider

\[ \rho_1 = u. \]

Then,

\[ J_1 = -\int u_t \, dx = \int -\alpha uu_x + u_{xxx} \, dx. \]

So,
Continuing with the KdV equation, consider

$$\rho_1 = u.$$  

Then,

$$J_1 = - \int u_t \, dx = \int -\alpha uu_x + u_{xxx} \, dx.$$  

So,

$$J_1 = -\frac{1}{2} \alpha u^2 + u_{xx}.$$
Software Demonstration
Issues

- Only works when \( X = X_0 - X_1 \lambda \)
Issues

- Only works when $X = X_0 - X_1 \lambda$
- Produces non-standard conservation laws
Only works when $X = X_0 - X_1 \lambda$

Produces non-standard conservation laws

Every other conservation law is trivial
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree.
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:
Conclusions

Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
Conclusions

Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
Conclusions

Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
  - with a number of possible forms
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
  - with a number of possible forms
- Build conservation laws given Lax pairs
Conclusions

Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for \textit{scaling invariant} equations
  - with a number of possible forms
- Build conservation laws given Lax pairs
  - of a \textit{specific form}
Conclusions

Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
  - with a number of possible forms

- Build conservation laws given Lax pairs
  - of a *specific form*
  - which help to narrow down possible Lax pairs
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
  - with a number of possible forms
- Build conservation laws given Lax pairs
  - of a *specific form*
  - which help to narrow down possible Lax pairs
- Despite the constraints, our methods are reliable and work on a large class of physically important equations.
Our goal was to generate Lax pairs and conservation laws easily and algorithmically - this has been done to some degree. We can:

- Construct Lax pairs
  - for *scaling invariant* equations
  - with a number of possible forms
- Build conservation laws given Lax pairs
  - of a *specific form*
  - which help to narrow down possible Lax pairs
- Despite the constraints, our methods are reliable and work on a large class of physically important equations.

In the future, we hope to expand these methods to a larger class of integrable equations.
Thanks!
Example: Short Pulse Equation

The Short Pulse Equation (SPE),

\[ u_{xt} + \alpha u + \beta (u^3)_{xx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \quad W(\alpha) = 4 \]

has a Lax pair with \( X \)-component

\[ X_{\text{SPE}} = \begin{bmatrix} \lambda & \lambda u_x \\ \frac{1}{\alpha} \lambda u_x & \lambda \end{bmatrix} \]
Example: Short Pulse Equation

The Short Pulse Equation (SPE),

\[ u_{xt} + \alpha u + \beta (u^3)_{xx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \quad W(\alpha) = 4 \]

has a Lax pair with \( X \)-component

\[ X_{SPE} = \begin{bmatrix} \lambda & \lambda u_x \\ \frac{1}{\alpha} \lambda u_x & \lambda \end{bmatrix} \]

This can be constructed with the method proposed in this talk.
The Short Pulse Equation (SPE),

\[ u_{xt} + \alpha u + \beta (u^3)_{xx} = 0, \]

with weights

\[ W \left( \frac{\partial}{\partial x} \right) = 1 \quad W \left( \frac{\partial}{\partial t} \right) = 3 \quad W(u) = 1 \quad W(\alpha) = 4 \]

has a Lax pair with \( X \)-component

\[ X_{SPE} = \begin{bmatrix} \lambda & \lambda u_x \\ \frac{1}{\alpha} \lambda u_x & \lambda \end{bmatrix} \]

- This can be constructed with the method proposed in this talk.
- We cannot build conservation laws from it (it is in WKI-form rather than AKNS-form).